A REVIEW OF GALACTIC DYNAMICS

P.C. van der Kruit

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Contents

1 Introduction.

The following as a description of fundamentals, methods and results in galactic dynamics. It was originally written (around 1990) as a summary of the principles and fundamental equations for students following my course on galaxies to give them the necessary background in case they had not yet had a course in galactic dynamics. Also it serves a reference to have important equations and results at hand.

I use the notation of the Saas-Fee Course *"The Milky Way as a Galaxy"* by Gerry Gilmore, Ivan King and myself. The text is not always original and has been taken or adapted from various sources. In addition to "The Milky Way as a Galaxy", I have used "Galactic Dynamics" by James Binney and Scott Tremaine, Oort's chapter on "Stellar Dynamics" in Stars and Stellar Systems, **V**, and various other publications and friends' lecture notes.

As a result of the origin of this text, the treatment is restricted to dynamics of flattened systems with axial symmetry and therefore generally aimed at disk galaxies. A general treatment of tri-axial systems is beyond the current scope of this overview, but may be attempted some time in the future.

2 The fundamentals.

2.1 The continuity equation.

Studies of galactic dynamics start with two fundamental equations. The first is the continuity equation, also called the *Liouville* or *collisionless Boltzman equation*. It states that obviously in any element of phase space the time derivative of the distribution function equals the number of stars entering it minus that leaving it, if no stars are created or destroyed. If the distribution function is $f(x,y,z,u,v,w,t)$ and Φ the potential then

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} - \frac{\partial \Phi}{\partial x}\frac{\partial f}{\partial u} - \frac{\partial \Phi}{\partial y}\frac{\partial f}{\partial v} - \frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial w} = 0. \tag{2.1}$$

There is another way of stating this, which does give some fundamental insight. Consider the equations of motion of an individual star:

$$\frac{dx}{dt} = u,\ \frac{dy}{dt} = v,\ \frac{dz}{dt} = w,\ \frac{du}{dt} = -\frac{\partial \Phi}{\partial x},\ \frac{dv}{dt} = -\frac{\partial \Phi}{\partial y},\ \frac{dw}{dt} = -\frac{\partial \Phi}{\partial z}. \tag{2.2}$$

We see by comparing these two equations that along the path of any star in phase space the total derivative of f is zero, or in other words the density in phase space is constant along any path a star can follow. So the flow of stars in 6-dimensional phase space is incompressible.

In most applications the system is assumed to be in equilibrium, so that f is not a function of time and the first term equals zero. The usual method of solving an equation like the Liouville equation is to form "subsidiary" equations, which in this case are simply the equations of motion of an individual star, that were already given above. These can be rearranged as follows

$$(dt =)\ \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = \frac{du}{-\partial \Phi/\partial x} = \frac{dv}{-\partial \Phi/\partial y} = \frac{dw}{-\partial \Phi/\partial z}. \tag{2.3}$$

The solution of these 5 independent ordinary differential equations is in the form of 5 independent integrals and then the general solution of the Liouville equation can be written as

$$f(x,y,z,u,v,w) = F(I_1, I_2, ..., I_5). \tag{2.4}$$

The I's are called the integrals of motion. So we see that the distribution function depends *only* on the integrals of motion. One can be identified as the energy, which is always conserved along an orbit:

$$I_1 = E = 1/2\ (u^2 + v^2 + w^2) + \Phi(x,y,z) = \text{constant}. \tag{2.5}$$

This is called an *isolating* integral of motion, because for particular values it isolates hyper-surfaces in phase space. The others in general are non-isolating and are only implicit in the numerical integration of an orbit. The important fact then is that the distribution function can be written as a function only of the isolating integrals. This is so, because in a steady state at any point on this hyper-surface, f must assume the same value, if a star with that energy is able to come in its orbit arbitrarily close to that point. This important statement is a fundamental one in galactic dynamics and is called *Jeans' theorem.*

In cylindrical coordinates the distribution function is $f(R,\theta,z,U,V,W,t)$ and the Liouville equation becomes

$$\frac{\partial f}{\partial t} + U\frac{\partial f}{\partial R} + \frac{V}{R}\frac{\partial f}{\partial \theta} + W\frac{\partial f}{\partial z} + \left(\frac{V^2}{R} - \frac{\partial \Phi}{\partial R}\right)\frac{\partial f}{\partial U} - \left(\frac{UV}{R} + \frac{1}{R}\frac{\partial \Phi}{\partial \theta}\right)\frac{\partial f}{\partial V} - \frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial W} = 0. \tag{2.6}$$

In the time-independent case, which I will consider exclusively from here on, and for axial symmetry this becomes

$$U\frac{\partial f}{\partial R} + W\frac{\partial f}{\partial z} - \left(\frac{\partial \Phi}{\partial R} - \frac{V^2}{R}\right)\frac{\partial f}{\partial U} - \frac{UV}{R}\frac{\partial f}{\partial V} - \frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial W} = 0. \tag{2.7}$$

In the literature one also often finds the velocities (U, V, W) notated as $(V_{\rm R}, V_\theta, V_{\rm z})$, while an old notation for R is ϖ, pronounced as *"pomega"* or *"curled pi"*.

In this case there is a second isolating integral, because the angular momentum in the direction of the symmetry axis z is also conserved along an orbit. This integral is

$$I_2 = J = RV. \tag{2.8}$$

Then Jeans' theorem can be written as

$$f(R, z, U, V, W) = F(E, J). \tag{2.9}$$

In the case of the Galaxy near the plane the R- and z-motions are likely to be decoupled. This means that the potential can be written as

$$\Phi(R, z) = \Phi_1(R) + \Phi_2(z) \tag{2.10}$$

and then there is a third integral for the z-motions:

$$I_3 = 1/2\, W^2 + \Phi_2(z), \tag{2.11}$$

which is a decoupled z-energy. There has been a long-standing problem with the question, whether in the Galaxy there is a third integral for the stellar motions. At small z it is approximately true, since the potential is separable in r and z, but for higher velocities it has only been possible to find a better approximate analytical description with the use of Stäckel potentials (see below).

2.2 Poisson's equation.

The second fundamental equation is Poisson's equation, which says that the gravitational potential derives from the combined gravitational forces of all the matter. It can be written as

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} \equiv \nabla^2 \Phi = 4\pi G \rho(x, y, z), \tag{2.12}$$

or in cylindrical coordinates

$$\frac{\partial^2 \Phi}{\partial R^2} + \frac{1}{R}\frac{\partial \Phi}{\partial R} + \frac{1}{R^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial z^2} = 4\pi G \rho(R, \theta, z). \tag{2.13}$$

For an axisymmetric case this reduces to (in cylindrical coordinates)

$$\frac{\partial K_{\rm R}}{\partial R} + \frac{K_{\rm R}}{R} + \frac{\partial K_{\rm z}}{\partial z} = -4\pi G \rho(R, z). \tag{2.14}$$

For completeness I also give these basic equations for the case of spherical symmetry, where we now have velocities $V_{\rm R}$, V_θ and V_ϕ:

$$\frac{\partial}{\partial R}(\nu\langle V_{\rm R}^2\rangle) + \frac{\nu}{R}\{2\langle V_{\rm R}^2\rangle - V_{\rm t}^2 - \langle(V_\theta - V_{\rm t})^2\rangle - \langle V_\phi^2\rangle\} = \nu K_{\rm R} \tag{2.15}$$

and

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial\Phi}{\partial R}\right) = 4\pi G\rho(R). \tag{2.16}$$

For plane-parallel layers the basic equations reduce to

$$\frac{d}{dz}\left\{\nu\langle W^2\rangle\right\} = \nu K_{\rm z} \tag{2.17}$$

and

$$\frac{dK_{\rm z}}{dz} = -4\pi G\rho(z). \tag{2.18}$$

In all of the above ν is the space density of a particular component. In case that we have to do with a self-gravitating system, ν can be equated to the total density ρ, that gives rise to the gravitational force through Poisson's equation. An important problem is to develop techniques to invert Poisson's equation, so that the potential can be calculated for a given density distribution (see below).

The two fundamental equations together completely describe the dynamics of a system. Especially when it is self-gravitating in principle a distribution function can be found that satisfies both equations. In practice this is seldom possible. Given a density distribution it is also often the case that no unique solution exists to the full distribution function. For special cases (such as isothermal models) solutions to the set of the two equations can be derived, but this is at the expense of making simplifying assumptions and loss of generality.

2.3 Hydrodynamical equations.

From the collisionless Boltzman equation follow the moment or hydrodynamical equations, which are obtained by multiplying the Liouville equation by a velocity-component (e.g. U) and then integrating over all velocities. For the radial direction we then find:

$$\frac{\partial}{\partial R}(\nu\langle U^2\rangle) + \frac{\nu}{R}\{\langle U^2\rangle - V_{\rm t}^2 - \langle(V - V_{\rm t})^2\rangle\} + \frac{\partial}{\partial z}(\nu\langle UW\rangle) = \nu K_{\rm R}. \tag{2.19}$$

By assumption we have taken here $V_{\rm t} = \langle V\rangle$ and $\langle U\rangle = \langle W\rangle = 0$. This can be rewritten as:

$$-K_{\rm R} = \frac{V_{\rm t}^2}{R} - \langle U^2\rangle\left[\frac{\partial}{\partial R}(\ln\nu\langle U^2\rangle) + \frac{1}{R}\left\{1 - \frac{\langle(V - V_{\rm t})^2\rangle}{\langle U^2\rangle}\right\}\right] + \langle UW\rangle\frac{\partial}{\partial z}(\ln\nu\langle UW\rangle). \tag{2.20}$$

The last term reduces in the symmetry plane to

$$\langle UW\rangle\frac{\partial}{\partial z}(\ln\nu\langle UW\rangle) = \frac{\partial}{\partial z}\langle UW\rangle \tag{2.21}$$

and may then be assumed zero. Note that in the literature we often find velocity dispersions notated as $\sigma_{\rm RR}$ for $\langle U^2\rangle^{1/2}$, etc. Since σ is the often used notation for the dispersion of a Gaussian and since the actual velocity distributions are not necessarily Gaussian, I prefer the notation as used above. Also the moment equations are independent of the higher order moments of the velocity distribution.

For the azimuthal direction the moment equation is seldom used, because it only contains cross-terms of the velocity tensor. It reads

$$\frac{2\nu}{R}\langle UV\rangle + \frac{\partial}{\partial R}(\nu\langle UV\rangle) + \frac{\partial}{\partial z}(\nu\langle VW\rangle) = 0. \tag{2.22}$$

In the vertical direction the moment equation becomes

$$\frac{\partial}{\partial z}(\nu\langle W^2\rangle) + \frac{\nu\langle UW\rangle}{R} + \frac{\partial}{\partial R}(\nu\langle UW\rangle) = \nu K_{\mathrm{z}}. \tag{2.23}$$

Equations (2.20), (2.23) and (2.14) together are the basic equations to describe the dynamics of axisymmetric systems and are therefore the starting point of any discussion of disk dynamics.

2.4 Virial theorem.

Alternatively, we may also multiply the collisionless Boltzman equation by a spatial coordinate and integrate over space. This second procedure also yields some fundamental insight in the global stability of self-gravitating systems. Write the coordinates as x_{i} with $i = 1, 2, 3$ and the position vector as $\mathbf{x}$. Then we get the tensor equation

$$\int x_{\mathrm{k}} \frac{\partial(\rho\overline{V_{\mathrm{j}}})}{\partial t} d^3\mathbf{x} = -\sum_{i=1}^{3} \int x_{\mathrm{k}} \frac{\partial(\rho\overline{V_{\mathrm{j}}V_{\mathrm{i}}})}{\partial x_{\mathrm{i}}} d^3\mathbf{x} - \int \rho x_{\mathrm{k}} \frac{\partial \Phi}{\partial x_{\mathrm{j}}} d^3\mathbf{x}. \tag{2.24}$$

The first term on the right can be written as the kinetic energy tensor

$$-\sum_{i=1}^{3} \int x_{\mathrm{k}} \frac{\partial(\rho\overline{V_{\mathrm{i}}V_{\mathrm{j}}})}{\partial x_{\mathrm{i}}} d^3\mathbf{x} = \sum_{i=1}^{3} \int \delta_{\mathrm{ki}} \rho \overline{V_{\mathrm{i}}V_{\mathrm{j}}} d^3\mathbf{x} = 2T_{\mathrm{kj}} = 2T_{\mathrm{jk}}, \tag{2.25}$$

which has a systematic and a random part

$$T_{\mathrm{jk}} = S_{\mathrm{jk}} + \tfrac{1}{2}R_{\mathrm{jk}} = \tfrac{1}{2}\int \rho\langle V_{\mathrm{j}}\rangle\langle V_{\mathrm{k}}\rangle d^3\mathbf{x} + \int \rho\langle V_{\mathrm{j}}V_{\mathrm{k}}\rangle d^3\mathbf{x}. \tag{2.26}$$

The second term on the right is the potential energy tensor

$$\Omega_{\mathrm{jk}} = \Omega_{\mathrm{kj}} = -\int \rho x_{\mathrm{j}} \frac{\partial \Phi}{\partial x_{\mathrm{k}}} d^3\mathbf{x}. \tag{2.27}$$

Average over the kj and jk components, then

$$\tfrac{1}{2}\frac{d}{dt}\int \rho(x_{\mathrm{k}}\langle V_{\mathrm{j}}\rangle + x_{\mathrm{j}}\langle V_{\mathrm{k}}\rangle) d^3\mathbf{x} = 2S_{\mathrm{jk}} + R_{\mathrm{jk}} + \Omega_{\mathrm{jk}}. \tag{2.28}$$

Define the inertia tensor as

$$I_{\mathrm{jk}} = \int \rho x_{\mathrm{j}} x_{\mathrm{k}} d^3\mathbf{x}. \tag{2.29}$$

Using the continuity equation if can be shown that

$$\frac{dI_{\mathrm{jk}}}{dt} = \int \rho(x_{\mathrm{k}}\langle V_{\mathrm{j}}\rangle + x_{\mathrm{j}}\langle V_{\mathrm{k}}\rangle) d^3\mathbf{x}, \tag{2.30}$$

and the result is the *tensor virial theorem*

$$\tfrac{1}{2}\frac{d^2 I_{\mathrm{jk}}}{dt^2} = 2S_{\mathrm{jk}} + R_{\mathrm{jk}} + \Omega_{\mathrm{jk}}. \tag{2.31}$$

For a system in equilibrium I_{jk} is time-independent. Taking the trace of the tensor equation yields the scalar virial theorem, which for a stable system is

$$2T + \Omega = 0. \tag{2.32}$$

2.5 Virialization and two-body relaxation time.

In all of the above we have taken a system in equilibrium. There are various processes for a galaxy to come into equilibrium and “virialize”, which means that the stellar velocity distribution randomizes. We can estimate the relaxation time due to two-body encounters as follows. Suppose that we have a cluster of radius R and mass M, made up of N stars with mass m, moving with a mean velocity V. If two stars pass at a distance r then the acceleration is about Gm/r^2. It lasts for about the period when the two stars are a distance r from the closest approach and therefore for a time $2r/V$. The total change in V^2 is then

$$\Delta V^2 \sim \left(\frac{2Gm}{rV}\right)^2. \tag{2.33}$$

The largest value of r is obviously R. For the smallest, one usually takes that where ΔV^2 is equal to V^2 itself, since then the approximation breaks down. It is not critical, since we will need the logarithm of the R divided by this $r_{\rm min}$. So we will take

$$r_{\rm min} = \frac{2Gm}{V^2}. \tag{2.34}$$

The density of stars is $3N/4\pi R^3$ and the surface density $3N/4\pi R^2 \sim N/\pi R^2$. The number of stars with impact parameter r is then the surface density times $2\pi r dr$. After crossing the cluster once the star has encountered all others and we can calculate the total change in V^2 by integrating over all r to find

$$(\Delta V^2)_{\rm tot} = \int_{r_{\rm min}}^{R} \left(\frac{2Gm}{rV}\right)^2 \frac{2Nr}{R^2} dr = \left(\frac{2Gm}{RV}\right)^2 2N \ln\Lambda, \tag{2.35}$$

where $\Lambda = R/r_{\rm min}$. The relaxation time is equal to the number of crossing times it takes for $(\Delta V^2)_{\rm tot}$ to be equal to V^2. Since a crossing time is of order R/V and since the virial theorem tells us that $V^2 \sim GNm/R$, we find

$$t_{\rm relax} \sim \frac{RN}{8V\ln\Lambda} \sim \left(\frac{R^3 N}{Gm}\right)^{1/2} \frac{1}{8\ln\Lambda}. \tag{2.36}$$

With the expression above for $r_{\rm min}$ we find that $\Lambda \sim N/2 \sim N$ and

$$t_{\rm relax} \sim \left(\frac{R^3}{GM}\right)^{1/2} \frac{N}{8\ln N}. \tag{2.37}$$

This ranges from about 10^9 years for globular clusters to 10^{12} years for clusters of galaxies. For galaxies we will need other mechanisms to stabilize them such as violent relaxation, where the stars virialize due to the rapidly changing gravitational potential field of the collapsing galaxy. The timescale of this is obiously the collapse time of the galaxy. Clusters of galaxies are probably not virialized and only approximately in equilibrium.

3 Stellar orbits.

3.1 Spherical potentials.

The equation of motion is in vector notation

$$\ddot{\mathbf{R}} = -\frac{d\Phi}{d\mathrm{R}}\hat{\mathbf{e}}_{\mathrm{R}}. \tag{3.1}$$

The angular momentum

$$\mathbf{R} \times \dot{\mathbf{R}} = \mathbf{L} \tag{3.2}$$

is constant and the orbit is in a plane. In polar coordinates in this plane we have

$$\ddot{R} - R\dot{\Phi}^2 = -\frac{d\Phi}{dR} \tag{3.3a}$$

$$R^2\dot{\Phi} = L. \tag{3.3b}$$

Integrating this we get

$$\tfrac{1}{2}\dot{R}^2 + \frac{1}{2}\frac{L^2}{R^2} = \Phi(R) = E, \tag{3.4}$$

where the energy E is constant. If $E < 0$ then the star is bound between radii $R_{\max}$ and $R_{\min}$, which are the roots of

$$\frac{1}{2}\frac{L^2}{R^2} + \Phi(R) = E. \tag{3.5}$$

The radial period is the time from $R_{\min}$ to $R_{\max}$ and back and follows from

$$T_{\mathrm{R}} = 2\int_{R_{\min}}^{R_{\max}} dt = 2\int_{R_{\min}}^{R_{\max}} \frac{dR}{\dot{R}} = 2\int_{R_{\min}}^{R_{\max}} \frac{dR}{\{2[E - \Phi(R)] - L^2/R^2\}^{1/2}}. \tag{3.6}$$

In the azimuthal direction the angle θ changes in the time T_{R} by

$$\Delta\theta = \int_0^{T_{\mathrm{R}}} \frac{d\theta}{dR} dR = 2\int_0^{T_{\mathrm{R}}} \left(\frac{L}{R^2}\right)\frac{dR}{\dot{R}}, \tag{3.7}$$

which can be evaluated further for the expression for T_{R} for a particular potential. The orbit is closed if

$$\Delta\theta = 2\pi\frac{m}{n}, \qquad (3.8$$

where m and n are integers. This is not generally true and the orbits then has the form of a rosette, which visits every point in $(R_{\min}, R_{\max})$. Even in the simple case of a spherical potential, the equation of motion of the orbit must be integrated numerically.

There are two special cases:

- *The harmonic oscillator*; this is the potential of a uniform sphere.

$$\Phi = \tfrac{1}{2}\Omega^2 R^2. \tag{3.9}$$

The orbits are closed ellipses centered on the origin and $\Delta\theta$ is equal to π in T_{R}.

- *The Keplerian potential*

$$\Phi = -\frac{GM}{R}. \tag{3.10}$$

The orbits are closed ellipses with one focus at the origin:

$$R = \frac{a(1-e^2)}{\{1+\cos(\theta-\theta_\circ)\}}, \tag{3.11}$$

where a and e are related to E and L by

$$a = \frac{L^2}{GM(1-e^2)}, \tag{3.12a}$$

$$E = -\frac{GM}{2a}. \tag{3.12b}$$

Further

$$R_{\rm max}, R_{\rm min} = a(1 \pm e) \tag{3.13}$$

and

$$T_{\rm R} = T_\theta = 2\pi\sqrt{\frac{a^3}{GM}} = T_{\rm R}(E). \tag{3.14}$$

Now $\Delta\theta = 2\pi$ in $T_{\rm R}$.

Galaxies have mass distributions somewhere between these two extremes, so we may expect that $\Delta\theta$ is in the range π to 2π in $T_{\rm R}$.

The Rosette orbit can be closed by observing it from a rotating frame (see below under resonances), when it is rotating at an angular velocity of

$$\Omega_{\rm p} = \frac{(\Delta\theta - 2\pi)}{T_{\rm R}}. \tag{3.15}$$

3.2 Axisymmetric potentials.

We now have a potential $\Phi = \Phi(R,z)$, that may be applicable to disk galaxies. The equations of motion now are

$$\ddot{R} - R\dot{\Phi}^2 = -\frac{\partial\Phi}{\partial R}, \tag{3.16a}$$

$$\frac{d}{dt}(R^2\dot{\Phi}) = 0, \tag{3.16b}$$

$$\ddot{z} = -\frac{\partial\Phi}{\partial z}. \tag{3.16c}$$

Integration of (3.16b) gives

$$L_{\rm z} = R^2\dot{\Phi}. \tag{3.17}$$

The motion in the meridional plane then can be described by an effective potential

$$\ddot{R} = -\frac{\partial\Phi_{\rm eff}}{\partial R}, \tag{3.18a}$$

$$\ddot{z} = -\frac{\partial\Phi_{\rm eff}}{\partial z}, \tag{3.18b}$$

where

$$\Phi_{\rm eff} = \Phi(R,z) + \frac{L_{\rm z}^2}{2R^2}. \tag{3.19}$$

The energy of the orbit is

$$E = \tfrac{1}{2}\dot{R}^2 + \tfrac{1}{2}\dot{z}^2 + \Phi_{\rm eff}(R,z). \tag{3.20}$$

The orbit is trapped *inside* the appropriate contour $E = \Phi_{\rm eff}$, which is called the zero-velocity curve. Only orbits with low $L_{\rm z}$ can approach the z-axis.

The minimum in $\Phi_{\rm eff}$ occurs for $\bigtriangledown\Phi_{\rm eff} = 0$, or at $z = 0$ and where

$$\frac{\partial\Phi}{\partial R} = \frac{L_{\rm z}^2}{R^3},$$

which corresponds to the circular orbit with $L = L_{\rm z}$. It si the highest angular momentum orbit that is possible for a given E, or in other words, it has all its kinetic energy in θ-motion.

3.3 Third integral and surface of section.

If E and $L_{\rm z}$ are the only two isolating integrals, the orbit would visit all points within the zero-velocity curves. In early numerical integrations it was found that there are limiting surfaces that seem to forbid the orbit to fill the whole volume within the zero-velocity curves. This behaviour is very common for orbits in axisymmetric potentials, when the combination $(E, L_{\rm z})$ is not too far form that of a circular orbit. A third integral is present, although in general its form cannot be explicitly written down.

For each orbit the energy $E(R, z, \dot{R}, \dot{z})$ is an integral, so only three of the four coordinates can be independent, say R, z and $\dot{R}$. The orbit can visit every point in $(R, z, \dot{R})$-space as far as allowed by E.

Now take a slice through $(R, z, \dot{R})$-space, say at $z = 0$. This is called a *surface of section*. The orbits' successive crossings of $z = 0$ generate a set of points inside the region $E = \frac{1}{2}\dot{R}^2 + \Phi_{\rm eff}(R, 0)$. If there is no other integral then these points fill the whole region. If there is another integral, then its surface $I_{\rm R}(R, z, \dot{R})$ cuts the plane in a *curve* $I_{\rm R}(R, 0, \dot{R})$ = constant. A *periodic* orbit is just a *point* (or a set of points) on the $(R, \dot{R})$ surface of section. Such curves and points are called *invariant*, because they are invariant under the mapping of the surface of section onto itself generated by the orbit.

Invariant points often have closed invariant curves around them on the surface of section. These represent *stable* periodic orbits. Ones where invariant curves cross are *unstable* periodic orbits.

3.4 Differential rotation.

From simple kinematics it can be derived that, seen from the sun, stars at small distances r have the following components of velocity as a result of the differential rotation of the Galaxy:

$$V_{\rm rad} = Ar \sin 2l \cos^2 b, \tag{3.21a}$$

$$\frac{V_{\rm tan}}{r} = 4.74\,\mu = \{A \cos 2l + B\}\cos^2 b, \tag{3.21b}$$

where l is galactic longitude. A and B are the Oort constants

$$A = \frac{1}{2}\left(\frac{V_{\rm rot}}{R} - \frac{dV_{\rm rot}}{dR}\right), \tag{3.22a}$$

$$B = -\frac{1}{2}\left(\frac{V_{\rm rot}}{R} + \frac{dV_{\rm rot}}{dR}\right). \tag{3.22b}$$

3.5 Epicycle orbits.

For small deviation from the rotation, stars move in epicyclic orbits. If R_0 is a fudicial distance from the centre and if the deviation $R - R_0$ is small compared to R_0, then we have in the radial direction

$$\frac{d^2R}{dt^2} = \frac{d^2}{dt^2}(R - R_0) = \frac{V^2}{R} - \frac{V_{\rm rot}^2}{R} = 4B(A - B)(R - R_0), \tag{3.23}$$

where the last approximation results from a Taylor expansion of $V_{\rm rot}$ at R_0 and ignoring higher order terms. Similarly we get for the tangential direction

$$\frac{d\theta}{dt} = \frac{V}{R} - \frac{V_{\rm rot,0}}{R_0} = -2\frac{A - B}{R_0}(R - R_0), \tag{3.24}$$

where θ is the angular tangential deviation seen from the galactic centre. These equations are easily integrated and it is then found that the orbit is described by

$$R - R_0 = \frac{U_0}{\kappa} \sin \kappa t, \tag{3.25a}$$

$$\theta R_0 = -\frac{U_0}{2B} \cos \kappa t \tag{3.25b}$$

and the orbital velocities by

$$U = U_0 \cos \kappa t, \tag{3.26a}$$

$$V - V_{\rm rot,0} = \frac{U_0 \kappa}{-2B} \sin \kappa t. \tag{3.26b}$$

The period in the epicycle equals $2\pi/\kappa$ and the epicyclic frequency is

$$\kappa = 2\{-B(A - B)\}^{1/2}. \tag{3.27}$$

For a flat rotation curve we have

$$\kappa = \sqrt{2}\frac{V_{\rm rot}}{R}. \tag{3.28}$$

Through the Oort constants and the epicyclic frequency, the parameters of the epicycle depend on the local forcefield, because these are all derived from the rotation velocity and its radial derivative. The direction of motion in the epicycle is opposite to that of galactic rotation.

The ratio of the velocity dispersions in the plane for the stars can also be calculated as

$$\frac{\langle V^2 \rangle}{\langle U^2 \rangle} = \frac{-B}{A - B}. \tag{3.29}$$

With this result equation (2.20) can then be reduced to the so-called *asymmetric drift* equation

$$V_{\rm rot}^2 - V_{\rm t}^2 = -\langle U^2 \rangle \left\{ R\frac{\partial}{\partial R}\ln \nu + R\frac{\partial}{\partial R}\ln\langle U^2 \rangle + \left[1 - \frac{B}{B - A}\right] \right\}. \tag{3.30}$$

If the asymmetric drift $(V_{\rm rot} - V_{\rm t})$ is small, the left-hand term can be approximated by

$$V_{\rm rot}^2 - V_{\rm t}^2 \approx 2V_{\rm rot}(V_{\rm rot} - V_{\rm t}). \tag{3.31}$$

3.6 Vertical motion.

For the vertical motion the equivalent approximation is also that of a harmonic oscillator. For a constant density with z we have

$$K_{\mathrm{z}} = \frac{d^2 z}{dt^2} = -4\pi G \rho_0 z. \tag{3.32}$$

Integration gives

$$z = \frac{W_0}{\lambda} \sin \lambda t \tag{3.33}$$

and

$$W = W_0 \cos \lambda t. \tag{3.34}$$

The period equals $2\pi/\lambda$ and the vertical frequency is

$$\lambda = (4\pi G \rho_0)^{1/2}. \tag{3.35}$$

For the solar neighbourhood we have $V_{\mathrm{rot}} \approx 220$ km/sec, $R \approx 8.5$ kpc and $\rho_0 \approx 0.1$ $\mathrm{M}_{\odot}$ pc^{-3}. Then the epicyclic period is about 1.7×10^8 years and the vertical period about 8.1×10^7 years. It would be interesting to see where in galaxies these two might be equal.

4 Ellipsoidal velocity distribution.

4.1 The Schwarzschild distribution.

The distribution of space velocities of the local stars can be described with the so-called ellipsoidal distribution. This was first introduced by Karl Schwarzschild and is therefore also sometimes called the Schwarzschild distribution. The distribution is Gaussian along the principal axes, but has different dispersions. This anisotropy was Schwarzschild's explanation of the "star-streams" that were discovered by Kapteyn.

The general equation for the Schwarzschild distribution is

$$f(R, z, U, V, W) = \frac{8\langle U^2\rangle\langle V^2\rangle\langle W^2\rangle}{\pi^{3/2}}\nu$$

$$\exp\left[-\frac{U^2}{2\langle U^2\rangle} - \frac{(V-V_{\rm t})^2}{2\langle V^2\rangle} - \frac{W^2}{2\langle W^2\rangle} - \frac{UV}{2\langle UV\rangle} - \frac{UW}{2\langle UW\rangle} - \frac{(V-V_{\rm t})W}{2\langle VW\rangle}\right]. \tag{4.1}$$

There is an interesting deduction that can be made from this ellipsoidal velocity distribution, as has been shown by Oort in 1928. If one puts the distribution in the asymmetric drift equation, adds the condition that $z = 0$ is a plane of symmetry, one gets an equation in terms of velocities and multiplications thereof that has to be identical. So after sorting the terms all of these need to be zero. This then gives

$$2\langle U^2\rangle = C_1 + \tfrac{1}{2}C_5 z^2, \tag{4.2}$$

$$2\langle V^2\rangle = C_1 + C_2 R^2 + \tfrac{1}{2}C_5 z^2, \tag{4.3}$$

$$2\langle W^2\rangle = C_4 + \tfrac{1}{2}C_5 z^2, \tag{4.4}$$

$$2\langle UW\rangle = -C_5 R z, \tag{4.5}$$

$$\langle UV\rangle = \langle VW\rangle = 0, \tag{4.6}$$

$$V_{\rm t} = \frac{C_3 R}{C_1 + C_2 R^2 + \frac{1}{2}C_5 z^2}. \tag{4.7}$$

The constants C_1 to C_5 are positive constants. The density distribution follows from

$$\frac{\partial \ln \nu}{\partial R} = 2C_1 K_{\rm R} + \frac{C_2^2 R^2 + (2C_1 C_3^2 - C_1 C_2)R}{(C_2 R^2 + C_1)^2} - \frac{C_1 R}{C_5 R^2 + 2C_4}, \tag{4.8}$$

at $z = 0$, and

$$\frac{\partial \ln \nu}{\partial z} = (C_5 R^2 + 2C_4)K_{\rm z} - C_5 z\left[RK_{\rm R} + \frac{2(C_2 + 2C_3^2)R^2 + C_5 z^2 + 2C_1}{(2C_2 R^2 + C_5 z^2 + 2C_1)^2} + \frac{1}{C_5 z^2 + 2C_1}\right]. \tag{4.9}$$

As it turns out, this does at best describe the solar neighbourhood approximately. The Schwarzschild distribution does e.g. not allow for high-velocity stars. Oort's derivation only holds if the stellar velocity distribution is exactly Gaussian. So, these equations are of historical interest only. However, it is interesting to see that Oort assumed that $C_5 = 0$. This uncoupled the radial and vertical motion (as for a third integral).

4.2 Properties of the velocity ellipsoid.

For the solar neighbourhood, but probably anywhere in galactic disks, the velocity distribution of the stars is very anisotropic.

- The *ratio of the radial versus tangential* velocity dispersions is governed by the local differential rotation and can be described by the epicycle approximation. For that we had

$$\frac{\langle V^2 \rangle}{\langle U^2 \rangle} = \frac{-B}{(A-B)}. \tag{4.10}$$

- The *ratio of the vertical to radial* velocity dispersion is unconstrained, as a result of the third integral. However, the existence of a third integral does not necessarily imply that the velocity distribution has to be anisotropic. However, if no third integral would exist, the velocity distribution would have to be isotropic, according to Jeans.

- The *long axis of the velocity ellipsoid in the plane* should point to the center. However, it does not in practice. This is called the "deviation of the vertex" and presumably is due to local irregularities in the Galactic gravitational field.

- The *long axis of the velocity ellipsoid perpendicular to the plane* has an unknown orientation. This has been a longstanding problem, also sometimes referred to as the "tilt" of the velocity ellipsoid. Oort assumed it to be pointing parallel to the Galactic plane ($C_5 = 0$), but later also assumed it to be pointing always towards the Galactic center. In the Poisson equation

$$\frac{\partial K_{\rm R}}{\partial R} + \frac{K_{\rm R}}{R} + \frac{\partial K_{\rm z}}{\partial z} = -4\pi G \rho(R,z) \tag{4.11}$$

for a flattened disk, the first two terms are near the plane $z = 0$

$$\frac{\partial K_{\rm R}}{\partial R} + \frac{K_{\rm R}}{R} \approx 2(A-B)(A+B). \tag{4.12}$$

For a flat rotation curve we have $A = -B$, so this is zero and the equation reduces to that for a plane-parallel case. On this basis one may expect the long axis to be parallel to the plane.

4.3 Higher order moment equations.

The hydrodynamical equations were obtained by multiplication of the Liouville equation with velocities and then integrating over all velocity space. These are useful equations, but the system is not complete (there is a "closure problem"): there are only three equations for eight unknowns. These unknows are as a function of position the density, the rotation velocity, the three velocity dispersions and three "cross-dispersions". In principle, taking higher order moments (by multiplying the Jeans equations with velocities and again integrating over all velocities) should make matters worse. However, with reasonable assumptions Vandervoort and in particular Cuddeford and Amendt have recently been able to make progress. In analogy to the second moment

$$\sigma_{\rm ab}(R,z) = \langle V_{\rm a} V_{\rm b} \rangle = \frac{1}{\nu} \int (V_{\rm a} - \langle V_{\rm a} \rangle)(V_{\rm b} - \langle V_{\rm b} \rangle) f d^3V, \tag{4.13}$$

define the third and fourth moments as

$$S_{\rm abc}(R,z) = \langle V_{\rm a} V_{\rm b} V_{\rm c} \rangle = \frac{1}{\nu} \int (V_{\rm a} - \langle V_{\rm a} \rangle)(V_{\rm b} - \langle V_{\rm b} \rangle)(V_{\rm c} - \langle V_{\rm c} \rangle) f d^3V, \tag{4.14}$$

$$T_{\rm abcd}(R,z) = \langle V_{\rm a}V_{\rm b}V_{\rm c}V_{\rm d}\rangle = \frac{1}{\nu}\int (V_{\rm a}-\langle V_{\rm a}\rangle)(V_{\rm b}-\langle V_{\rm b}\rangle)(V_{\rm c}-\langle V_{\rm c}\rangle)(V_{\rm d}-\langle V_{\rm d}\rangle) f d^3V. \quad (4.15)$$

For a Gaussian the "skewness" (e.g. $S_{\rm RRR}/(\sigma_{\rm RR})^{3/2}$) is zero, since it is completely symmetric. The fourth moment is related to the "kurtosis" (e.g. $T_{\rm RRRR}/(\sigma_{\rm RR})^2$), which decribes how peaked the distribution is and a Gaussian has a kurtosis of 3. The assumptions of Cuddeford and Amendt were:

– All parameters can be expanded in terms of a small parameter ϵ, which is the ratio of the radial velocity dispersion to the rotation velocity.

– The ordering scheme of these remains such that only terms in the leading order have to be taken.

– The velocity distributions are Gaussian (Schwarzschild) up to one more order than required by the equations.

This means that the assumption is that we have a cool, highly flattened and quasi-isothermal system. Then the system can be closed and five more equations result after a lot of algebra:

$$\langle VW\rangle = 0, \quad (4.16)$$

$$\frac{\partial \ln \nu}{\partial z}\left[\langle W^2\rangle\frac{\partial\langle W^2\rangle}{\partial R} + 2\langle UW\rangle\frac{\partial\langle UW\rangle}{\partial R} + \langle UW\rangle\frac{\partial\langle W^2\rangle}{\partial z} + 2\langle W^2\rangle\frac{\partial\langle UW\rangle}{\partial z}\right]$$
$$= -\frac{\langle UW\rangle}{R^3}\frac{\partial(R^2V_{\rm t}^2)}{\frac{1}{2}R} - \frac{1}{R}\frac{\partial(V_{\rm t}^2\langle W^2\rangle)}{\partial z}, \quad (4.17)$$

$$\langle W^2\rangle\frac{\partial\langle W^2\rangle}{\partial z} + \langle UW\rangle\frac{\partial\langle W^2\rangle}{\partial R} = 0, \quad (4.18)$$

$$\frac{\partial \ln \nu}{\partial z}\left[\langle UW\rangle\frac{\partial\langle U^2\rangle}{\partial R} + 2\langle U^2\rangle\frac{\partial\langle UW\rangle}{\partial R} + \langle W^2\rangle\frac{\partial\langle U^2\rangle}{\partial z} + 2\langle UW\rangle\frac{\partial\langle UW\rangle}{\partial z}\right]$$
$$= -\frac{4V_{\rm t}^2}{R^2}\left(\langle U^2\rangle - \langle V^2\rangle\right) - \frac{2}{R}\frac{\partial(V_{\rm t}^2\langle U^2\rangle)}{\partial R} - \frac{2}{R}\frac{\partial(V_{\rm t}^2\langle UW\rangle)}{\partial z}, \quad (4.19)$$

$$\frac{1}{4R}\frac{\partial(R^2V_{\rm t}^2)}{\partial R}\left[\langle W^2\rangle\langle UW\rangle\frac{\partial\langle U^2\rangle}{\partial R} + \langle W^2\rangle^2\frac{\partial\langle U^2\rangle}{\partial z} + 2\left(\langle U^2\rangle\langle W^2\rangle - 2\langle UW\rangle^2\right)\frac{\partial\langle UW\rangle}{\partial R}\right.$$
$$\left.\langle W^2\rangle\langle UW\rangle\frac{\partial\langle UW\rangle}{\partial z} - 2\langle UW\rangle\langle U^2\rangle\frac{\partial\langle W^2\rangle}{\partial R} - 2\langle UW\rangle^2\frac{\partial\langle W^2\rangle}{\partial z}\right]$$
$$= -\langle W^2\rangle\left[\langle W^2\rangle\frac{\partial\langle V^2\rangle}{\partial z} + \frac{\langle UW\rangle}{R^2}\frac{\partial(R^2\langle V^2\rangle)}{\partial R}\right]V_{\rm t}^2. \quad (4.20)$$

These equations can be used to derive further information on the velocity ellipsoid in cool, flattened galaxies (i.e. in disks). There are a few applications.

The first is the *tilt of the velocity ellipsoid*. From the equations it can be found that the tilt can be described as follows by the derivative of the cross-dispersion term near the plane

$$\frac{\partial\langle UW\rangle}{\partial z}(R,0) = \lambda(R)\left(\frac{\langle U^2\rangle - \langle W^2\rangle}{R}\right)(R,0), \quad (4.21)$$

with

$$\lambda(R) = \left[R^2\frac{\partial^3\Phi}{\partial R\partial z^2}\left(3\frac{\partial\Phi}{\partial R} + R\frac{\partial^2\Phi}{\partial R^2} - 4R\frac{\partial^2\Phi}{\partial z^2}\right)^{-1}\right](R,0). \quad (4.22)$$

For a flat rotation curve this gives

$$\lambda(R,0) = \left(\frac{2\pi G R^3}{V_t^2 - 8\pi G R^2 \rho}\frac{\partial \rho}{\partial R}\right)(R,0). \tag{4.23}$$

The second application has to do with the radial dependence of velocity dispersions. A solution of the equations then has the following form

$$f_1(R)\left(\frac{\partial \langle U^2\rangle}{\partial R}\right)(R,0) + f_2(R)\langle U^2\rangle(R,0) = f_3(R). \tag{4.24}$$

The complication is in the functions f:

$$f_1 = 4(\alpha+\beta)(2\alpha+\beta)(4\alpha+\beta)\frac{\partial\langle W^2\rangle}{\partial R}, \tag{4.25}$$

$$f_2 = (4\alpha+\beta)(4\alpha^2+3\alpha\beta+2\beta^2)\frac{\partial^2\langle W^2\rangle}{\partial R^2} + \frac{3}{R}\alpha\beta(4\alpha+\beta)\frac{\partial\langle W^2\rangle}{\partial R} -$$
$$\frac{2}{\langle W^2\rangle}(12\alpha^3+15\alpha^2\beta+14\alpha\beta^2+2\beta^3)\frac{\partial^2\langle W^2\rangle}{\partial R^2} + \frac{2}{R}(3-4\gamma)(\alpha+\beta)(2\alpha+\beta)(4\alpha+\beta)\frac{\partial\langle W^2\rangle}{\partial R} +$$
$$\alpha\beta(4\alpha+7\beta)\left[\frac{1}{\gamma}\frac{d\gamma}{dR} + \frac{4}{R}(\gamma-1)\right]\frac{\partial\langle W^2\rangle}{\partial R}, \tag{4.26}$$

$$f_3 = -\alpha(4\alpha+\beta)(4\alpha+7\beta)\langle W^2\rangle\frac{\partial^2\langle W^2\rangle}{\partial R^2} + 2\alpha^2(4\alpha+13\beta)\left(\frac{\partial\langle W^2\rangle}{\partial R}\right)^2 + \frac{3}{R}\alpha\beta(4\alpha+\beta)\langle W^2\rangle\frac{\partial\langle W^2\rangle}{\partial R} -$$
$$\frac{4}{R}(3-4\gamma)(4\alpha+\beta)\alpha(\alpha+\beta)\langle W^2\rangle\frac{\partial\langle W^2\rangle}{\partial R} + \alpha\beta(4\alpha+7\beta)\left[\frac{1}{\gamma}\frac{d\gamma}{dR} + \frac{4}{R}(\gamma-1)\right]\langle W^2\rangle\frac{\partial\langle W^2\rangle}{\partial R}. \tag{4.27}$$

The parameters α, β and γ are related to the potential and therefore to the local kinematics:

$$\alpha = -\left(\frac{\partial^2\Phi}{\partial z^2}\right)(R,0) = -\lambda^2, \tag{4.28}$$

where λ is the vertical frequency,

$$\beta = \left(\frac{\partial^2\Phi}{\partial R^2}\right)(R,0) + \frac{3}{R}\left(\frac{\partial\Phi}{\partial R}\right)(R,0) = \frac{1}{R^3}\left(\frac{\partial(R^2V_t^2)}{\partial R}\right)(R,0) = -\kappa^2, \tag{4.29}$$

with κ the epicyclic frequency,

$$\gamma = \frac{1}{4}\left\{R\left(\frac{\partial^2\Phi}{\partial R^2}\right)\left(\frac{\partial\Phi}{\partial z}\right)^{-1} + 3\right\}(R,0) = \left(\frac{\langle V^2\rangle}{\langle W^2\rangle}\right)(R,0), \quad (4.30$$

which is the anisotropy in the velocity distribution.

This can be solved for a given potential; the most realistic solution is with a logarithmic-exponential potential

$$\Phi(R,z) = A\ln R - BR - C^2 z\exp\left(-\frac{R}{h}\right), \tag{4.31}$$

which has

$$\left(\frac{\partial^2\Phi}{\partial z^2}\right)(R,0) = 2C\exp\left(-\frac{R}{h}\right) \tag{4.32}$$

and thus an exponential density profile.

The resulting distributions show:

– the radial velocity dispersion $\langle U^2 \rangle$ to decrease more or less exponentially with radius,

– the velocity anisotropy $\langle U^2 \rangle / \langle W^2 \rangle$ to be roughly constant in the inner regions at least, and

– Toomre Q to be constant with radius, except near the center.

A further application is the following equation, which can be derived from the new set of hydrodynamic equations

$$\left(\frac{\partial^2 \langle W^2 \rangle}{\partial z^2} \right) (R, 0) = -\lambda(R) \left[\left(\frac{\langle U^2 \rangle - \langle W^2 \rangle}{R} \right) \frac{\partial \ln \langle W^2 \rangle}{\partial R} \right] (R, 0). \tag{4.33}$$

Since $\lambda(R) > 0$; $\langle U^2 \rangle > \langle W^2 \rangle$ and $\langle W^2 \rangle$ decreasing with R, the righthand side of the equation has to be positive. That means that $\langle W^2 \rangle$ has a minimum in the plane. So disks are not strictly isothermal in z and numerical values suggest less peaked in density than the exponential function.

The final application gives a more accurate estimate of the velocity anisotropy in the plane through

$$\frac{\langle V^2 \rangle}{\langle U^2 \rangle} = \frac{1}{2} \left\{ 1 + \frac{\partial \ln V_{\rm t}}{\partial \ln R} - \frac{S_{\theta\theta\theta}}{V_{\rm t} \langle U^2 \rangle} + \frac{1}{\nu R V_{\rm t} \langle U^2 \rangle} \frac{\partial R^2 \nu S_{\rm RR\theta}}{\partial R} + \right.$$
$$\left. \frac{R}{V_{\rm t} \langle U^2 \rangle} \frac{\partial S_{\rm R\theta z}}{\partial z} + \frac{V_{\rm t}^2 - V_{\rm rot}^2}{V_{\rm t} \langle U^2 \rangle^2} S_{\rm RR\theta} + \frac{T_{\rm RR\theta\theta}}{\langle U^2 \rangle^2} \right\}. \tag{4.34}$$

In practice this can be approximated as

$$\frac{\langle U^2 \rangle}{\langle V^2 \rangle} = \frac{1}{2} \left(1 + \frac{\partial \ln V_{\rm t}}{\partial \ln R} + \frac{T_{\rm RR\theta\theta}}{\langle U^2 \rangle^2} \right). \tag{4.35}$$

This constitutes a small correction to the classical result

$$\frac{\langle U^2 \rangle}{\langle V^2 \rangle} = \frac{1}{2} \left(1 + \frac{\partial \ln V_{\rm t}}{\partial \ln R} \right) = \frac{-B}{A - B}. \tag{4.36}$$

5 Isothermal solutions and related results.

For simple geometries, such as spherical density distributions or density distributions on stratified layers and isothermal velocity distributions (that is equal velocity dispersions at all positions), full solutions for the distribution function to the set of two fundamental equations can be obtained.

5.1 Isothermal sphere.

The equations for a (non-rotating) isothermal sphere are

$$\frac{1}{R^2}\frac{\partial}{\partial R}(R^2 K_{\rm R}) = -4\pi G\rho(R) \tag{5.1}$$

and

$$\langle V^2\rangle \frac{\partial \nu}{\partial R} = \nu K_{\rm R}. \tag{5.2}$$

The solution is given by

$$\rho(R) = \rho_0 \exp(-\Phi), \tag{5.3}$$

where Φ follows from a numerical integration of

$$\exp(-\Phi) = \frac{1}{\chi^2}\frac{d}{d\chi}\left(\chi^2\frac{d\Phi}{d\chi}\right) \tag{5.4a}$$

$$\chi = \left(\frac{\langle V^2\rangle}{4\pi G\rho_0}\right)^{1/2} R. \tag{5.4b}$$

For large R this becomes

$$\rho(R) = \frac{\langle V^2\rangle}{2\pi G}R^{-2}. \tag{5.5}$$

The "core radius" is

$$r_0 = \left(\frac{4\pi G\rho_0}{9\langle V^2\rangle}B\right)^{-1/2}. \tag{5.6}$$

King-models are adapted isothermal spheres with a tidal radius $R_{\rm t}$ and a corresponding upper boundary in the velocity distribution. The total mass is

$$M(R_{\rm t}) = \frac{2}{G}\langle V^2\rangle r_0\, f\left(\frac{R_{\rm t}}{r_0}\right) \tag{5.7}$$

and the central surface density

$$\sigma_0 = \rho_0 r_0\, g\left(\frac{R_{\rm t}}{r_0}\right). \tag{5.8}$$

The functions f and g can only be calculated numerically and are given in the literature. From the above it follows that the velocity dispersion is

$$\langle V^2\rangle^{1/2} \propto \frac{\rho_0 M(R_{\rm t})}{f\left(\frac{R_{\rm t}}{r_0}\right) g\left(\frac{R_{\rm t}}{r_0}\right)}. \tag{5.9}$$

Since for elliptical galaxies it is observed that $\log(R_{\rm t}/r_0)$ is about constant at 2.2 and the central surface brightness also ("Fish's law"), we find for a constant M/L the Faber-Jackson relation, which is the equivalent of the Tully-Fisher relation for spirals:

$$L^{1/4} \propto \langle V^2\rangle^{1/2}. \tag{5.10}$$

5.2 Isothermal sheet.

For an isothermal sheet the basic equations become

$$\frac{\partial K_{\rm z}}{\partial z} = -4\pi G\rho(z) \tag{5.11}$$

and

$$\langle W^2\rangle \frac{\partial \nu}{\partial z} = \nu K_{\rm z}. \tag{5.12}$$

In the self-gravitating case we may write $\nu = \rho$. The two equations can be combined into

$$-4\pi G\rho(z) = \langle W^2\rangle \frac{d^2}{dz^2}\left\{\ln \frac{\rho(z)}{\rho(0)}\right\}. \tag{5.13}$$

The solution is

$$\rho(z) = \frac{\langle W^2\rangle}{2\pi G z_0^2}\,\mathrm{sech}^2\left(\frac{z}{z_0}\right). \tag{5.14}$$

The corresponding surface density is

$$\sigma = 2z_0\rho_0 \tag{5.15}$$

and the relation to the velocity dispersion is

$$\langle W^2\rangle = \pi G\sigma z_0. \tag{5.16}$$

The vertical force results from integration of Poisson's equation as

$$K_{\rm z} = -2\frac{\langle W^2\rangle}{z_0}\,\tanh\left(\frac{z}{z_0}\right). \tag{5.17}$$

Usefull approximations are

$$\mathrm{sech}^2\left(\frac{z}{z_0}\right) = \exp\left(\frac{z^2}{z_0^2}\right) \qquad \text{for} \quad z \ll z_0, \tag{5.18a}$$

$$\mathrm{sech}^2\left(\frac{z}{z_0}\right) = 4\,\exp\left(\frac{2z}{z_0}\right) \qquad \text{for} \quad z \gg z_0. \tag{5.18b}$$

For a second isothermal component of negligible mass and different velocity dispersion in this force-field we find

$$\rho_{\rm II}(z) = \rho_{\rm II}(0)\mathrm{sech}^{2p}\left(\frac{z}{z_0}\right), \tag{5.19}$$

where

$$p = \frac{\langle W^2\rangle}{\langle W^2\rangle_{\rm II}}. \tag{5.20}$$

5.3 Exponential and sech-distributions.

The isothermal z-distribution is only an approximate description of the vertical mass distribution in disks of galaxies. There is a range of generations of stars, each with their own velocity dispersion.

Often used is the exponential distribution, since it is a convenient fitting function. Since the velocity dispersion now varies with z we have to write the equation in terms of the velocity dispersion in he plane $\langle W^2\rangle_0^{1/2}$. The equations corresponding to this case are:

$$\rho(z) = \frac{\langle W^2\rangle_0}{2\pi G Z_{\rm e}^2}\exp\left(-\frac{z}{z_{\rm e}}\right), \tag{5.21}$$

$$\sigma = 2z_{\rm e}\rho_0, \tag{5.22}$$

$$\langle W^2\rangle_0 = \pi G\sigma z_{\rm e}, \tag{5.23}$$

$$K_{\rm z} = -2\pi G\sigma\left\{1-\exp\left(-\frac{z}{z_{\rm e}}\right)\right\}. \tag{5.24}$$

If an *isothermal* component of negligible mass moves in this force field, then

$$\rho_{\rm II}(z) = \rho_{\rm II}(0)\exp\left[-\frac{2pz}{z_{\rm e}} + 2p\left\{1-\exp\left(-\frac{z}{z_{\rm e}}\right)\right\}\right], \tag{5.25}$$

where now

$$p = \frac{\langle W^2\rangle_0}{\langle W^2\rangle_{\rm II}}. \tag{5.26}$$

As an intermediate case between the isothermal solution and the exponetial it is also possible to use the sech-distribution. This corresponds probably closest to reality. The equations then are:

$$\rho(z) = \frac{2\langle W^2\rangle_{\rm II}}{\pi^3 G z_{\rm e}^2}\mathrm{sech}\left(\frac{z}{z_{\rm e}}\right), \tag{5.27}$$

$$\sigma = \pi\rho_0 z_{\rm e}, \tag{5.28}$$

$$\langle W^2\rangle_0 = \frac{\pi^2}{2}G\sigma z_{\rm e}, \tag{5.29}$$

$$K_{\rm z} = -4G\sigma\arctan\left\{\sinh\left(\frac{z}{z_{\rm e}}\right)\right\}. \tag{5.30}$$

For the second isothermal component we now get

$$\rho_{\rm II}(z) = \rho_{\rm II}(0)\exp\left\{-\frac{8}{\pi^2}pI\left(\frac{z}{z_{\rm e}}\right)\right\}, \tag{5.31}$$

where

$$I(y) = \int_0^y \arctan(\sinh x)dx. \tag{5.32}$$

This integral can be evaluated easily by numerical methods or through a series expansion.

6 Potential theory.

6.1 General axisymmetric theory.

Much attention has been paid to inverting Poisson's equation (for the axisymmetric case)

$$\frac{\partial^2 \Phi}{\partial R^2} + \frac{1}{R}\frac{\partial \Phi}{\partial R} + \frac{\partial^2 \Phi}{\partial z^2} = 4\pi G \rho(R,z), \tag{6.1}$$

so that the potential (and the forces) can be calculated when the density distribution is given. This is a limited problem in that it does not involve the continuity equation and the distribution function and therefore is not a general solution for a dynamical system, such as the isothermal solutions above.

At the basis lies the Hankel transform, which in the radial direction for the density is

$$\tilde{\rho}(k,z) = \int_0^\infty u J_0(ku)\rho(u,z)du, \tag{6.2}$$

where J_0 is the Bessel function of the first kind. The important property, why this is useful, is that the transform can be inverted to

$$\rho(u,z) = \int_0^\infty k J_0(kR)\tilde{\rho}(k,z)dk.$$

If we take this transform in the radial direction for both sides of the Poisson equation we get

$$-k^2\tilde{\Phi}(k,z) + \frac{\partial^2}{\partial z^2}\tilde{\Phi}(k,z) = 4\pi G\tilde{\rho}(k,z). \tag{6.3}$$

This linear non-homogeneous ordinary differential equation can be solved to give

$$\tilde{\Phi}(k,z) = -\frac{2\pi G}{k}\int_{-\infty}^\infty \exp\left(-k|z-v|\right)\tilde{\rho}(k,v)dv. \tag{6.4}$$

Using this, Poisson's equation can then be inverted to

$$\Phi(R,z) = -2\pi G\int_0^\infty \int_{-\infty}^\infty J_0(kR)\tilde{\rho}(k,v)e^{-k|z-v|}dv\,dk. \tag{6.5}$$

Then

$$\Phi(R,z) = -2\pi G\int_0^\infty \int_0^\infty \int_{-\infty}^\infty uJ_0(kR)J_0(ku)\rho(u,v)e^{-k|z-v|}dv\,du\,dk. \tag{6.6}$$

The forces follow by taking the negative derivatives of the potential in the radial and vertical directions.

$$K_{\rm R}(R,z) = -\frac{\partial\Phi(R,z)}{\partial R} = -2\pi G\int_0^\infty \int_0^\infty \int_{-\infty}^\infty ukJ_1(kR)J_0(ku)\rho(u,v)e^{-k|z-v|}dv\,du\,dk, \tag{6.7}$$

and

$$K_{\rm z}(R,z) = -\frac{\partial\Phi(R,z)}{\partial z} = -2\pi G\int_0^\infty \int_0^\infty \int_{-\infty}^\infty uJ_0(kR)J_0(ku)\rho(u,v){\rm sign}(z-v)e^{-k|z-v|}dv\,du\,dk. \tag{6.8}$$

The integrations are somewhat simpler, if the density is separable as

$$\rho(R,z) = \rho_{\rm R}(R)\rho_{\rm z}(z). \tag{6.9}$$

6.2 Exponential disks.

There are various ways of proceeding from here. The first is by taking an analytical form for the density distribution. Kuijken and Gilmore have done this for exponential disks. If the radial density distribution is exponential

$$\rho_{\rm R}(R) = \rho_0 \exp\,(-R/h), \tag{6.10}$$

then the Hankel transform becomes

$$\int_0^\infty \rho_0 J_0(ku) u e^{-u/h} du = \frac{\rho_0 h^2}{(k^2h^2+1)^{3/2}} \tag{6.11}$$

and the potential

$$\Phi(R,z) = -2\pi G h^2 \int_0^\infty \int_{-\infty}^\infty \frac{J_0(kR)}{(k^2h^2+1)^{3/2}} \rho_{\rm z}(v) e^{-k|z-v|} dv\, dk. \tag{6.12}$$

Kuijken and Gilmore first solve for an exponential z-distribution:

$$\rho_{\rm z} = \exp\,(-|z|/z_{\rm e}). \tag{6.13}$$

First note that if $\rho_{\rm z}(z)$ is symmetric around $z = 0$, then

$$I_{\rm z}(k,z) = \int_{-\infty}^\infty \rho_{\rm z}(v) e^{-k|z-v|} dv$$

$$= 2e^{k|z|} \int_0^{|z|} \rho_{\rm z}(v) \cosh(kv) dv + 2\cosh(kz) \int_{|z|}^\infty \rho_{\rm z}(v) e^{-kv} dv$$

$$= e^{-k|z|} \int_0^{|z|} \rho_{\rm z}(v) e^{kv} dv + e^{k|z|} \int_{|z|}^\infty \rho_{\rm z}(v) e^{-kv} dv + e^{-k|z|} \int_0^\infty \rho_{\rm z}(v) e^{-kv} dv. \tag{6.14}$$

Solving this for the exponential z-distribution gives

$$\Phi(R,z) = -4\pi G \rho_0 h^2 z_{\rm e} \int_0^\infty \frac{J_0(kR)}{(k^2h^2+1)^{3/2}} \frac{e^{-k|z|} - z_{\rm e} k e^{-|z|/z_{\rm e}}}{1 - k^2 z_{\rm e}^2} dk. \tag{6.15}$$

The possible term for which the denominator is zero ($kz_{\rm e} = 1$) is still finite; the last quotient is then

$$\frac{1}{2z_{\rm e}k}(1 + k|z|) e^{-k|z|}. \tag{6.16}$$

The forces are

$$K_{\rm R}(R,z) = -4\pi G \rho_0 h^2 z_{\rm e} \int_0^\infty k \frac{J_1(kR)}{(k^2h^2+1)^{3/2}} \frac{e^{-k|z|} - z_{\rm e} k e^{-|z|/z_{\rm e}}}{1 - k^2 z_{\rm e}^2} dk, \tag{6.17}$$

and

$$K_{\rm z}(R,z) = -4\pi G \rho_0 h^2 z_{\rm e} \int_0^\infty k \frac{J_0(kR)}{(k^2h^2+1)^{3/2}}\, {\rm sign}(z) \frac{e^{-k|z|} - e^{-|z|/z_{\rm e}}}{1 - k^2 z_{\rm e}^2} dk. \tag{6.18}$$

Next they go further and assume that the density distribution is given by

$$\rho(R,z) = \rho_0 \exp\,(-R/h)\, {\rm sech}^n(z/nz_{\rm e}). \tag{6.19}$$

For $n = 0$ we have again the exponential z-distribution with vertical, exponential scaleheight z_e. For $n = 2$ we have the locally isothermal disk of van der Kruit and Searle and for $n = 1$ the "sech-disk" proposed by van der Kruit. Kuijken and Gilmore then show that the potential can be written as

$$\Phi(R,z) = -4\pi G\rho_0 h^2 z_e 2^n \int_0^\infty J_0(kR)(k^2h^2+1)^{-3/2}\times$$

$$\sum_{m=0}^{\infty} \binom{-n}{m} \frac{(1+2m/n)\exp(-k|z|) - z_e k \exp[-(1+2m/n)|z|/z_e]}{(1+2m/n)^2 - k^2 z_e^2} dk. \tag{6.20}$$

The possible term, for which $m = n(kz_e - 1)/2$, has a zero denominator and must be written as

$$\frac{1}{2z_e k}\binom{-n}{m}(1+k|z|)e^{-k|z|}. \tag{6.21}$$

The binomial with the upper coefficient negative can be written as follows

$$\binom{-n}{m} = \frac{(-n)(-n-1)........(-n-m+1)}{m!}$$

$$= (-1)^m \binom{m+n-1}{n-1} = (-1)^m \frac{(m+n-1)!}{(n-1)!m!}. \tag{6.22}$$

So the potential is in this case expressed as a sum of those for exponential z-distributions. This is essentially related to the fact that the sech is written as a sum of exponentials:

$$\text{sech}\, x = 2\sum_{j=0}^{\infty}(-1)^j e^{-(2j+1)|x|}. \tag{6.23}$$

This well-known expansion suffers from the fact that it does not work for $x = 0$, because then the terms are alternatingly +1 and –1. This does not necessarily make it unsuitable, because after integration each term gets divided by $-(2j+1)$ and the series will converge even for $x = 0$. However, it may remain slow for small x. For example the sum for $x = 0$

$$2\sum_{j=0}^{\infty}\frac{(-1)^j}{2j+1} = \frac{\pi}{2} \tag{6.24}$$

takes 32 steps to reach an accuracy of 1%.

Similar expressions as above can be found for the forces, but this will not be fully written out here.

6.3 Rotation curves from a general surface density distribution.

Casertano has derived an expression for the potential in the plane in order to find the rotation curve of a disk with a general density distribution. He uses the radial force in the plane and performs the integration over k first (rather than over u). The equation for the radial force in the plane for a symmetrical z-distribution is

$$K_R(R,0) = -4\pi G \int_0^\infty \int_0^\infty \int_0^\infty uk J_1(kR) J_0(ku)\rho(u,v)e^{-kv} dv\, du\, dk. \tag{6.25}$$

It helps to have the same order Bessel functions and get rid of the linear factor k by integrating by parts

$$\int_0^\infty uJ_0(ku)\rho(u,v)du = \frac{u}{k}J_1(uk)\rho(u,v)\bigg|_0^\infty - \frac{1}{k}\int_0^\infty uJ_1(uk)\frac{\partial\rho(u,v)}{\partial u}du. \tag{6.26}$$

Then

$$K_{\rm R}(R,0) = -4\pi G \int_0^\infty \int_0^\infty \int_0^\infty u J_1(kR) J_1(uk) \frac{\partial \rho(u,v)}{\partial u} e^{-kv} dv\, dk\, du, \tag{6.27}$$

and this can be solved to give

$$K_{\rm R}(R,0) = 8G \int_0^\infty \int_0^\infty \sqrt{\frac{u}{Rp}} \frac{\partial \rho(u,v)}{\partial u} [K(p) - E(p)] du\, dv, \tag{6.28}$$

where

$$p = x - \sqrt{x^2 - 1}, \quad x = \frac{R^2 + u^2 + v^2}{2Ru}. \tag{6.29}$$

K and E are the complete elliptic integrals of the second and first kind respectively for which good approximations are known. For the z-dependence of the density one can take an exponential or the isothermal distribution.

Casertano's work can be extended to the potential, vertical force and the radial force out of the plane. First start with $K_{\rm R}$ at arbitrary z. At a general position we had

$$K_{\rm R}(R,z) = -2\pi G \int_0^\infty \int_0^\infty \int_{-\infty}^\infty uk J_1(kR) J_0(ku) \rho(u,v) e^{-k|z-v|} dv\, du\, dk. \tag{6.30}$$

As Casertano we can do the integration over k (after integration by parts) and obtain

$$\int_0^\infty J_1(kR) J_1(uk) e^{-k|z-v|} dk = \frac{(2-p^2)K(p) - 2E(p)}{\pi p \sqrt{Ru}}, \tag{6.31}$$

where

$$p = 2 \frac{\sqrt{Ru}}{\sqrt{(z-v)^2 + (R+u)^2}}. \tag{6.32}$$

This is the same as Casertano found (except that he had $z = 0$), but he chose to rework it further to the form above. The formula for p has a singularity at $R = u = z = 0$. Note however that for $R = u = 0$ we already have $p = 0$ for all z, so that we should take $p = 0$ also for $z = 0$. Of course this only occurs when evaluating the force in the center.

The radial force now becomes

$$K_{\rm R}(R,z) = 2G \int_0^\infty \int_{-\infty}^\infty \frac{(2-p^2)K(p) - 2E(p)}{p\sqrt{Ru}} \frac{\partial \rho(u,v)}{\partial u} du\, dv. \tag{6.33}$$

For the vertical force and the potential itself we have a product of Bessel functions of equal order before the integration by parts, but this of different order after that. When then the integration over k is done, we get expressions which contain the Heuman Lambda function. This can be rewritten only in forms that involve incomplete elliptic integrals of the first and second kind or the elliptic integral of the third kind, but these are much more difficult to evaluate numerically. Also the integrals over u must then be written as the sum of two different integrals, one from 0 to R and one from R to ∞. So it is better to start with the forms before the integration by parts.

For the vertical force we start with

$$K_{\rm z}(R,z) = -2\pi G \int_0^\infty \int_0^\infty \int_{-\infty}^\infty u J_0(kR) J_0(ku) \rho(u,v) \mathrm{sign}(z-v) e^{-k|z-v|} dv\, du\, dk. \tag{6.34}$$

The integration over k yields

$$\int_0^\infty k J_0(kR) J_0(ku) e^{-k|z-v|} dk = \frac{|z-v| p^3}{4\pi (1-p^2)\sqrt{(uR)^3}} E(p), \tag{6.35}$$

and we get

$$K_z(R,z) = -\frac{G}{2}\int_0^\infty \int_{-\infty}^\infty \mathrm{sign}(z-v)\frac{u|z-v|p^3E(p)}{(1-p^2)\sqrt{(uR)^3}}\rho(u,v)dv\,du. \tag{6.36}$$

For the potential we start with

$$\Phi(R,z) = -2\pi G\int_0^\infty \int_0^\infty \int_{-\infty}^\infty uJ_0(kR)J_0(ku)\rho(u,v)e^{-k|z-v|}dv\,du\,dk. \tag{6.37}$$

The integration over k now yields

$$\int_0^\infty J_0(kR)J_0(ku)e^{-k|z-v|}dk = \frac{p}{\pi\sqrt{uR}}K(p). \tag{6.38}$$

The potential then is given by

$$\Phi(R,z) = -2G\int_0^\infty \int_{-\infty}^\infty \frac{upK(p)}{\sqrt{uR}}\rho(u,v)dv\,du. \tag{6.39}$$

Kent has used yet another method by starting from a straightforward integration of all contributions to the forces over the whole volume of the system. This goes as follows

$$\frac{\partial\Phi}{\partial R} = G\int_0^\infty \int_0^{2\pi} \int_{-\infty}^\infty \rho(u,v)\frac{u(R-u\cos\theta)}{[R^2+u^2+(z-v)^2-2Ru\cos\theta]^{3/2}}dv\,d\theta\,du, \tag{6.40}$$

$$\frac{\partial\Phi}{\partial z} = G\int_0^\infty \int_0^{2\pi} \int_{-\infty}^\infty \rho(u,v)\frac{u(z-v)}{[R^2+u^2+(z-v)^2-2Ru\cos\theta]^{3/2}}dv\,d\theta\,du. \tag{6.41}$$

This formulation suffers from the fact that the denominators have singularities for $R = u$. Kent avoids this by changing to another independent variable. As expected he also ends up with elliptic integrals. The expression for the radial force is similar to the one found by Casertano. The one for the vertical force contains two terms, each with an elliptic integral of the third kind. The expressions will not be repeated here, but it is clear that the ones derived above are better usable in practice.

7 Other potentials.

There are in the literature many particular potentials that can be used to describe galaxies, but are not isothermal. The most important ones will be summarized here. These are not solutions of the Liouville and Poisson equation. Rather they are convenient expressions for the potential or density distribution that can be inserted analytically in Poisson's equation. Equilibrium solutions exist for rotating, incompressible fluids, such as the **Maclaurin** spheroids for the axisymmetric case. Tri-axial solutions also exist as **Jacobi** ellipsoids and **Riemann** ellipsoids. These will not be given here.

7.1 Plummer model.

This was originally used to describe globular clusters. The potential has the simple spherical form

$$\Phi(R) = -\frac{GM}{\sqrt{R^2+a^2}}. \tag{7.1}$$

The corresponding density distribution is

$$\rho(R) = \left(\frac{3M}{4\pi a^3}\right)\left(1+\frac{R^2}{a^2}\right)^{-5/2}. \tag{7.2}$$

7.2 Kuzmin model.

This derives from the potential

$$\Phi(R,z) = -\frac{GM}{\sqrt{R^2+(a+|z|)^2}}. \tag{7.3}$$

This is an axisymmetric potential that can be used to describe very flat disks. The corresponding surface density is

$$\sigma(R) = \frac{aM}{2\pi(R^2+a^2)^{3/2}}. \qquad (7.4$$

7.3 Toomre models.

These are models that derive from the Kuzmin model by differentiating with respect to a^2. The n-th model follows after $(n-1)$ differentiations:

$$\sigma_{\rm n}(R) = \sigma(0)\left(1+\frac{R^2}{4n^2a^2}\right). \tag{7.5}$$

The corresponding potential can be derived by differentiating the potential an equal number of times. It can be seen that Toomre's model 1 (which has $n=1$) is Kuzmin's model. The limiting case of $n\rightarrow\infty$ becomes a Gaussian surface density model.

7.4 Logarithmic potentials.

These are made to provide rotation curves that are not Keplerian for large R. Since these can be made flattened they provide an alternative to the simple isothermal sphere. The potential is

$$\Phi(R,z) = \frac{V_0^2}{2}\ln\left(r_0^2+R^2+\frac{z^2}{c^2}\right). \tag{7.6}$$

V_0 is the rotation velocity for large radii and c controls the flattening of the isopotential surfaces ($c \le 1$). The density distribution is

$$\rho(R,z) = \frac{V_0^2}{4\pi G c^2} \frac{(2c^2+1)r_0^2 + R^2 + 2z^2[1-1/(2c^2)]}{(r_0^2 + R^2 + z^2/c^2)^2}. \tag{7.7}$$

At large radii $R \gg r_0$ the isodensity surfaces have a flattening

$$\left(\frac{b}{a}\right)^2 = c^4(2 - c^{-2}). \tag{7.8a}$$

In the inner regions $R \ll r_0$ it is

$$\left(\frac{b}{a}\right)^2 = \frac{1 = 4c^2}{2 + 3c^{-2}}. \tag{7.8b}$$

The rotation curve is

$$V_{\rm rot} = \frac{V_0 R}{\sqrt{r_0^2 + R^2}}. \tag{7.9}$$

7.5 Oblate spheroids.

Assume that all iso-density surfaces are confocal ellipsoids with axis ratio c/a and therefore excentricity

$$e = \sqrt{1 - \frac{c^2}{a^2}}. \tag{7.10}$$

Let the density along the major axis be $\rho(R)$. Define

$$\alpha^2(R,z) = R^2 + \frac{z^2}{1-e^2}$$

Then inside the spheroid the forces and potential are

$$K_{\rm R} = -\frac{4\pi G\sqrt{1-e^2}}{e^3} R \int_0^{\sin^{-1} e} \rho(\alpha)\sin^2\beta d\beta. \tag{7.11a}$$

$$K_{\rm z} = -\frac{4\pi G\sqrt{1-e^2}}{e^3} z \int_0^{\sin^{-1} e} \rho(\alpha)\tan^2\beta d\beta. \tag{7.11b}$$

$$\Phi(R,z) = \frac{4\pi G\sqrt{1-e^2}}{e} \left[\int_0^{\delta} \rho(\alpha)\alpha\beta d\alpha + \sin^{-1} e \int_{\delta}^{a} \rho(\alpha)\alpha d\alpha\right]. \tag{7.11c}$$

Here

$$\delta^2 = R^2 + \frac{z^2}{1-e^2}, \tag{7.12}$$

and

$$\alpha^2 = \frac{R^2\sin^2\beta + z^2\tan^2\beta}{e^2}. \tag{7.13}$$

Outside the spheroid we have

$$K_{\rm R} = -\frac{4\pi G\sqrt{1-e^2}}{e^3} R \int_0^{\gamma} \rho(\alpha)\sin^2\beta d\beta, \tag{7.14a}$$

$$K_z = -\frac{4\pi G\sqrt{1-e^2}}{e^3} z \int_0^{\gamma} \rho(\alpha) \tan^2 \beta d\beta, \tag{7.14b}$$

$$\Phi(R,z) = \frac{4\pi G\sqrt{1-e^2}}{e} \int_0^{a} \rho(\alpha)\alpha\beta d\alpha. \tag{7.14c}$$

Here γ follows from

$$R^2 \sin^2 \gamma + z^2 \tan^2 \gamma = a^2 e^2. \tag{7.15}$$

7.6 Infinitesimally thin disks.

This is analogous to the treatment of general disk potentials above. The potential can be written as

$$\Phi(R,z) = -2\pi G \int_0^{\infty} \exp\left(-k|z|\right) J_0(kR) \int_0^{\infty} \sigma r J_0(kr) r \; dr \; dk \tag{7.16}$$

and the rotation velocity

$$V_c^2(R) = -R \int_0^{\infty} S(k) J_1(kR) \; k \; dk, \tag{7.17}$$

where

$$S(k) = -2\pi G \int_0^{\infty} J_0(kR)\sigma(R) dR. \tag{7.18}$$

Also it may be useful to calculate the surface density corresponding to a known rotation curve $V_c(R)$. Using the inversion of (7.16) it can be shown that

$$\sigma(R) = \frac{1}{\pi^2 G} \left[\frac{1}{R} \int_0^R \frac{dV_c^2}{dr} K\left(\frac{r}{R}\right) dr \; + \; \int_R^{\infty} \frac{1}{r} \frac{dV_c^2}{dr} K\left(\frac{R}{r}\right) dr \right], \tag{7.19}$$

where K is the complete elliptic integral. Note that there is a contribution from the part of the disk beyond R. This also holds for disks with finite thickness as long as the density distribution is not described by spheroids. Only in the case that isodensity surfaces are spheroids do the forces at radii larger than R cancel. In the general case the rotation curve of a disk depends on the surface density at all radii.

7.7 Mestel disk.

This has the surface density distribution

$$\sigma(R) = \sigma_0 \frac{R_0}{R}. \tag{7.20}$$

The corresponding rotation curve is flat and has

$$V_c^2(R) = 2\pi G \sigma_0 R_0 = \frac{GM(R)}{R}, \tag{7.21}$$

where $M(R)$ is the mass interior to R.

7.8 Exponential disk.

The surface density is

$$\sigma(R) = \sigma_0 \exp\left(\frac{R}{h}\right). \tag{7.22}$$

The corresponding potential is from (7.16)

$$\Phi(R,0) = -\pi G\sigma_0 R\left[I_0\left(\frac{R}{2h}\right)K_1\left(\frac{R}{2h}\right) - I_1\left(\frac{R}{2h}\right)K_0\left(\frac{R}{2h}\right)\right]. \tag{7.23}$$

Here I and K are the modified Bessel functions. The rotation curve is using (7.17)

$$V_{\rm c}^2(R) = 4\pi G\sigma_0 h\left(\frac{R}{2h}\right)^2\left[I_0\left(\frac{R}{2h}\right)K_0\left(\frac{R}{2h}\right) - I_1\left(\frac{R}{2h}\right)K_1\left(\frac{R}{2h}\right)\right]. \tag{7.24}$$

The total potential energy of the disk is

$$\Omega \approx -11.6 G\sigma_0^2 h^3. \tag{7.25}$$

7.9 Disks with finite thickness.

This has been treated in section 6. Here I will only give the result for the exponential sech-disk with density distribution

$$\rho(R,z) = \rho_0 \exp\left(-\frac{R}{h}\right)\,\mathrm{sech}\left(\frac{z}{z_{\rm e}}\right). \tag{7.26}$$

Then

$$\Phi(R,z) = -8\pi G\rho_0 h^2 z_{\rm e}\int_0^\infty \frac{J_0(kR)}{(k^2h^2+1)^{3/2}}\sum_{m=0}^{\infty}(-1)^m\frac{(1+2m)e^{-k|z|} - kz_{\rm e}e^{-(1+2m)|z|/z_{\rm e}}}{(1+2m)^2 - k^2z_{\rm e}^2}dk. \tag{7.27}$$

8 Stäckel potentials.

Stäckel potentials are potentials that can be written as separable functions in ellipsoidal coordinate systems. I will here only treat the axisymmetric case with oblate density distributions (which means a prolate potential distribution), which applies to disk galaxies. In that case the coordinate system is spheroidal and it can be seen as a further generalisation of the axisymmetric, plane-parallel case, where the potential is separable in R and z.

8.1 Coordinate system.

The new coordinate system is (λ, ϕ, ν). The relation with the axisymmetric system (r, ϕ, z) is, that λ and ν are the two roots for τ of

$$\frac{r^2}{\tau+\alpha} + \frac{z^2}{\tau+\gamma} = 1, \tag{8.1}$$

with

$$0 \leq \nu \leq \lambda. \tag{8.2}$$

The constants α and γ are sometimes also given in the form

$$\alpha = -a^2, \quad \gamma = -c^2. \tag{8.3}$$

These correspond to a focal distance

$$\Delta = (|\gamma - \alpha|)^{1/2} = (|a^2 - c^2|)^{1/2}. \tag{8.4}$$

Note that λ and ν have a dimension of *length*^{2}. The coordinate surfaces are spheroids for constant λ and hyperboloids for constant ν with the z-axis as rotation axis. The case for flattened disks obtains, when $-\alpha > -\gamma$, so that $-\gamma = c^2 \leq \nu \leq -\alpha = a^2 \leq \lambda$. Spheroids of constant λ then are prolate, while the hyperboloids of constant ν have two sheets. On each meridional plane of constant ϕ we then have elliptical coordinates (λ, ν) with foci on the z-axis at $z = \pm\Delta$. Note that the mass distribution is oblate, although the coordinate system is prolate.

Other relations between the two coordinate systems are

$$r^2 = \frac{(\lambda+\alpha)(\nu+\alpha)}{\alpha-\gamma}, \tag{8.5a}$$

$$z^2 = \frac{(\lambda+\gamma)(\nu+\gamma)}{\gamma-\alpha}, \tag{8.5b}$$

and

$$\lambda, \nu = \tfrac{1}{2}(r^2 + z^2 - \gamma - \alpha) \pm \tfrac{1}{2}\sqrt{(r^2 - z^2 + \gamma - \alpha)^2 + 4r^2 z^2}. \tag{8.6}$$

Also

$$\lambda + \nu = r^2 + z^2 - \alpha - \gamma, \tag{8.7a}$$

$$\lambda\nu = \alpha\gamma - \gamma r^2 - \alpha z^2. \tag{8.7b}$$

Note that ν and λ occupy different, but contiguous parts of the positive real line. In the plane we have $\nu = -\gamma$, $\lambda = r^2 - \alpha$ and on the z-axis $\nu = z^2 - \gamma$, $\lambda = -\alpha$ for $0 \leq |z| \leq \Delta$ and $\nu = -\alpha$, $\lambda = z^2 - \gamma$ for $|z| \geq \Delta$.

8.2 The potential and the density distribution.

Now suppose that the potential Φ, which is minus the usual potential Φ and therefore always positive, can be separated as follows

$$\Phi(\lambda,\nu) = \frac{(\lambda+\gamma)G(\lambda)-(\nu+\gamma)G(\nu)}{\lambda-\nu}. \tag{8.8}$$

Such potentials are called (axi-symmetric) Stäckel potentials. For models with a finite mass M the potential should tend to zero for large radii, which means that for $\lambda \to \infty$ we get

$$G(\lambda) \sim \frac{GM}{\lambda^{1/2}}. \tag{8.9}$$

The density ρ, which is defined such that $\rho\, dx\, dy\, dz$ is the mass in the volume element $dx\, dy\, dz$, can be calculated from Poisson's equation, which has the complicated form

$$\pi G\rho(\lambda,\nu)(\nu-\lambda) = (\lambda+\alpha)(\lambda+\gamma)\frac{\partial^2\Phi}{\partial\lambda^2} + \left(\tfrac{3}{2}\lambda+\tfrac{1}{2}\alpha+\gamma\right)\frac{\partial\Phi}{\partial\lambda} -$$

$$(\nu+\alpha)(\nu+\gamma)\frac{\partial^2\Phi}{\partial\nu^2} - \left(\tfrac{3}{2}\nu+\tfrac{1}{2}\alpha+\gamma\right)\frac{\partial\Phi}{\partial\nu}. \tag{8.10}$$

The Kuzmin equations give the properties, when the density on the z-axis are given. Assume that this density is $\varphi(\tau)$, where $\tau = \lambda, \nu$ and note from above that on the z-axis we always have $\tau = z^2 - \gamma$ for all z. Then the density is

$$\rho(z) = \varphi(z^2-\gamma) = \varphi(\tau). \tag{8.11}$$

Define the primitive function of $\varphi(\tau)$ as

$$\psi(\tau) = \int_{-\gamma}^{\tau} \varphi(\sigma)\, d\sigma. \tag{8.12}$$

Then

$$\rho(\lambda,\nu) = \left(\frac{\lambda+\alpha}{\lambda-\nu}\right)^2\varphi(\lambda) - 2\frac{(\lambda+\alpha)(\nu+\alpha)}{(\lambda-\nu)^2}\frac{\psi(\lambda)-\psi(\nu)}{\lambda-\nu} + \left(\frac{\nu+\alpha}{\lambda-\nu}\right)^2\varphi(\nu). \tag{8.13}$$

The total mass is

$$M = 2\pi\int_{-\gamma}^{\infty}\frac{\sigma+2\gamma-\alpha}{\sqrt{\sigma+\gamma}}\varphi(\sigma)\, d\sigma = 4\pi\int_0^{\infty}(z^2+\Delta^2)\varphi(z)\, dz. \tag{8.14}$$

The potential follows from

$$G(\tau) = 2\pi G\psi(\infty) - \frac{2\pi G}{\sqrt{\tau+\gamma}}\int_{-\gamma}^{\tau}\frac{\sigma+\alpha}{2(\sigma+\gamma)^{3/2}}\psi(\sigma)\, d\sigma. \tag{8.15}$$

8.3 Velocities and angular momentum.

In order to convert velocities we write

$$\cos\Theta = \left[\frac{(\nu+\alpha)(\lambda+\gamma)}{(\alpha-\gamma)(\lambda-\nu)}\right]^{1/2}, \tag{8.16a}$$

$$\sin\Theta = \left[\frac{(\lambda+\alpha)(\nu+\gamma)}{(\gamma-\alpha)(\lambda-\nu)}\right]^{1/2}. \tag{8.16b}$$

Velocities are related for the oblate mass models ($\gamma - \alpha > 0$) as

$$V_{\rm r} = V_\lambda \cos\Theta - V_\nu \sin\Theta, \tag{8.17a}$$

$$\text{sign}\ (z)\ V_{\rm z} = V_\lambda \sin\Theta + V_\nu \cos\Theta, \tag{8.17b}$$

and

$$V_\lambda = V_{\rm r}\cos\Theta + \text{sign}\ (z)\ V_{\rm z}\sin\Theta, \tag{8.18a}$$

$$V_\nu = -V_{\rm r}\sin\Theta + \text{sign}\ (z)\ V_{\rm z}\cos\Theta. \tag{8.18b}$$

Note that V_λ and V_ν are the velocities in the local Cartesian system and do *not* describe the changes in the coordinates λ and ν.

For the momenta we need the coefficients of the coordinate system

$$P^2 = \frac{\lambda-\nu}{4(\lambda+\alpha)(\lambda+\gamma)}, \tag{8.19a}$$

$$R^2 = \frac{\nu-\lambda}{4(\nu+\alpha)(\nu+\gamma)}. \tag{8.19b}$$

The momenta then are

$$p_\lambda = PV_\lambda, \quad p_\phi = rV_\phi, \quad p_\nu = RV_\nu. \tag{8.20}$$

The angular momenta are

$$L_{\rm x} = y\dot{z} - z\dot{y} = rV_{\rm z}\sin\phi - z(V_{\rm r}\sin\phi + V_\phi\cos\phi), \tag{8.21a}$$

$$L_{\rm y} = z\dot{x} - x\dot{z} = -rV_{\rm z}\cos\phi + z(V_{\rm r}\cos\phi - V_\phi\sin\phi), \tag{8.21b}$$

$$L_{\rm z} = x\dot{y} - y\dot{x} = rV_\phi. \tag{8.21c}$$

The total angular momentum L is

$$L^2 = (r^2+z^2)V_\phi^2 + (rV_{\rm z} - zV_{\rm r})^2. \tag{8.22}$$

8.4 Integrals of motion and the orbits.

The Hamiltonian H is

$$H = \frac{p_\lambda^2}{2P^2} + \frac{p_\phi^2}{2r^2} + \frac{p_\nu^2}{2R^2} - \Phi(\lambda,\nu). \tag{8.23}$$

It can then be shown that there are three integrals of motion, namely

$$E = -H, \tag{8.24a}$$

$$I_2 = \tfrac{1}{2}L_{\rm z}^2, \tag{8.24b}$$

$$I_3 = \tfrac{1}{2}(L_{\rm x}^2 + L_{\rm y}^2) + (\gamma-\alpha)\left[\tfrac{1}{2}V_{\rm z}^2 - z^2\frac{G(\lambda)-G(\nu)}{\lambda-\nu}\right]. \tag{8.24c}$$

The equations of motion then are

$$p_\lambda^2 = \frac{1}{2(\lambda+\alpha)}\left[G(\lambda) - \frac{I_2}{\lambda+\alpha} - \frac{I_3}{\lambda+\gamma} - E\right], \tag{8.25a}$$

$$p_\phi^2 = 2I_2, \tag{8.25b}$$

$$p_\nu^2 = \frac{1}{2(\nu+\alpha)}\left[G(\nu) - \frac{I_2}{\nu+\alpha} - \frac{I_3}{\nu+\gamma} - E\right]. \tag{8.25c}$$

In the meridional plane the orbits are restricted to the area defined by

$$-\gamma \le \nu \le \nu_0, \quad \lambda_1 \le \lambda \le \lambda_2, \tag{8.26}$$

where the turning points ν_0, λ_1 and λ_2 are the values for ν and λ for which respectively V_ν and V_λ are zero. The case $\nu = -\gamma$ corresponds to $z = 0$. The turning points are the three solutions $\tau_1 \le \tau_2 \le \tau_3$ of

$$G(\tau) - \frac{I_2}{\tau+\alpha} - \frac{I_3}{\tau+\gamma} - E = 0, \tag{8.27}$$

where in general there should be one solution $\tau_1 \le -\alpha$, which is ν_0, and two $-\alpha \le \tau_2 \le \tau_3$, which are λ_1 and λ_2. In the case of an oblate mass distribution (prolate coordinate system) all orbits are "short axis tubes", bounded by two prolate spheroids and one hyperboloid of one sheet.

Stäckel potentials were used for galactic dynamics by Eddington in 1915, who showed that the velocity dispersion tensor is diagonal in the coordinate system in which the potential is separable. Hence, the principal axes of the velocity ellipsoid everywhere line up with the local coordinate system. It is this property, plus the fact that three isolating integrals of motion can be written explicitly, that makes Stäckel potentials most useful. In the tri-axial case studied by de Zeeuw, it can be shown that stellar orbits can be classified in four families, namely box, inner long axis, outer long axis and short axis tube orbits. In axisymmetric potentials only the last family exists as a set of stable orbits.

8.5 Stäckel models for the Galaxy.

A few Stäckel models have appeared in the literature, but these have a number of shortcomings. They will be reviewed here briefly.

• Statler (1989) used a model, which is useful in *local* applications. He starts from a flat rotation curve, so that with rotation velocity V_c

$$-r\frac{d\Psi(r,0)}{dr} = -r\frac{d\lambda}{dr}\frac{d\Psi(\lambda,-\gamma)}{d\lambda} = V_\mathrm{c}^2. \tag{8.28}$$

Note that in the plane $\Psi(r,0) = G(\lambda)$. Also $r^2 = \lambda + \alpha$ and then

$$G(\lambda) = -\frac{V_\mathrm{c}^2}{2}\ln(\lambda+\alpha) + \text{constant}. \tag{8.29}$$

Now let the flat rotation curve extend out to $\lambda = \Lambda = r_\mathrm{halo}^2 - \alpha$ and let the mass within this radius be M. Then

$$GM = V_\mathrm{c}^2 r_\mathrm{halo} = V_\mathrm{c}^2(\Lambda+\alpha)^{1/2}. \tag{8.30}$$

Then for $\lambda \ge \Lambda$ we need

$$G(\lambda) = \frac{GM}{(\lambda+\alpha)^{1/2}}, \tag{8.31}$$

so that

$$G(\Lambda) = V_\mathrm{c}^2. \tag{8.32}$$

Then we get for $\lambda \leq \Lambda$

$$G(\lambda) = \frac{1}{2} V_{\mathrm{c}}^2 \left[2 - \ln \left(\frac{\lambda + \alpha}{\Lambda + \alpha} \right) \right]. \tag{8.33}$$

In order to describe the z-distribution Statler proposed

$$G(\nu) = \alpha V_{\mathrm{c}}^2 \frac{\Delta^2}{(\nu + \gamma)(\nu + \alpha)} \left[\left(\frac{S^2}{C^2} + S \frac{\nu + \gamma}{\Delta^2} \right)^{1/2} - \frac{S}{C} \right]. \tag{8.34}$$

This obviously will not work on that part of the z-axis, where $\nu = -\alpha$ or $|z| \geq \Delta$. In the plane, where $\nu = -\gamma$, we have

$$(\nu + \gamma) G(\nu) = 0. \tag{8.35}$$

As Statler states in his paper, the constant C relates to the density in the plane and the constant S to the integrated surface density. Both monotonically, but not proportionally. In a sense then S/C is something like the vertical scaleheight, which can be taken independent of galactocentric radius. However, S is only locally constant and really $G(\nu)$ has an R-dependence implicit in the constant S. So Statler's formula is only a local approximation and it ignores the local radial density gradient. This model can only be used locally, because $G(\tau)$ is not continuous at $\tau = -\alpha$.

In Statler's model $\Delta = 0.1$ kpc.

• Sommer-Larsen and Zhen (1990) build the Galaxy from two components. The disk (and the central bulge) are the oblate perfect ellipsoid. This has

$$\rho(r, z) = \rho_0 \left(1 + \frac{r^2}{a^2} + \frac{z^2}{c^2} \right)^{-2}. \tag{8.36}$$

Then

$$G(\tau) = \frac{2GM}{\pi} (\tau + \gamma)^{-1/2} \arctan \sqrt{\frac{\tau + \gamma}{-\gamma}}, \tag{8.37}$$

where the total mass M is

$$M = \pi^2 a^2 c \rho_0. \tag{8.38}$$

Again this is only useful *locally*, because the radial profile is not exponential and the thickness not constant. The surface density is

$$\sigma(r) = \frac{\pi \rho_0 c}{2} \left(1 + \frac{r^2}{a^2} \right)^{-3/2}, \tag{8.39}$$

and the equivalent thickness

$$z_0 = \frac{\pi c}{2} \left(1 + \frac{r^2}{a^2} \right)^{1/2} \tag{8.40}$$

The dark halo is the $s = 2$ model of de Zeeuw, Peletier & Franx, which has

$$\rho(0, z) \propto \frac{1}{z^2 + c^2}, \tag{8.41}$$

for which

$$G(\tau) = 4\pi G \rho_0 c^2 \left[\ln \frac{\Delta^2 + c^2}{c^2} - \frac{\tau + \gamma + \Delta^2}{2(\tau + \gamma)} \ln \frac{\tau + c^2 + \gamma}{c^2} + \right.$$

$$\frac{\Delta^2+c^2}{c}\left(\frac{1}{\sqrt{\tau+\gamma}}\arctan\sqrt{\frac{\tau+\gamma}{c^2}}-\frac{1}{\Delta}\arctan\frac{\Delta}{c}\right)\Bigg]. \tag{8.42}$$

For large r this gives a flat rotation curve with an amplitude

$$V_{\rm c}^2 = 4\pi G\rho_0 c^2. \tag{8.43}$$

Sommer-Larsen and Zhen use $\Delta = 4.0$ kpc. The difficulty is, that the disk component is flaring to increasing thickness with galactocentric radius, which is in contradiction with observation and is not exponential.

The use of the *"flattened isochrone"* (Evans *et al.*, 1990) instead of the perfect ellipsoid does not improve things. The general equation

$$G(\tau) = \frac{GM(\tau+\gamma)}{\sqrt{(-\alpha)}+\sqrt{\tau}} \tag{8.44}$$

is used in the extreme case $\gamma = 0$. Then the density distribution has a central axis ratio zero and this ratio becomes $(3/8)^{1/4} \approx 0.78$ at large radii. Then we have

$$\rho(\lambda,\nu) = \frac{Ma^2}{2\pi}\frac{X+2aY+a^2}{Y^3(X+aY+a^2)^2} \tag{8.45}$$

with

$$X = a|z| \quad \text{and} \quad Y^2 = r^2 + (a+|z|)^2. \tag{8.46}$$

The surface density is

$$\sigma(r) = \frac{Ma}{\pi}\frac{1}{Y(Y+a)^2}, \tag{8.47}$$

which is $\propto r^{-3}$ for large r. The equivalent thickness is

$$z_0 = \frac{2Y^2}{2Y+a}, \tag{8.48}$$

which again is $\propto r$ for large r. So again the surface density is not at all exponential and the disk flares at larger galactocentric radii.

- The model of Dejonghe and de Zeeuw (1988) uses the function

$$f(\tau) = (\tau+\gamma)G(\tau), \tag{8.49}$$

so that the potential is

$$\Psi(\lambda,\nu) = \frac{f(\lambda)-f(\nu)}{\lambda-\nu}. \tag{8.50}$$

The function $f(\tau)$, which is monotonically increasing, is then expanded into an interpolating formula

$$\ln f(\tau) = \sum_{i=0}^{k} A_i t^i, \tag{8.51}$$

where

$$t = \frac{2\ln\tau - \ln\lambda_m - \ln(-\gamma)}{\ln\lambda_m - \ln(-\gamma)}. \tag{8.52}$$

They use $\Delta = 0.88$ kpc and 12 constants A_i. Sommer-Larson and Zhen have shown, that the model does not adequately represent the effect of the disk, mainly because of the choice of a small value for Δ.

9 Instabilities and related topics.

9.1 Resonances.

First we will look at resonances. The most important ones are between epicyclic frequency and some other frequency that we will call *pattern speed* Ω_p. The *inner Lindblad resonance* occurs for

$$\Omega_\mathrm{p} = \Omega_\mathrm{rot}(R) - \frac{\kappa}{2}, \tag{9.1}$$

where $\Omega_\mathrm{rot}(R)$ is the angular rotation speed. This resonance occurs at the radius, where –in a rotating frame with angular velocity Ω_p– the particle goes through 2 epicycles in the same time is it goes once around the centre. The resulting orbit in that frame then is closed and has an oval shape. It goes back to Lindblad's discovery that the property $\Omega_\mathrm{rot}(R) - \kappa/2$ in the inner Galaxy is roughly constant with R. The pattern speed may be identified with that of the rotating frame in which the spiral *pattern* (not the spiral arms as physical structures themselves) is stationary or with the body rotation of a bar or oval distortion.

Equivalently we have the *outer Lindblad resonance*

$$\Omega_\mathrm{p} = \Omega_\mathrm{rot}(R) + \frac{\kappa}{2} \tag{9.2}$$

and *co-rotation*

$$\Omega_\mathrm{p} = \Omega_\mathrm{rot}(R). \tag{9.3}$$

Higher order Lindblad resonances (involving κ/n) sometimes also play a role.

9.2 Jeans instability.

Now we will look at local stability. We then start with the Jeans instability in a homogeneous medium. There are various ways of describing it to within an order of magnitude. The first is to make use of the "virial theorem"

$$2\,T_\mathrm{kin} + \Omega = 0 \tag{9.4}$$

for stability against gravitational contraction. In a uniform, isothermal sphere the kinetic energy is

$$T_\mathrm{kin} = 1/2\; M\langle V^2\rangle \tag{9.5}$$

and the potential energy

$$\Omega = -\frac{3}{5}\frac{GM^2}{R}. \tag{9.6}$$

So the sphere is unstable, when its mass M is larger than the Jeans mass M_Jeans, which then comes out as

$$M_\mathrm{Jeans} = \left(\frac{5}{3G}\right)^{3/2}\left(\frac{3}{4\pi}\right)^{1/2}\left(\frac{\langle V^2\rangle^3}{\rho}\right)^{1/2}. \tag{9.7}$$

A method that gives roughly the same result, but is easier to adapt to the two-dimensional case is the following, which starts by calculating the free-fall time of a homogeneous sphere. Anywhere the equation of motion is

$$\frac{d^2r}{dt^2} = -\frac{G\;M(r)}{r^2} = -\frac{4\pi}{3}G\rho r. \tag{9.8}$$

Solve this and apply for $r = 0$, then

$$t_{\rm ff} = \left(\frac{3\pi}{32G\rho}\right)^{1/2}. \tag{9.9}$$

The free-fall time is independent of the initial radius and depends only on the density. Now, if there were no gravity a star will move out to the radius of the sphere R in a time

$$t = \frac{R}{\langle V^2\rangle^{1/2}}. \tag{9.10}$$

For marginal stability the two have to be equal and it follows that the Jeans length is

$$R_{\rm Jeans} = \left(\frac{3\pi}{32}\frac{\langle V^2\rangle}{G\rho}\right)^{1/2}. \tag{9.11}$$

This can be done in a similar manner for an infinitely flat disk. The equation of motion now is (writing immediately R for the radius of the circular area considered)

$$\frac{d^2R}{dt^2} = -\pi G\sigma. \tag{9.12}$$

The free-fall time then becomes

$$t_{\rm ff} = \left(\frac{2R}{\pi G\sigma}\right)^{1/2} \tag{9.13}$$

and the Jeans length

$$R_{\rm Jeans} = \frac{2\langle V^2\rangle}{\pi G\sigma}. \tag{9.14}$$

9.3 Toomre criterion.

Now look at the differentially rotating disk. For each element with radius R_0, the average angular velocity Ω equals obviously B as a result of differential rotation and the specific angular momentum R_0^2B. This gives rise to a centrifugal force $F_{\rm cf} = R_0\Omega^2$. Now let it contract to a radius R. The angular momentum is conserved, so the angular velocity becomes $\Omega = R_0^2B/R^2$ and the centrifugal force $F_{\rm cf} = R\Omega^2 = R_0^4B^2/R^3$. If the contraction is dR, then we have

$$\frac{dF_{\rm cf}}{dR} = -\frac{3R_0^4B^2}{R^4}. \tag{9.15}$$

The gravitational attraction is approximately (within a factor less than two, since we now have a flat distribution) $F_{\rm grav} = -G\pi R_0^2\sigma/R^2$ and

$$\frac{dF_{\rm grav}}{dR} = \frac{2\pi GR_0^2\sigma}{R^3}. \tag{9.16}$$

The critical condition then is that at $R = R_0$ the two must compensate each other and the critical radius then is

$$R_{\rm crit} = \frac{2\pi G\sigma}{3B^2}. \tag{9.17}$$

So for lengthscales larger than this critical radius the disk is stabilized by differential rotation. Toomre realized then that the disk is stable only if this critical radius is smaller than the local Jeans length and therefore the velocity dispersion has to exceed a critical value, which then is

$$\langle V^2\rangle^{1/2}_{\rm crit} = \frac{\pi}{\sqrt{3}}\frac{G\sigma}{B}. \tag{9.18}$$

In practice $-B \approx A$ and then the equation can be written as

$$\langle V^2\rangle^{1/2}_{\rm crit} \approx 2\pi \left(\frac{2}{3}\right)^{1/2} \frac{G\sigma}{\kappa} = 5.13\frac{G\sigma}{\kappa}. \qquad (9.19)$$

This is *Toomre*'s criterion, who used a more precise analysis to arrive at a constant of 3.36.

9.4 Goldreich–Lynden-Bell criterion.

This can also be extended to the criterion, that Goldreich and Lynden-Bell derived for stability of gaseous disks of finite thickness against sheared instabilities. With the equations of the isothermal sheet we can express the velocity dispersion as a function of ρ_0 and z_0. Then for the critical case

$$\frac{\pi}{6}G\frac{\sigma^2}{z_0^2} \approx \rho_0 B^2. \qquad (9.20)$$

Equating σ/z_0 and ρ_0 to a mean density $\overline{\rho}$ and using $(B - A) \approx 2B$, we get

$$\frac{\pi}{3}G\frac{\overline{\rho}}{B(B-A)} \sim 1. \qquad (9.21)$$

From a detailed discussion Goldreich and Lynden-Bell found for stability

$$\frac{\pi G\overline{\rho}}{4B(B-A)} \lesssim 1. \qquad (9.22)$$

These sheared instabilities were proposed as a possible mechanism for the formation of spiral structure. More recently, Toomre has studied the process in stellar disks and finds an instability based on shear due to differential rotation, that he called "*swing amplification*". This process is prevented when

$$X = \frac{R\kappa^2}{2\pi m G\sigma} \gtrsim 3, \qquad (9.23)$$

where m is the number of arms. For $-B \approx A$ (a flat rotation curve) this can be written as

$$\frac{QV_{\rm rot}}{\langle U^2\rangle^{1/2}} \gtrsim 3.97\ m. \qquad (9.24)$$

This is Toomre's local stability citerion if the velocity dispersion is replaced by 0.22 $V_{\rm rot}/m$.

9.5 Global stability.

For global stability there is a global condition due to Efstathiou, Lake and Negroponte from numerical experiments, which reads

$$Y = V_{\rm rot}\left(\frac{h}{GM_{\rm disk}}\right)^{1/2} \gtrsim 1.1. \qquad (9.25).$$

For a pure exponential disk without any dark halo $Y = 0.59$. For a flat rotation curve it is then easy to show that the condition implies that within the disk radius of 4 to 5 scalelengths the mass in the halo should exceed that of the disk by a factor of about 3.5.

For a flat rotation curve and an exponential disk Y can be rewritten as

$$Y = 0.615\ \left\{\frac{QRV_{\rm rot}}{h\langle U^2\rangle^{1/2}}\right\}^{1/2}\ \exp\left(\frac{R}{2h}\right) \qquad (9.26)$$

and this gives

$$\frac{QV_{\rm rot}}{\langle U^2\rangle^{1/2}} \gtrsim 7.91. \tag{9.27}$$

Comparing this to (9.24) we see that for spirals that are stable against global modes, swing amplification is possible for all modes with $m \geq 2$, at least at those radii where the rotation curve is flat.

Ostriker and Peebles have also found from numerical experiments a general condition for global stability. Stability occurs only when the ratio of kinetic energy in rotation S to the potential energy Ω is less than a certain value. This is related to the criterion above. They found that stability requires

$$t = \frac{S}{|\Omega|} \lesssim 0.14. \tag{9.28}$$

The virial theorem says that $2S + R + \Omega = 0$ and since $R/S > 0$, we would have expected t to have the range 0 to 0.5 available. The criterion translates into $R/S \gtrsim 5$, while for the local Galactic disk it is about 0.15. So disk galaxies require additional material with high random motion in order to conform to the criterion, either in the disk itself (e.g. the stars in the central region) or in the dark halo.

9.6 Tidal radii.

Globular clusters have tidal radii due to the force field of the galaxy. These radii can be estimated as follows. Assume two point masses M (the Galaxy) and m (the cluster) and a separation R in a circular orbit (the following can be adapted to elliptical orbits as well with R the smallest separation). The angular velocity of the globular cluster around the center of gravity is

$$\Omega = \left[\frac{G(M+m)}{R^3}\right]^{1/2}, \tag{9.29}$$

while the center of gravity is at a distance $MR/(M+m)$ from the cluster. Take a star at distance r from the center of the cluster in the direction of M and calculate where the total force on that star is zero. Thus

$$\frac{M}{(R-r)^2} - \frac{M}{r^2} - \frac{M+m}{R^3}\left(\frac{MR}{M+m} - r\right) = 0. \tag{9.30}$$

Since r is much less than R we may expand the first term

$$\frac{M}{(R-r)^2} \approx \frac{M}{R^2}\left(1 + 2\frac{r}{R}\right). \tag{9.31}$$

Then the tidal radius is the solution for r of this equation

$$r_{\rm tidal} \sim R\left(\frac{m}{3M}\right)^{1/3}, \tag{9.32}$$

where m has been taken small compared to M.

9.7 Dynamical friction.

As a star moves through a background of other stars, the small deflections will give a small overdensity behind the star and consequently induce a drag. Suppose that a body of mass m moves in a circular orbit with radius R through a background of bodies with mass M at a speed

V_c and assume that the background is an isothermal sphere with V_c the circular speed (and $V_c/2$ the velocity dispersion). Then the loss of angular momentum is about

$$\frac{dJ}{dR} \sim -0.4 \frac{Gm^2}{R} \ln \Lambda, \tag{9.33}$$

where

$$\Lambda = \frac{R_c V_c^2}{G(m+M)}. \tag{9.34}$$

R_c is the core radius of the isothermal sphere (the typical lengthscale of the background density distribution). The timescale of dynamical friction for the body to spiral into the center is then

$$t_{df} \sim \frac{R^2 V_c}{Gm \ln \Lambda}. \tag{9.35}$$

This timescale is large and only relevant for globular clusters in the inner few kpc of the halo or for galaxies in the central parts of clusters of galaxies. The effect may contribute to respectively the formation of galactic nuclei and the central cD or gE galaxies in clusters through cannibalism. The timescale for the Magellanic Clouds to be drawn in into the Galaxy is about another 10^{10} years.