Astrophysical Hydrodynamics – Assignment 4

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In this assignment we will solve Laplace's Partial Differential Equation (PDE) in spherical polar coordinates using the method of separation of variables. These type of problems are known as boundary value problems and arise in diverse areas of Physics, starting from Electrostatics to Quantum Mechanics. Then we make use of plotting softwares like, matplotlib to visualize the solutions.

1 The Immersed Moving Sphere

Consider a sphere with radius a which moves with a velocity u in an ideal, incompressible fluid, e.g. a protoplanet swiping through the planetary disk or a person wading through a crowded tunnel. We will estimate the velocity of the fluid, $\boldsymbol{v}(\boldsymbol{x})$, induced by the sphere. Ignore gravity (or take gravity constant in the volume). If we assume that the vorticity vanishes, $\boldsymbol{\nabla} \times \boldsymbol{v} = 0$, then fluid motion is a potential flow so $\boldsymbol{v}$ can be written in terms of a potential $\boldsymbol{v} = \boldsymbol{\nabla}\Psi$. The aim of this exercise is to find this potential by use of Laplace's equation

$$\nabla^2 \Psi = 0\,, \tag{1}$$

combined with the boundary conditions

$$\lim_{r\to\infty} \Psi = 0\,, \tag{2}$$

$$v_N(a) = u_N\,, \tag{3}$$

with v_N the velocity component in the direction of the normal of the sphere.

1. Using spherical coordinates, write down the Laplace equation. Then use the *ansatz* $\Psi = R(r)\Theta(\theta)$ to split Laplace's equation into a radial and an angular part. Both sides of the equation you then end up with must be a constant of opposite sign i.e. the equation you will find is of the form

$$-f(\Theta;\theta) = g(R;r) = \text{constant}\,. \tag{4}$$

2. Focus on the angular side of the equation. Solutions for Θ can be found from the boundary condition on the sphere, since

$$v_r = \partial_r \Psi = \Theta(\theta)\partial_r R(r)\,. \tag{5}$$

   Thus, how should $v_r(a)$ vary with θ in order to satisfy the boundary conditions?

3. Focus on the radial part. Try a power law solution,

$$R(r) = Br^{-n}\,. \tag{6}$$

   Why should n be positive? Solve the radial equation and determine n.

4. Finally, write down the potential Ψ in terms of r and θ. Apply the boundary conditions to derive the proportionality constant(s).

In the next exercise we will continue with this physical situation.

2 Streamlines of the Immersed Moving Sphere

In the previous exercise, we solved the flow $\boldsymbol{v}$ around a sphere with radius R moving with velocity u through a fluid. The solution for the potential in the (average) restframe of the fluid was

$$\Psi = -\frac{1}{2}R^3 r^{-2} u \cos\theta \,, \tag{7}$$

with θ the zenith or colatitude w.r.t. the direction of movement of the sphere and r the radial distance to the sphere. In this exercise we will plot the streamlines of the fluid. This is best done in the restframe of the sphere, with the fluid flowing around it. This means that the velocity of the fluid at infinity is $-u$ and the potential becomes

$$\Psi = -\left(\frac{1}{2}R^3 r^{-2} + r\right) u \cos\theta \,. \tag{8}$$

1. Streamlines are defined by the equation

$$\frac{dy}{dx} = \frac{v_y}{v_x} \tag{9}$$

   or equivalently in spherical coordinates by

$$\frac{r d\theta}{dr} = \frac{v_\theta}{v_r} \,. \tag{10}$$

   Give an expression for the streamlines in terms of r and θ, i.e. integrate eqn. (10).

2. Consider a streamline through the point ($r = r_0$, $\theta = \pi/2$), with $r_0 > R$. Show this streamline obeys

$$\sin^2\theta = \frac{r}{r_0}\frac{r_0^3 - R^3}{r^3 - R^3} \tag{11}$$

   and argue that $r > r_0$.

3. Plot (using Python, Gnuplot, Matlab, Mathematica, or any other visualization software) the streamlines passing the sphere at e.g. $r_0 = \{1.01R, 1.1R, 1.5R\}$. Please send me the plotting code and images via email.

References

1. Arfken G. B. & Weber H. J., Mathematical Methods for Physicists, Elsevier
2. Landau L.D. and Lifshitz E. M., Course of Theoretical Physics, Vol. 6: Fluid Mechanics