THE THEORY OF CHEMICAL EVOLUTION

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1 Star formation.

The discussion below assumes that we are considering a particular volume of a galaxy or a cluster or galaxy as a whole. Although not mentioned explicitely, most properties therefore can have the additional dimension pc^{-3} or so.

The number of stars formed with mass between M and $M + dM$ at time t is

$$\Psi(t)\Phi(M) \tag{1}$$

with for example

$$\Phi(M) = xM_{\mathrm{L}}^{x}M^{-(1+x)} \quad \text{for} \quad M_{\mathrm{L}} \leq M \leq M_{\mathrm{U}}. \tag{2}$$

For practicle purposes we may take $M_{\mathrm{U}} = \infty$. The total star formation at time t in mass per dt is then

$$A(t) = \int_0^\infty M\Psi(t)\Phi(M)\, dM = \Psi(t)\frac{x}{x-1}M_{\mathrm{L}}. \tag{3}$$

Here $\Phi(M)$ is the **Initial Mass Function (IMF)** and $\Psi(t)$ the **Star Formation Rate (SFR)**. $A(t)$ is the total star formation.

2 Gas mass.

The gas mass $M_{\mathrm{g}}(t)$ evolves according to

$$\frac{dM_{\mathrm{g}}(t)}{dt} = -A(t) + R(t) + f(t) - g(t). \tag{4}$$

$R(t)$ = mass returned to ISM from evolved stars.
$f(t)$ = infow of gas with a certain abundance.
$g(t)$ = outflow of gas with the current (ISM) abundance.

Each star resides on the main sequence for a time τ_{M} and then ejects all its mass, except for a quantity M_{R}. Thus, if M_{t} is the stellar mass for which $\tau_{\mathrm{M}} = t$,

$$R(t) = \int_{M_{\mathrm{t}}}^\infty (M - M_{\mathrm{R}})\Psi(t - \tau_{\mathrm{M}})\Phi(M)\, dM. \tag{5}$$

In the case of the **Instantaneous Recycling Approximation (IRA)** we have $\tau_{\mathrm{M}} = 0$ for $M > M_{\mathrm{t}}$ and $\tau_{\mathrm{M}} = \infty$ for $M < M_{\mathrm{t}}$. Then

$$R(t) = \Psi(t)\int_{M_{\mathrm{t}}}^\infty (M - M_{\mathrm{R}})\Phi(M)\, dM = A(t)R, \tag{6}$$

where

$$R = (x-1)M_{\mathrm{L}}^{x-1}\left(\frac{M_{\mathrm{t}}^{1-x}}{x-1} - \frac{M_{\mathrm{R}}}{x}M_{\mathrm{t}}^{-x}\right). \tag{7}$$

For each IMF and when M_{R} is known (really as a function of the original stellar mass), R can be calculated. For example, when $M_{\mathrm{L}} = 0$, $M_{\mathrm{t}} = 1M_\odot$, $M_{\mathrm{R}} = 0.8M_\odot$ and $x = 1.35$, we get $R \approx 0.2$.

So we have as fundamental equation

$$\frac{dM_{\mathrm{g}}(t)}{dt} = -(1 - R)A(t) + f(t) - g(t), \tag{8}$$

or alternatively

$$\frac{dM_{\rm g}(t)}{dt} = -\frac{dM_*(t)}{dt} + f(t) - g(t). \tag{9}$$

$M_*(t)$ is the total mass in stars and stellar remnants at time t.

3 Heavy elements in the gas.

Define the abundance of the gas at time t as $Z(t)$. The the total amount of heavy elements (metals) is

$$(M_{\rm Z})_{\rm g}(t) = M_{\rm g}(t)Z(t). \tag{10}$$

Then it follows:

$$\frac{d(M_{\rm Z})_{\rm g}(t)}{dt} = -Z(t)A(t) + \int_{M_{\rm t}}^{\infty}(M - M_{\rm R})Z(t-\tau_{\rm M})\Phi(M)\Psi(t-\tau_{\rm M})\, dM +$$

$$\int_{M_{\rm t}}^{\infty} P_{\rm Z}(M)M\Phi(M)\Psi(t-\tau_{\rm M})\, dM \ + Z_{\rm f}f(t) - Z(t)g(t). \tag{11}$$

Here $P_{\rm M}$ is the fraction of the total mass of a star of mass M, that is being expelled in the form of metals synthesized in that star.

In the case of the IRA we have as the second fundamental equation

$$\frac{d(M_{\rm Z})_{\rm g}(t)}{dt} = \frac{d}{dt}\left[M_{\rm g}(t)Z(t)\right] = -(1-R)Z(t)A(t) + P_{\rm Z}A(t) + Z_{\rm f}f(t) - Z(t)g(t). \tag{12}$$

Here

$$P_{\rm Z} = \frac{x-1}{xM_{\rm L}}\int_1^{\infty} P_{\rm Z}(M)M\Phi(M)dM. \tag{13}$$

For realistic situations we have $P_{\rm Z}$ = 0.005 - 0.05.

Often in use is also the so-called **"yield"**

$$y = \frac{P_{\rm Z}}{1-R}. \tag{14}$$

Then the alternative equation reads

$$\frac{d(M_{\rm Z})_{\rm g}(t)}{dt} = -Z(t)\frac{dM_*(t)}{dt} + y\frac{dM_*(t)}{dt} + Z_{\rm f}f(t) - Z(t)g(t). \tag{15}$$

In most applications $Z_{\rm f} = 0$; for galactic disks it is reasonable to assume $g(t) = 0$.

4 Radio-active elements in the gas.

Element X has an abundance $X(t)$, which is simply the equivalent to $Z(t)$ for all metals. If $Z_{\rm f} = 0$ and $g(t) = 0$ the form of eq. (12) for this element becomes

$$\frac{d(M_{\rm X})_{\rm g}(t)}{dt} = \frac{d}{dt}\left[X(t)M_{\rm g}(t)\right] = -\lambda_{\rm X}(M_{\rm X})_{\rm g}(t) - (1-R)X(t)A(t) + P_{\rm X}A(t). \tag{16}$$

Here $\lambda_{\rm X}$ is the decay-constant for that element. Now the number of atoms of X (in the volume that we are considering), which each have atomic weight $A_{\rm X}$, is

$$N_{\rm X} = \frac{(M_{\rm X})_{\rm g}(t)}{A_{\rm X}m_{\rm H}} = \frac{X(t)M_{\rm g}(T)}{A_{\rm X}m_{\rm H}}. \tag{17}$$

Therefore

$$\begin{aligned}\frac{dN_{\rm X}}{dt} &= -\lambda_{\rm X} N_{\rm X}(t) - \frac{1-R}{M_{\rm g}(t)} N_{\rm X}(t) A(t) + \frac{P_{\rm X}}{A_{\rm X} m_{\rm H}} A(t) \\ &= -\lambda_{\rm X} N_{\rm X}(t) - \omega_{\rm g}(t) + P A(t), \end{aligned} \tag{18}$$

where we have defined

$$\omega_{\rm g} = A(t) \frac{1-R}{M_{\rm g}(t)} \tag{19}$$

and

$$P = \frac{P_{\rm X}}{A_{\rm X} m_{\rm H}}. \tag{20}$$

Since we have

$$\frac{dM_{\rm g}(t)}{dt} = -(1-R)A(t) + f(t), \tag{21}$$

it follows that

$$\omega_{\rm g}(t) = -\frac{1}{M_{\rm g}(t)} \frac{dM_{\rm g}(t)}{dt} + \frac{f(t)}{M_{\rm g}(t)}. \tag{22}$$

Thus

$$\frac{dN_{\rm X}}{dt} = \left[-\lambda_{\rm X} - \omega_{\rm g}(t)\right] N_{\rm X}(t) + P A(t). \tag{23}$$

This differential equation can be solved to give

$$N_{\rm X}(t) \exp\ \{\lambda_{\rm X} t + \nu(t)\} = N_{\rm X}(0) + P \int_0^t A(t') \exp\ \{\lambda_{\rm X} t' + \nu(t')\}\ dt', \tag{24}$$

where

$$\nu(t) = \int_0^t \omega_{\rm g}(t')\ dt' = \ln \frac{M_{\rm g}(0)}{{\rm g}(t)} + \int_0^t \frac{f(t')}{M_{\rm g}(t')} dt'. \tag{25}$$

Now, if the solar system formed at time, say, t and if this was preceeded by a period Δ, in which no further heavy elements were added to the gas cloud, then the abundance of the gas at the time of the formation of the solar system must have been

$$\begin{aligned} N_{\rm x}(t+\Delta) = N_{\rm X}(t) \exp\ (-\lambda_{\rm X}\Delta) \\ = P \exp\ \{-\lambda_{\rm X} t - \nu(t) - \lambda_{\rm X}\Delta\} \left[\frac{N_{\rm X}(0)}{P} + \int_0^t A(t') \exp\ \{\lambda_{\rm X} t' + \nu(t')\}\ dt'\right]. \end{aligned} \tag{26}$$

Write

$$A_0 = \frac{N_0}{P} \tag{27}$$

and consider long-lived elements, such that $\lambda_{\rm X} t \ll 1$ and therefore $\exp\ (\lambda_{\rm X} t) = 1 + \lambda_{\rm X} t$ when terms of higher order are neglected. Then

$$N_{\rm X}(t+\Delta) = P \exp\ \{-\nu(t) - \Delta\lambda_{\rm X}\}(1 - \lambda_{\rm X} t)(A_0 + D + \lambda_{\rm X} D t_\nu), \tag{28}$$

where

$$D = \int_0^t A(t') \exp\ \{\nu(t')\}\ dt' \tag{29}$$

and

$$t_\nu = \frac{1}{D} \int_0^t A(t') t' \exp\ \{\nu(t')\}\ dt'. \tag{30}$$

For two isotopes i and j,both with long decay-times, we can then measure

$$\bar{t} = \frac{1}{\lambda_{\mathrm{i}} - \lambda_{\mathrm{j}}} \ln \left[\frac{P_{\mathrm{i}}}{P_{\mathrm{j}}} \frac{N_{\mathrm{j}}(t+\Delta)}{N_{\mathrm{i}}(t+\Delta)} \right] - \Delta. \tag{31}$$

The ratio $P_{\mathrm{i}}/P_{\mathrm{j}}$ follows from the theory of nucleosynthesis (in practice we have to do with r-process elements) and $N_{\mathrm{j}}/N_{\mathrm{i}}$ must be measured by laboratory analysis of meteorites. Δ can be measured with the use of short-lived elements ($\approx 10^8$ years according to ^{129}I and ^{244}Pu, but $\approx 2 \times 10^6$ years according to ^{26}Al). The isotope combinations that are being used for this cosmonucleochronology are (^{235}U, ^{238}U), (^{238}U, ^{232}Th) and (^{187}Re, ^{187}Os). The best answer at present is $\approx (2-4) \times 10^9$ years.

5 Age of the heavy elements.

The property $\bar{t}$ that we found for two radio-active elements still needs to be interpreted in terms of the synthesis history of the metals. This will be done in this section. We will find that $\bar{t}$ equals the average age of the heavy elements.

Define $p(\tau, t)\, d\tau$ as the fraction of metals present at time t and formed between τ and $\tau + d\tau$, where of course we have $\tau < t$. Now we had eq. (12), which reads without the outflow and for an inflow with unenriched gas

$$\frac{d}{dt}\left[M_{\mathrm{g}}(t)Z(t)\right] = -(1-R)Z(t)A(t) + P_{\mathrm{Z}}A(t). \tag{32}$$

The equivalent of this for the heavy elements only is

$$\frac{d}{dt}\left[M_{\mathrm{Z}}(t)p(\tau,t)\right] = -(1-R)Z(t)A(t)p(\tau,t) + P_{\mathrm{Z}}A(t)\delta(t-\tau). \tag{33}$$

From these two equation it follows that

$$\frac{dp(\tau,t)}{dt} = \frac{P_{\mathrm{Z}}A(t)}{M_{\mathrm{g}}(t)Z(t)}\left[\delta(t-\tau) - p(\tau,t)\right]. \tag{34}$$

Integrate this over time from τ to t and then differentiate with respect to t:

$$\frac{dp(\tau,t)}{dt} = -\frac{P_{\mathrm{Z}}A(t)}{M_{\mathrm{g}}(t)Z(t)}p(\tau,t) \tag{35}$$

and

$$p(\tau,\tau) = \left[\frac{P_{\mathrm{Z}}A(t)}{M_{\mathrm{g}}(t)Z(t)}\right]_{\mathrm{t}=\tau}. \tag{36}$$

Before we proceed we first need to search for a solution for $Z(t)$. According to eq. (8) and (12) we may write

$$\frac{dZ(t)}{dt}M_{\mathrm{g}}(t) = P_{\mathrm{Z}}A(t) - Z(t)f(t), \tag{37}$$

or

$$\frac{dZ(t)}{dt} + \frac{Z(t)f(t)}{M_{\mathrm{g}}(t)} = \frac{P_{\mathrm{Z}}A(t)}{M_{\mathrm{g}}(t)} = \frac{P_{\mathrm{Z}}\omega_{\mathrm{g}}(t)}{1-R}. \tag{38}$$

This is the same differential equation as we had above, so the solution is

$$Z(t) = Z_0 \exp\left\{-\theta(t)\right\} + \frac{P_{\mathrm{Z}}}{1-R}\exp\left\{-\theta(t)\right\}\phi(t), \tag{39}$$

where

$$\theta(t) = \int_0^t \frac{f(t')}{M_{\rm g}(t')}\, dt' \tag{40}$$

and

$$\phi(t) = \int_0^t \omega_{\rm g}(t') \exp\left\{\theta(t')\right\} dt'. \tag{41}$$

Now

$$\frac{dp(\tau,t)}{dt} = -\frac{P_{\rm Z}A(t)}{M_{\rm g}(t)Z(t)}p(\tau,t) = -\frac{P_{\rm Z}\omega_{\rm g}(t)}{(1-R)Z(t)}p(\tau,t). \tag{42}$$

Thus

$$\frac{dp(\tau,t)}{p(\tau,t)} = -\frac{P_{\rm Z}\omega_{\rm g}(t)}{(1-R)Z(t)}dt \tag{43}$$

and

$$p(\tau,t) = p(\tau,\tau) \exp\left\{\int_\tau^t \frac{P_{\rm Z}\omega_{\rm g}(t')}{(1-R)Z(t')}dt'\right\}. \tag{44}$$

Now also

$$\frac{P_{\rm Z}\omega_{\rm g}(t)}{(1-R)Z(t)} = \frac{P_{\rm Z}\omega_{\rm g}(t)\exp\left\{\theta(t)\right\}}{(1-R)Z_0 + P_{\rm Z}\phi(t)}. \tag{45}$$

Integrate this with $\phi(t)$ as variable, then

$$\int_\tau^t \frac{P_{\rm Z}\omega_{\rm g}(t')}{(1-R)Z(t')}dt' = -\ln\left[\frac{(1-R)Z_0}{P_{\rm Z}} + \phi(t')\right]_{\rm t'=\tau}^{t'=t}. \tag{46}$$

The solution then is

$$p(\tau,t) = \frac{P_{\rm Z}\omega_{\rm g}(t)\exp\left\{\theta(\tau)\right\}}{(1-R)Z_0 + P_{\rm Z}\phi(t)}. \tag{47}$$

The average epoch of formation of the heavy elements is by definition

$$t_{\rm Z} \equiv \int_0^t \tau p(\tau,t)\, d\tau. \tag{48}$$

So

$$t_{\rm Z} = \left[\frac{(1-R)Z_0}{P_{\rm Z}} + \phi(t)\right]^{-1} \int_0^t \tau\omega_{\rm g}(\tau)\exp\left\{\theta(\tau)\right\} d\tau. \tag{49}$$

To find $\omega_{\rm g}(t)$ we first need $A(t)$ and $M_{\rm g}(t)$. Now

$$\frac{dM_{\rm g}(t)}{dt} = -(1-R)A(t) + f(t), \tag{50}$$

so that

$$\frac{1}{M_{\rm g}(t)} = \frac{\omega_{\rm g}(t)}{A(t)(1-R)} = \omega_{\rm g}(t)\left[f(t) - \frac{dM_{\rm g}(t)}{dt}\right]^{-1}, \tag{51}$$

or

$$-\frac{dM_{\rm g}(t)}{M_{\rm g}(t)} = \left[-\frac{f(t)}{M_{\rm g}(t)} + \omega_{\rm g}(t)\right] dt. \tag{52}$$

Integrate this over time from 0 to t:

$$-\ln M_{\rm g}(t) = -\int_0^t \frac{f(t')}{M_{\rm g}(t')}dt' + \int_0^t \omega_{\rm g}(t')\, dt' = -\theta(t) + \nu(t). \tag{53}$$

Then

$$M_{\rm g}(t) = M_{\rm g}(0)\ \exp\left\{\theta(t) - \nu(t)\right\} \tag{54}$$

and

$$A(t) = \frac{M_{\rm g}(0)}{1-R}\omega_{\rm g}(t)\ \exp\left\{\theta(t) - \nu(t)\right\}. \tag{55}$$

The property D, defined in eq. (29), then is

$$D = \int_0^t A(t')\ \exp\left\{\nu(t')\right\}\,dt' = \frac{M_{\rm g}(0)}{1-R}\phi(t). \tag{56}$$

Then the final result for τ_ν, defined in eq. (30), is

$$\tau_\nu = \frac{1}{D}\int_0^t t'A(t')\ \exp\left\{\nu(t')\right\}\,dt' = \frac{1}{\phi(t)}\int_0^t t'\omega_{\rm g}(t')\ \exp\left\{\theta(t')\right\}\,dt'. \tag{57}$$

For the radio-active elements we have written

$$A_0 = \frac{N_{\rm X}Z_0}{P}$$

and equivalently we write

$$A_0 = \frac{M_{\rm g}(0)Z_0}{P_{\rm Z}}, \tag{58}$$

so that

$$\phi(t) = \frac{Z_0(1-R)D}{P_{\rm Z}A_0}. \tag{59}$$

Then

$$\frac{(1-R)Z_0}{P_{\rm Z}} + \phi(t) = \frac{\omega_{\rm g}(0)}{A_0}(A_0+D) = \frac{1-R}{M_{\rm g}(0)}(A_0+D) = (A_0+D)\frac{\phi(t)}{D}. \tag{60}$$

So eq. (49) becomes

$$t_{\rm Z} = \frac{1}{\phi(t)}\left(1+\frac{A_0}{D}\right)^{-1}\int_0^t \tau\omega_{\rm g}(\tau)\ \exp\left\{\theta(\tau)\right\}\,d\tau \tag{61}$$

and therefore

$$t_{\rm Z} = t_\nu\left(1+\frac{A_0}{D}\right)^{-1}. \tag{62}$$

If the time is $T_\odot$, then it follows from the definition of $t_{\rm Z}$ that the average age at that time is equal to $T_\odot - t_{\rm Z}$. With the definition of $\bar{t}$ in eq. (31) and with eq. (28) it is easily found that, for Δ small, we get

$$\bar{t} = T_\odot - t_\nu\left(1+\frac{A_0}{D}\right)^{-1}. \tag{63}$$

So we see that $\bar{t} = T_\odot - t_{\rm Z}$ is the average age of the heavy elements. Of course we measure that $\bar{t}$ for which $T_\odot$ is the time of formation of the solar system.

6 Summary.

♠ **Evolution of the gas mass:**

This is eq. (8):

$$\frac{dM_{\rm g}(t)}{dt} = -(1-R)A(t) + f(t) - g(t). \tag{64}$$

♠ **Evolution of the abundance:**

This is eq. (12):

$$\frac{d(M_{\rm Z})_{\rm g}(t)}{dt} = \frac{d}{dt}\left[M_{\rm g}(t)Z(t)\right] = -(1-R)Z(t)A(t) + P_{\rm Z}A(t) + Z_{\rm f}f(t) - Z(t)g(t). \tag{65}$$

♠ **Average age of the heavy elements:**

This is eq. (63):

$$\bar{t} = T_\odot - t_\nu \left(1 + \frac{A_0}{D}\right)^{-1}, \tag{66}$$

where

$$t_\nu = \frac{1}{D}\int_0^{T_\odot} A(t)t \exp\left\{\nu(t)\right\} dt, \tag{67}$$

$$D = \int_0^{T_\odot} A(t) \exp\left\{\nu(t)\right\} dt, \tag{68}$$

$$\nu(t) = \int_0^t \omega_{\rm g}(t')\, dt', \tag{69}$$

$$\omega_{\rm g}(t) = A(t)\frac{1-R}{M_{\rm g}(t)}, \tag{70}$$

$$A_0 = \frac{M_{\rm g}(0)Z_0}{P_{\rm Z}}. \tag{71}$$

♠ **Mass in stars and stellar remnants:**

$$\frac{dM_*(t)}{dt} = (1-R)A(t). \tag{72}$$

♠ **"Yield" of heavy elements:**

$$y = \frac{P_{\rm Z}}{1-R}. \tag{73}$$