Astrophysical Hydrodynamics – Assignment 3

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February 21, 2017

The main goal of this assignment is to get used to with the curvilinear co-ordinate systems. So, the first problem is for revising the properties of coordinate transformations, differential vector operators and so on so forth. Next two problems of this assignment are continuation of isentropic, non-viscid fluid flow from previous problem set. In the fourth problem we see how Euler's equation changes in the presence of magnetic field and we derive a fundamental equation of MHD. Then we consider hydrostatics and it's application to stellar structure.

1 Vector analysis in curvilinear coordinates

In transforming from Cartesian coordinates (x, y, z) to a general coordinate (q_1, q_2, q_3), the square of the differential distance element can be written as:

$$ds^2 = d\boldsymbol{r} \cdot d\boldsymbol{r} = d\boldsymbol{r}^2 = \sum_{ij} \frac{\partial \boldsymbol{r}}{\partial q_i} \cdot \frac{\partial \boldsymbol{r}}{\partial q_j} dq_i dq_j = \sum_{ij} g_{ij} dq_i dq_j \tag{1}$$

here g_{ij} is called *metric* which describes the transformation from Cartesian coordinates to another set of coordinates q_1, q_2 and q_3:

$$g_{ij}(q_1, q_2, q_3) = \frac{\partial x}{\partial q_i}\frac{\partial x}{\partial q_j} + \frac{\partial y}{\partial q_i}\frac{\partial y}{\partial q_j} + \frac{\partial z}{\partial q_i}\frac{\partial z}{\partial q_j} = \frac{\partial \boldsymbol{r}}{\partial q_i} \cdot \frac{\partial \boldsymbol{r}}{\partial q_j}\,. \tag{2}$$

We will only consider orthogonal systems, which means that the metric simplifies to a diagonal tensor g_{ii}:

$$g_{ii} = \left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^2 . \tag{3}$$

We further define the scale factors $h_i = \sqrt{g_{ii}}$. Now we can write down a number of results for general coordinate transformations of this form:

$$d\boldsymbol{r} = \sum_i h_i dq_i \hat{\boldsymbol{q}}_i \tag{4}$$

$$ds^2 = \sum_i (h_i dq_i)^2 \tag{5}$$

$$dS_{ij} = h_i h_j dq_i dq_j \tag{6}$$

$$dV = h_1 h_2 h_3 dq_1 dq_2 dq_3\,. \tag{7}$$

In the transformed orthogonal curvilinear coordinate system the gradient, divergence, Laplacian and curl are written as:

$$\nabla\phi = \sum_i \frac{1}{h_i}\frac{\partial\phi}{\partial q_i}\hat{\boldsymbol{q}}_i \tag{8}$$

$$\nabla\cdot\boldsymbol{V}(q_1,q_2,q_3) = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}(V_1h_2h_3) + \frac{\partial}{\partial q_2}(V_2h_3h_1) + \frac{\partial}{\partial q_3}(V_3h_1h_2)\right] \tag{9}$$

$$\nabla^2\phi(q_1,q_2,q_3) = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\phi}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial q_3}\right)\right] \tag{10}$$

$$\nabla\times\boldsymbol{V}|_i = \epsilon_{ijk}\frac{1}{h_jh_k}\left[\frac{\partial}{\partial q_j}(h_kV_k) - \frac{\partial}{\partial q_k}(h_jV_j)\right]\hat{\boldsymbol{q}}_i\,. \tag{11}$$

Determine the scale factors and write down the line element, surface element, volume element and vector differential operators in these following two coordinate systems:

1. Circular cylindrical coordinates:

$$\rho = \sqrt{x^2+y^2} \qquad x = \rho\cos\phi \tag{12}$$

$$\phi = \arctan\frac{y}{x} \qquad y = \rho\sin\phi \tag{13}$$

$$z = z \qquad z = z\,, \tag{14}$$

   where $(0 \le \rho < \infty)$, $(0 \le \phi \le 2\pi)$ and $(-\infty < z < \infty)$.

2. Spherical polar coordinate system:

$$r = \sqrt{x^2+y^2+z^2} \qquad x = r\sin\theta\cos\phi \tag{15}$$

$$\theta = \arccos\frac{z}{\sqrt{x^2+y^2+z^2}} \qquad y = r\sin\theta\sin\phi \tag{16}$$

$$\phi = \arctan\frac{y}{x} \qquad z = r\cos\theta\,, \tag{17}$$

   where $(0 \le r < \infty)$, $(0 \le \theta \le \pi)$ and $(0 \le \phi \le 2\pi)$.

2 Kelvin's circulation theorem

The circulation Γ around a closed curve C is defined as follows:

$$\Gamma = \oint_C \boldsymbol{u}\cdot\boldsymbol{dr} \tag{18}$$

1. If the curve C moves with the fluid (assumed to be inviscid and isentropic), show that Γ is a constant:

$$\frac{D\Gamma}{Dt} = 0 \tag{19}$$

   This is known as "Kelvin's circulation theorem".

3 Rotating cylindrical fluid

1. This problem is about a smooth circular cylinder of radius a and height h that contains a fluid of uniform density ρ_c, rotating uniformly with angular velocity Ω about its axis of symmetry. Compute the vorticity $\boldsymbol{\omega}$.

2. Show that in cylindrical polar coordinates, (ρ, ϕ, z), the velocity field given by

$$\boldsymbol{u} = \left(0, \rho\Omega(t), \frac{z}{h}\right), \tag{20}$$

applied for an appropriate time, describes a stretching of the cylinder to height of $2h$ while keeping the density and rotation uniform.

For incompressible flow it is found that stretching vortex lines leads to an increase in their strength. Use the vorticity equation to show that this is not the case here.

3. Show that the flow field given by

$$\boldsymbol{u} = \left(-\frac{\rho}{a}, \rho\Omega(t), 0\right), \tag{21}$$

applied for an appropriate time, describes decreasing the radius of the cylinder to $\frac{a}{2}$ while keeping the density and the rotation uniform. Use the vorticity equation to show that in this case the vorticity does change.

4 Magnetohydrodynamics: Cauchy momentum equation

Including the Lorentz force term $\boldsymbol{J} \times \boldsymbol{B}$, Euler's equation in the presence of magnetic field becomes,

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{\nabla P}{\rho} + \boldsymbol{f} + \frac{\boldsymbol{J} \times \boldsymbol{B}}{\rho} \tag{22}$$

1. Using Maxwell's equation show that (neglect displacement current) the above equation can be written in the following form:

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \boldsymbol{f} - \frac{1}{\rho}\nabla\left(P + \frac{B^2}{2\mu_0}\right) + \frac{(\boldsymbol{B} \cdot \nabla)\boldsymbol{B}}{\mu_0 \rho} \tag{23}$$

The term $\frac{B^2}{2\mu_0}$ corresponds to magnetic pressure, introduced by the magnetic field. The last term above is related to the tension along the magnetic field lines, that is, the magnetic field is associated to an isotropic pressure and also to a tension along the field lines.

5 Hydrostatics: Stellar structure

This question gives some examples of static fluids (no velocities). In that case, Euler's equation becomes

$$\boldsymbol{\nabla} P = \rho \mathrm{g}\,, \tag{24}$$

where P is the pressure of the fluid and $\rho\mathrm{g}$ the force acting on the fluid.

1. Consider a very large, very massive fluid such that self-gravity becomes important (any idea what we are talking about here?). The Poisson equation – familiar from mechanics, stellar dynamics and/or stellar evolution courses – relates the gravitational potential ϕ to the density ρ

$$\nabla^2 \phi(\boldsymbol{r}) = 4\pi G_N \rho(\boldsymbol{r})\,, \tag{25}$$

where G_N is Newton's constant. What is $\boldsymbol{g}$ in this scenario? Use the assumption of spherical symmetry to find (one of) the equations of stellar structure

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G_N \rho\,. \tag{26}$$

N.B. It's not only in stars that this equation is used. Another example are clusters of galaxies. Then ρ is the total matter density: baryonic (stellar, gas) and dark.

2. In order to solve this, we need a relation between P and ρ, called the *equation of state*. Often this is a polytropic relation

$$P \propto \rho^{1+1/n}\,, \tag{27}$$

where n is called the polytropic index. Low mass white dwarf stars are well approximated as $n = 1.5$ polytropes, red giants with $n = 3$, the giant planets Jupiter and Saturn with $n = 1$ and the small planets with a constant density (no relation between P and ρ). Assume a planet or a star has radius R and mass M. The pressure at the center is P_c, and the pressure at the surface is $P_R = 0$ (which in fact can be used as a definition of the surface).

Show that for the gaseous planets, $P = \alpha\rho^2$ the above differential equation turns into the following form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\rho}{\partial r}\right) + \beta^2\rho = 0\,. \tag{28}$$

where $\beta^2 = \frac{2\pi G_N}{\alpha}$.

3. Derive an expression for the pressure $P(r)$ in Mercury as function of radius in terms of M and R (and G etc.). What is the central pressure?

References

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