# THE THEORY OF CHEMICAL EVOLUTION

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## 1 Star formation.

The discussion below assumes that we are considering a particular volume of a galaxy or a cluster or galaxy as a whole. Although not mentioned explicitely, most properties therefore can have the additional dimension $\mathrm{pc}^{-3}$ or so.

The number of stars formed with mass between $M$ and $M+dM$ at time $t$ is

$$\Psi(t)\Phi(M) \tag{1}$$

with for example

$$\Phi(M) = xM_{\mathrm{L}}^{x}M^{-(1+x)} \quad \text{for} \quad M_{\mathrm{L}} \leq M \leq M_{\mathrm{U}}. \tag{2}$$

For practicle purposes we may take $M_{\mathrm{U}} = \infty$. The total star formation at time $t$ in mass per $dt$ is then

$$A(t) = \int_0^\infty M\Psi(t)\Phi(M)\,dM = \Psi(t)\frac{x}{x-1}M_{\mathrm{L}}. \tag{3}$$

Here $\Phi(M)$ is the **Initial Mass Function (IMF)** and $\Psi(t)$ the **Star Formation Rate (SFR)**. $A(t)$ is the total star formation.

## 2 Gas mass.

The gas mass $M_{\mathrm{g}}(t)$ evolves according to

$$\frac{dM_{\mathrm{g}}(t)}{dt} = -A(t) + R(t) + f(t) - g(t). \tag{4}$$

$R(t) =$ mass returned to ISM from evolved stars.
$f(t) =$ infow of gas with a certain abundance.
$g(t) =$ outflow of gas with the current (ISM) abundance.

Each star resides on the main sequence for a time $\tau_{\mathrm{M}}$ and then ejects all its mass, except for a quantity $M_{\mathrm{R}}$. Thus, if $M_{\mathrm{t}}$ is the stellar mass for which $\tau_{\mathrm{M}} = t$,

$$R(t) = \int_{M_{\mathrm{t}}}^\infty (M - M_{\mathrm{R}})\Psi(t-\tau_{\mathrm{M}})\Phi(M)\,dM. \tag{5}$$

## 6 Summary.

♠ **Evolution of the gas mass:**
This is eq. (8):

$$\frac{dM_{\rm g}(t)}{dt} = -(1-R)A(t) + f(t) - g(t). \tag{64}$$

♠ **Evolution of the abundance:**
This is eq. (12):

$$\frac{d(M_{\rm Z})_{\rm g}(t)}{dt} = \frac{d}{dt}\left[M_{\rm g}(t)Z(t)\right] = -(1-R)Z(t)A(t) + P_{\rm Z}A(t) + Z_{\rm f}f(t) - Z(t)g(t). \tag{65}$$

♠ **Average age of the heavy elements:**
This is eq. (63):

$$\bar{t} = T_\odot - t_\nu \left(1 + \frac{A_0}{D}\right)^{-1}, \tag{66}$$

where

$$t_\nu = \frac{1}{D}\int_0^{T_\odot} A(t)t \exp\{\nu(t)\}\, dt, \tag{67}$$

$$D = \int_0^{T_\odot} A(t) \exp\{\nu(t)\}\, dt, \tag{68}$$

$$\nu(t) = \int_0^t \omega_{\rm g}(t')\, dt', \tag{69}$$

$$\omega_{\rm g}(t) = A(t)\frac{1-R}{M_{\rm g}(t)}, \tag{70}$$

$$A_0 = \frac{M_{\rm g}(0)Z_0}{P_{\rm Z}}. \tag{71}$$

♠ **Mass in stars and stellar remnants:**

$$\frac{dM_*(t)}{dt} = (1-R)A(t). \tag{72}$$

♠ **"Yield" of heavy elements:**

$$y = \frac{P_{\rm Z}}{1-R}. \tag{73}$$



In the case of the **Instantaneous Recycling Approximation (IRA)** we have $\tau_{\rm M} = 0$ for $M > M_{\rm t}$ and $\tau_{\rm M} = \infty$ for $M < M_{\rm t}$. Then

$$R(t) = \Psi(t)\int_{M_{\rm t}}^\infty (M - M_{\rm R})\Phi(M)\, dM = A(t)R, \tag{6}$$

where

$$R = (x-1)M_{\rm L}^{x-1}\left(\frac{M_{\rm t}^{1-x}}{x-1} - \frac{M_{\rm R}}{x}M_{\rm t}^{-x}\right). \tag{7}$$

For each IMF and when $M_{\rm R}$ is known (really as a function of the original stellar mass), $R$ can be calculated. For example, when $M_{\rm L} = 0$, $M_{\rm t} = 1M_\odot$, $M_{\rm R} = 0.8M_\odot$ and $x = 1.35$, we get $R \approx 0.2$.

So we have as fundamental equation

$$\frac{dM_{\rm g}(t)}{dt} = -(1-R)A(t) + f(t) - g(t), \tag{8}$$

or alternatively

$$\frac{dM_{\rm g}(t)}{dt} = -\frac{dM_*(t)}{dt} + f(t) - g(t). \tag{9}$$

$M_*(t)$ is the total mass in stars and stellar remnants at time $t$.

## 3 Heavy elements in the gas.

Define the abundance of the gas at time $t$ as $Z(t)$. The the total amount of heavy elements (metals) is

$$(M_{\rm Z})_{\rm g}(t) = M_{\rm g}(t)Z(t). \tag{10}$$

Then it follows:

$$\frac{d(M_{\rm Z})_{\rm g}(t)}{dt} = -Z(t)A(t) + \int_{M_{\rm t}}^\infty (M - M_{\rm R})Z(t-\tau_{\rm M})\Phi(M)\Psi(t-\tau_{\rm M})\, dM + \int_{M_{\rm t}}^\infty P_{\rm Z}(M)M\Phi(M)\Psi(t-\tau_{\rm M})\, dM + Z_{\rm f}f(t) - Z(t)g(t). \tag{11}$$

Here $P_{\rm M}$ is the fraction of the total mass of a star of mass $M$, that is being expelled in the form of metals synthesized in that star.

In the case of the IRA we have as the second fundamental equation

$$\frac{d(M_{\rm Z})_{\rm g}(t)}{dt} = \frac{d}{dt}\left[M_{\rm g}(t)Z(t)\right] = -(1-R)Z(t)A(t) + P_{\rm Z}A(t) + Z_{\rm f}f(t) - Z(t)g(t). \tag{12}$$

Here

$$P_{\rm Z} = \frac{x-1}{xM_{\rm L}} \int_1^\infty P_{\rm Z}(M) M \Phi(M) dM. \tag{13}$$

For realistic situations we have $P_{\rm Z} = 0.005$ - $0.05$.

Often in use is also the so-called **"yield"**

$$y = \frac{P_{\rm Z}}{1-R}. \tag{14}$$

Then the alternative equation reads

$$\frac{d(M_{\rm Z})_{\rm g}(t)}{dt} = -Z(t)\frac{dM_*(t)}{dt} + y\frac{dM_*(t)}{dt} + Z_{\rm f} f(t) - Z(t)g(t). \tag{15}$$

In most applications $Z_{\rm f} = 0$; for galactic disks it is reasonable to assume $g(t) = 0$.

## 4 Radio-active elements in the gas.

Element X has an abundance $X(t)$, which is simply the equivalent to $Z(t)$ for all metals. If $Z_{\rm f} = 0$ and $g(t) = 0$ the form of eq. (12) for this element becomes

$$\frac{d(M_{\rm X})_{\rm g}(t)}{dt} = \frac{d}{dt}\left[X(t)M_{\rm g}(t)\right] = -\lambda_{\rm X}(M_{\rm X})_{\rm g}(t) - (1-R)X(t)A(t) + P_{\rm X}A(t). \tag{16}$$

Here $\lambda_{\rm X}$ is the decay-constant for that element. Now the number of atoms of X (in the volume that we are considering), which each have atomic weight $A_{\rm X}$, is

$$N_{\rm X} = \frac{(M_{\rm X})_{\rm g}(t)}{A_{\rm X}m_{\rm H}} = \frac{X(t)M_{\rm g}(T)}{A_{\rm X}m_{\rm H}}. \tag{17}$$

Therefore

$$\begin{aligned}\frac{dN_{\rm X}}{dt} &= -\lambda_{\rm X}N_{\rm X}(t) - \frac{1-R}{M_{\rm g}(t)}N_{\rm X}(t)A(t) + \frac{P_{\rm X}}{A_{\rm X}m_{\rm H}}A(t) \\ &= -\lambda_{\rm X}N_{\rm X}(t) - \omega_{\rm g}(t) + PA(t),\end{aligned} \tag{18}$$

where we have defined

$$\omega_{\rm g} = A(t)\frac{1-R}{M_{\rm g}(t)} \tag{19}$$



The property $D$, defined in eq. (29), then is

$$D = \int_0^t A(t') \exp\{\nu(t')\}\, dt' = \frac{M_{\rm g}(0)}{1-R}\phi(t). \tag{56}$$

Then the final result for $\tau_\nu$, defined in eq. (30), is

$$\tau_\nu = \frac{1}{D}\int_0^t t' A(t') \exp\{\nu(t')\}\, dt' = \frac{1}{\phi(t)}\int_0^t t'\omega_{\rm g}(t') \exp\{\theta(t')\}\, dt'. \tag{57}$$

For the radio-active elements we have written

$$A_0 = \frac{N_{\rm X}Z_0}{P}$$

and equivalently we write

$$A_0 = \frac{M_{\rm g}(0)Z_0}{P_{\rm Z}}, \tag{58}$$

so that

$$\phi(t) = \frac{Z_0(1-R)D}{P_{\rm Z}A_0}. \tag{59}$$

Then

$$\frac{(1-R)Z_0}{P_{\rm Z}} + \phi(t) = \frac{\omega_{\rm g}(0)}{A_0}(A_0+D) = \frac{1-R}{M_{\rm g}(0)}(A_0+D) = (A_0+D)\frac{\phi(t)}{D}. \tag{60}$$

So eq. (49) becomes

$$t_{\rm Z} = \frac{1}{\phi(t)}\left(1+\frac{A_0}{D}\right)^{-1}\int_0^t \tau\omega_{\rm g}(\tau)\exp\{\theta(\tau)\}\, d\tau \tag{61}$$

and therefore

$$t_{\rm Z} = t_\nu\left(1+\frac{A_0}{D}\right)^{-1}. \tag{62}$$

If the time is $T_\odot$, then it follows from the definition of $t_{\rm Z}$ that the average age at that time is equal to $T_\odot - t_{\rm Z}$. With the definition of $\bar{t}$ in eq. (31) and with eq. (28) it is easily found that, for $\Delta$ small, we get

$$\bar{t} = T_\odot - t_\nu\left(1+\frac{A_0}{D}\right)^{-1}. \tag{63}$$

So we see that $\bar{t} = T_\odot - t_{\rm Z}$ is the average age of the heavy elements. Of course we measure that $\bar{t}$ for which $T_\odot$ is the time of formation of the solar system.

Integrate this with $\phi(t)$ as variable, then

$$\int_\tau^t \frac{P_Z \omega_g(t')}{(1-R)Z(t')} dt' = -\ln\left[\frac{(1-R)Z_0}{P_Z} + \phi(t')\right]_{t'=\tau}^{t'=t}. \quad (46)$$

The solution then is

$$p(\tau,t) = \frac{P_Z \omega_g(t) \exp\{\theta(\tau)\}}{(1-R)Z_0 + P_Z\phi(t)}. \quad (47)$$

The average epoch of formation of the heavy elements is by definition

$$t_Z \equiv \int_0^t \tau p(\tau,t)\, d\tau. \quad (48)$$

So

$$t_Z = \left[\frac{(1-R)Z_0}{P_Z} + \phi(t)\right]^{-1} \int_0^t \tau \omega_g(\tau) \exp\{\theta(\tau)\}\, d\tau. \quad (49)$$

To find $\omega_g(t)$ we first need $A(t)$ and $M_g(t)$. Now

$$\frac{dM_g(t)}{dt} = -(1-R)A(t) + f(t), \quad (50)$$

so that

$$\frac{1}{M_g(t)} = \frac{\omega_g(t)}{A(t)(1-R)} = \omega_g(t)\left[f(t) - \frac{dM_g(t)}{dt}\right]^{-1}, \quad (51)$$

or

$$-\frac{dM_g(t)}{M_g(t)} = \left[-\frac{f(t)}{M_g(t)} + \omega_g(t)\right] dt. \quad (52)$$

Integrate this over time from 0 to $t$:

$$-\ln M_g(t) = -\int_0^t \frac{f(t')}{M_g(t')}dt' + \int_0^t \omega_g(t')\, dt' = -\theta(t) + \nu(t). \quad (53)$$

Then

$$M_g(t) = M_g(0) \exp\{\theta(t) - \nu(t)\} \quad (54)$$

and

$$A(t) = \frac{M_g(0)}{1-R}\omega_g(t) \exp\{\theta(t) - \nu(t)\}. \quad (55)$$



and

$$P = \frac{P_X}{A_X m_H}. \quad (20)$$

Since we have

$$\frac{dM_g(t)}{dt} = -(1-R)A(t) + f(t), \quad (21)$$

it follows that

$$\omega_g(t) = -\frac{1}{M_g(t)}\frac{dM_g(t)}{dt} + \frac{f(t)}{M_g(t)}. \quad (22)$$

Thus

$$\frac{dN_X}{dt} = [-\lambda_X - \omega_g(t)]\, N_X(t) + PA(t). \quad (23)$$

This differential equation can be solved to give

$$N_X(t) \exp\{\lambda_X t + \nu(t)\} = N_X(0) + P\int_0^t A(t') \exp\{\lambda_X t' + \nu(t')\}\, dt', \quad (24)$$

where

$$\nu(t) = \int_0^t \omega_g(t')\, dt' = \ln\frac{M_g(0)}{{}_g(t)} + \int_0^t \frac{f(t')}{M_g(t')}dt'. \quad (25)$$

Now, if the solar system formed at time, say, $t$ and if this was preceeded by a period $\Delta$, in which no further heavy elements were added to the gas cloud, then the abundance of the gas at the time of the formation of the solar system must have been

$$N_x(t+\Delta) = N_X(t) \exp(-\lambda_X \Delta)$$

$$= P \exp\{-\lambda_X t - \nu(t) - \lambda_X\Delta\}\left[\frac{N_X(0)}{P} + \int_0^t A(t') \exp\{\lambda_X t' + \nu(t')\}\, dt'\right]. \quad (26)$$

Write

$$A_0 = \frac{N_0}{P} \quad (27)$$

and consider long-lived elements, such that $\lambda_X t \ll 1$ and therefore $\exp(\lambda_X t) = 1 + \lambda_X t$ when terms of higher order are neglected. Then

$$N_X(t+\Delta) = P \exp\{-\nu(t) - \Delta\lambda_X\}(1-\lambda_X t)(A_0 + D + \lambda_X D t_\nu), \quad (28)$$

where

$$D = \int_0^t A(t') \exp\{\nu(t')\}\, dt' \quad (29)$$

and

$$t_\nu = \frac{1}{D}\int_0^t A(t')t' \exp\{\nu(t')\}\, dt'. \tag{30}$$

For two isotopes $i$ and $j$,both with long decay-times, we can then measure

$$\bar{t} = \frac{1}{\lambda_\mathrm{i} - \lambda_\mathrm{j}} \ln\left[\frac{P_\mathrm{i}}{P_\mathrm{j}}\frac{N_\mathrm{j}(t+\Delta)}{N_\mathrm{i}(t+\Delta)}\right] - \Delta. \tag{31}$$

The ratio $P_\mathrm{i}/P_\mathrm{j}$ follows from the theory of nucleosynthesis (in practice we have to do with r-process elements) and $N_\mathrm{j}/N_\mathrm{i}$ must be measured by laboratory analysis of meteorites. $\Delta$ can be measured with the use of short-lived elements ($\approx 10^8$ years according to $^{129}$I and $^{244}$Pu, but $\approx 2\times10^6$ years according to $^{26}$Al). The isotope combinations that are being used for this cosmonucleochronology are ($^{235}$U, $^{238}$U), ($^{238}$U, $^{232}$Th) and ($^{187}$Re, $^{187}$Os). The best answer at present is $\approx (2-4)\times 10^9$ years.

## 5 Age of the heavy elements.

The property $\bar{t}$ that we found for two radio-active elements still needs to be interpreted in terms of the synthesis history of the metals. This will be done in this section. We will find that $\bar{t}$ equals the average age of the heavy elements.

Define $p(\tau,t)\, d\tau$ as the fraction of metals present at time $t$ and formed between $\tau$ and $\tau + d\tau$, where of course we have $\tau < t$. Now we had eq. (12), which reads without the outflow and for an inflow with unenriched gas

$$\frac{d}{dt}\left[M_\mathrm{g}(t)Z(t)\right] = -(1-R)Z(t)A(t) + P_\mathrm{Z}A(t). \tag{32}$$

The equivalent of this for the heavy elements only is

$$\frac{d}{dt}\left[M_\mathrm{Z}(t)p(\tau,t)\right] = -(1-R)Z(t)A(t)p(\tau,t) + P_\mathrm{Z}A(t)\delta(t-\tau). \tag{33}$$

From these two equation it follows that

$$\frac{dp(\tau,t)}{dt} = \frac{P_\mathrm{Z}A(t)}{M_\mathrm{g}(t)Z(t)}\left[\delta(t-\tau) - p(\tau,t)\right]. \tag{34}$$



Integrate this over time from $\tau$ to $t$ and then differentiate with respect to $t$:

$$\frac{dp(\tau,t)}{dt} = -\frac{P_\mathrm{Z}A(t)}{M_\mathrm{g}(t)Z(t)}p(\tau,t) \tag{35}$$

and

$$p(\tau,\tau) = \left[\frac{P_\mathrm{Z}A(t)}{M_\mathrm{g}(t)Z(t)}\right]_{\mathrm{t}=\tau}. \tag{36}$$

Before we proceed we first need to search for a solution for $Z(t)$. According to eq. (8) and (12) we may write

$$\frac{dZ(t)}{dt}M_\mathrm{g}(t) = P_\mathrm{Z}A(t) - Z(t)f(t), \tag{37}$$

or

$$\frac{dZ(t)}{dt} + \frac{Z(t)f(t)}{M_\mathrm{g}(t)} = \frac{P_\mathrm{Z}A(t)}{M_\mathrm{g}(t)} = \frac{P_\mathrm{Z}\omega_\mathrm{g}(t)}{1-R}. \tag{38}$$

This is the same differential equation as we had above, so the solution is

$$Z(t) = Z_0 \exp\{-\theta(t)\} + \frac{P_\mathrm{Z}}{1-R}\exp\{-\theta(t)\}\phi(t), \tag{39}$$

where

$$\theta(t) = \int_0^t \frac{f(t')}{M_\mathrm{g}(t')}\, dt' \tag{40}$$

and

$$\phi(t) = \int_0^t \omega_\mathrm{g}(t')\exp\{\theta(t')\}\, dt'. \tag{41}$$

Now

$$\frac{dp(\tau,t)}{dt} = -\frac{P_\mathrm{Z}A(t)}{M_\mathrm{g}(t)Z(t)}p(\tau,t) = -\frac{P_\mathrm{Z}\omega_\mathrm{g}(t)}{(1-R)Z(t)}p(\tau,t). \tag{42}$$

Thus

$$\frac{dp(\tau,t)}{p(\tau,t)} = -\frac{P_\mathrm{Z}\omega_\mathrm{g}(t)}{(1-R)Z(t)}dt \tag{43}$$

and

$$p(\tau,t) = p(\tau,\tau)\exp\left\{\int_\tau^t \frac{P_\mathrm{Z}\omega_\mathrm{g}(t')}{(1-R)Z(t')}dt'\right\}. \tag{44}$$

Now also

$$\frac{P_\mathrm{Z}\omega_\mathrm{g}(t)}{(1-R)Z(t)} = \frac{P_\mathrm{Z}\omega_\mathrm{g}(t)\exp\{\theta(t)\}}{(1-R)Z_0 + P_\mathrm{Z}\phi(t)}. \tag{45}$$