Voronoi Tessellations and the Cosmic Web: Spatial Patterns and Clustering across the Universe

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Abstract

The spatial cosmic matter distribution on scales of a few up to more than a hundred Megaparsec[1] *displays a salient and pervasive foamlike pattern. Voronoi tessellations are a versatile and flexible mathematical model for such weblike spatial patterns. They would be the natural asymptotic result of an evolution in which low-density expanding void regions dictate the spatial organization of the Megaparsec Universe, while matter assembles in high-density filamentary and wall-like interstices between the voids. We describe the results of ongoing investigations of a variety of aspects of cosmologically relevant spatial distributions and statistics within the framework of Voronoi tessellations. Particularly enticing is the finding of a profound scaling of both clustering strength and clustering extent for the distribution of tessellation nodes, suggestive for the clustering properties of galaxy clusters. Cellular patterns may be the source of an intrinsic "geometrically biased" clustering.*

1. Introduction: the Cosmic Web

Macroscopic patterns in nature are often due the collective action of basic, often even simple, physical processes. These may yield a surprising array of complex and genuinely unique physical manifestations. The macroscopic organization into complex spatial patterns is one of the most striking. The rich morphology of such systems and patterns represents a major source of information on the underlying physics. This has made them the subject of a major and promising area of inquiry.

One of the most striking examples of a physical system displaying a salient geometrical morphology, and the largest in terms of sheer size, is the Universe as a whole. The past few decades have revealed that on scales of a few up to more than a hundred Megaparsec, galaxies conglomerate into intriguing weblike patterns that pervade throughout the observable cosmos. Revealed through the painstaking efforts of redshift survey campaigns, it has completely revised our view of the matter distribution on these cosmological scales. The spatial distribution of galaxies is far from homogeneous. Instead, we recognize a weblike arrangement. Galaxies aggregate in striking geometric patterns, outlined by huge *filamentary* and *sheetlike* structures, the sizes of the most conspicuous ones frequently exceeding $100h^{-1}\mathrm{Mpc}$. Within and around these anisotropic features we find a variety of density condensations, ranging from modest groups of a few galaxies up to massive compact *clusters* of galaxies. The latter represent the most prominent density enhancements in our universe and usually mark the dense intersections of filaments and sheets. They stand out as the most massive and most recently formed (individual) objects in the Universe. Complementing this cosmic inventory leads to the existence of large *voids*, enormous regions with sizes in the range of $20-50h^{-1}\mathrm{Mpc}$ that are practically devoid of any galaxy, usually roundish in shape and occupying the major share of space in the Universe.

Of utmost significance for our inquiry into the issue of cosmic structure formation is the fact that the prominent structural components of the galaxy distribution – clusters, filaments, walls and voids – are not merely randomly and independently scattered features. On the contrary, they have arranged themselves in a seemingly highly organized and structured fashion, woven into an intriguing *weblike* tapestry that permeates the whole of the explored Universe. The weblike spatial arrangement of galaxies and mass into elongated filaments, sheetlike walls and dense compact clusters, the existence of large near-empty void regions and the hierarchical nature of this mass distribution – marked by substructure over a wide range of scales and densities – are three major characteristics of what we have come to know as the *Cosmic Web*. Its appearance is most

[1]The main measure of length in astronomy is the parsec. Technically a parsec is the distance at which we would see the distance Earth-Sun at an angle of 1 arcsec. It is equal to 3.262 lightyears $= 3.086 \times 10^{13}$km. Cosmological distances are substantially larger, so that a Megaparsec ($= 10^6\ pc$) is the regular unit of distance. Usually this goes along with h, the cosmic expansion rate (Hubble parameter) H in units of 100 km/s/Mpc ($h \approx 0.71$).

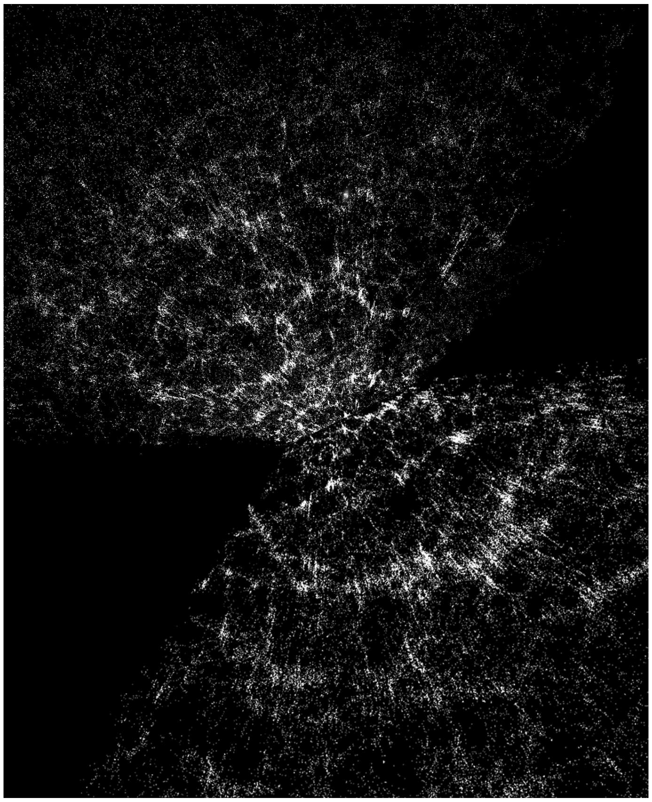

Figure 1. Image of the weblike galaxy distribution in the SDSS galaxy redshift survey (3rd data release, 374,767 galaxies).

dramatically illustrated by the most recently produced maps of the nearby cosmos. The 2dF – two-degree field – Galaxy Redshift Survey (e.g. [4]) mapped the spatial distribution of nearly 250,000 galaxies in two narrow sections through the local Universe, out to a depth of more than $300h^{-1}\mathrm{Mpc}$. The SDSS survey (fig. 1, see e.g. [12]), is mapping the location of up to a million galaxies, covering nearly a quarter of the sky and reaching out to a depth of beyond $500h^{-1}\mathrm{Mpc}$.

2. Voronoi Models & the Cosmic Web

In the cosmological context *Voronoi Tessellations* represent the *Asymptotic Frame* for the ultimate matter distribution distribution in any cosmic structure formation scenario. The Voronoi tessellation is the *skeleton* of the cosmic matter distribution, identifying the structural frame around which matter will gradually assemble during the emergence of cosmic structure.

Voronoi tessellations are a versatile and flexible mathematical model for foamlike spatial patterns. They would be the natural result of an evolution in which expanding *voids* dictate the spatial organization of the Megaparsec Universe, with matter assembling in the high-density filamentary and wall-like interstices between the voids. According to recent work this is indeed what may be expected in standard scenarios of cosmic structure formation. Voronoi models would represent the asymptotic limit for which the population of voids would correspond to one single void size and excess expansion rate. In this paper we seek to sketch the background and ramifications of this idea.

The premise is that some primordial cosmic process generated a random (Gaussian) density fluctuation field. The troughs, minima, in this field will become the centres of expanding voids. Matter will flow away until it runs into its surroundings and encounters similar material flowing out of adjacent voids. The spatial distribution of the density troughs in the primordial density field is dependent on the specific cosmological structure formation scenario at hand.

Within the cellular Voronoi skeleton the interior of the Voronoi *cells* corresponds to voids. The Voronoi *planes* are identified with walls of galaxies. The *edges* delineating the rim of each wall are identified with the filaments in the galaxy distribution. The most outstanding structural elements are the *vertices*, corresponding to the very dense compact nodes within the cosmic web, the rich clusters of galaxies. In general, what is denoted as a flattened *supercluster* will consist of an assembly of various connecting walls in the Voronoi foam. The elongated *superclusters* or *filaments* usually consist of a few connected edges. This may be clearly appreciated from Fig. 5.

2.1. Voronoi Virtues

Cosmologically, the great virtue of the *Voronoi foam* is that it provides a conceptually simple model for a cellular or foamlike distribution of galaxies. By using these geometrically constructed models one is not restricted by the resolution or number of particles. A cellular structure can be generated over a part of space beyond the reach of any N-body experiment. Even though the model does not and cannot addres the galaxy distribution on small scales, it is nevertheless a useful prescription for the spatial distribution of the walls and filaments themselves. In all, it makes the Voronoi model particularly suited for studying the properties of galaxy clustering in spatial cellular patterns. Its ease and versatility of construction, and its flexibility with respect to defining cosmological parameters, has made it into an ideal tool for statistical studies and tests of structure finding and identification techniques.

3. Gravitational Instability

Comprising features on a typical scale of tens of Megaparsec, the cosmic web offers a direct link to the matter dis-

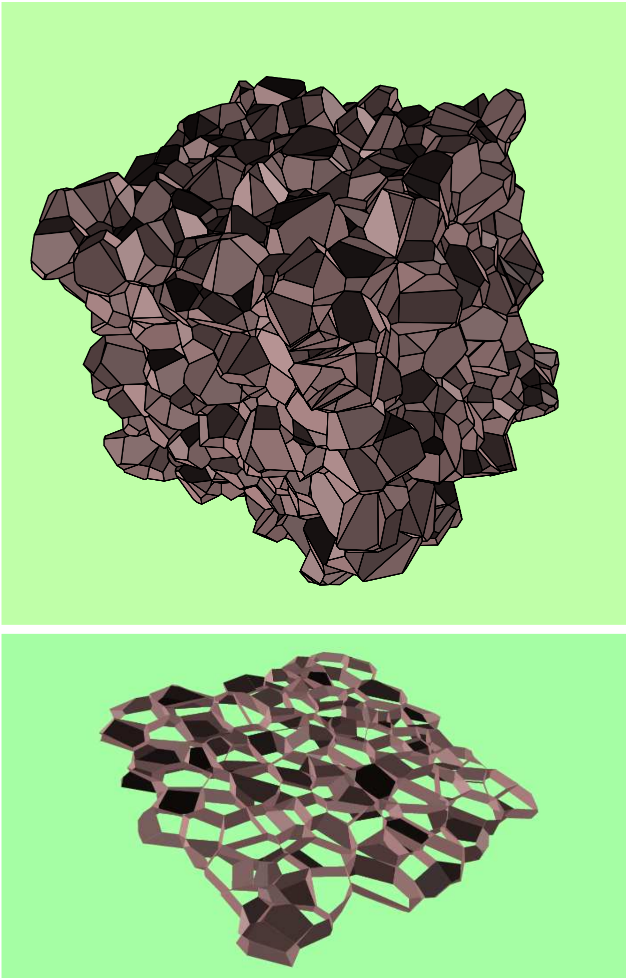

Figure 2. A full 3-D tessellation comprising 1000 Voronoi cells/polyhedra generated by 1000 Poissonian distributed nuclei. Courtesy: Jacco Dankers

tribution in the primordial Universe and contains a wealth of information on the cosmic structure formation process[2]. It thus represents a key to unravelling one of the most pressing enigmas in modern astrophysics, the rise of the wealth and variety of structure in the present-day Universe from an almost perfectly smooth, virtually featureless, pristine cosmos.

The generally accepted theoretical framework for the formation of structure is that of *gravitational instability*. The formation and moulding of structure is ascribed to the gravitational growth of tiny initial density- and velocity deviations from the global cosmic density and expansion. Overdense regions will initially expand slightly less rapid than the cosmic background, reach a maximum size, turn around and ultimately condense – dependent on the scale of the density perturbation – into a recognizable astrophysical object. Underdense regions, on the other hand, do have a gravity deficit with respect to the surrounding Universe.

Three fundamental aspects of the ensuing nonlinear gravitational clustering process determine the morphology of the resulting matter distribution. The *first* is *hierarchical* clustering: the first objects to form are small compact objects which subsequently merge with their surroundings into ever larger features. The *second* fundamental aspect concerns *anisotropic gravitational collapse*. Aspherical overdensities, on any scale and in any scenario, will contract such that they become increasingly anisotropic. At first they turn into a flattened 'pancake', rapidly followed by contraction into an elongated filament, possibly to finally collapse to become a galaxy or a cluster. The *third* manifest feature of the Megaparsec Universe is the marked and dominant presence of large roundish underdense regions, the *voids* (see fig. 3).

3.1. Voids

Inspired by early computer calculations, [5] pointed out that for the understanding of the formation of the large coherent patterns pervading the Universe it is worthwhile to direct attention to the complementary evolution of underdense regions. The gravity deficit in their interior makes them expand with respect to the background Universe. Meanwhile we see the matter assembling in ever dense *planar* and/or *filamentary* interstices.

They form in and around density troughs in the primordial density field. Because of their lower interior gravity they will expand faster than the rest of the Universe, while their internal matter density rapidly decreases as matter evacuates their interior (see fig. 3). They evolve in the nearly empty void regions with sharply defined boundaries marked by filaments and walls. Their essential role in the organization of the cosmic matter distribution got recognized early after their discovery. Recently, their emergence and evolution has been explained within the context of hierarchical gravitational scenarios [10].

As the voids expand they start to take up an increasingly major fraction of the cosmic volume. [5] made the additional interesting observation that the outward expansion of voids is accompanied by a tendency of voids to assume a spherical geometry. While the above properties refer to the evolution of isolated voids, we know that in reality they will be surrounded by other density structures and that they will themselves contain a range of smaller scale features and ob-

[2] According to our latest insights the Universe is 13.7Gyr old. With the COBE and WMAP microwave background telescopes we have observed tiny disturbances in the early Universe, when the Universe was around 379.000 yrs old. These perturbations were no larger than a factor 10^{-5}.

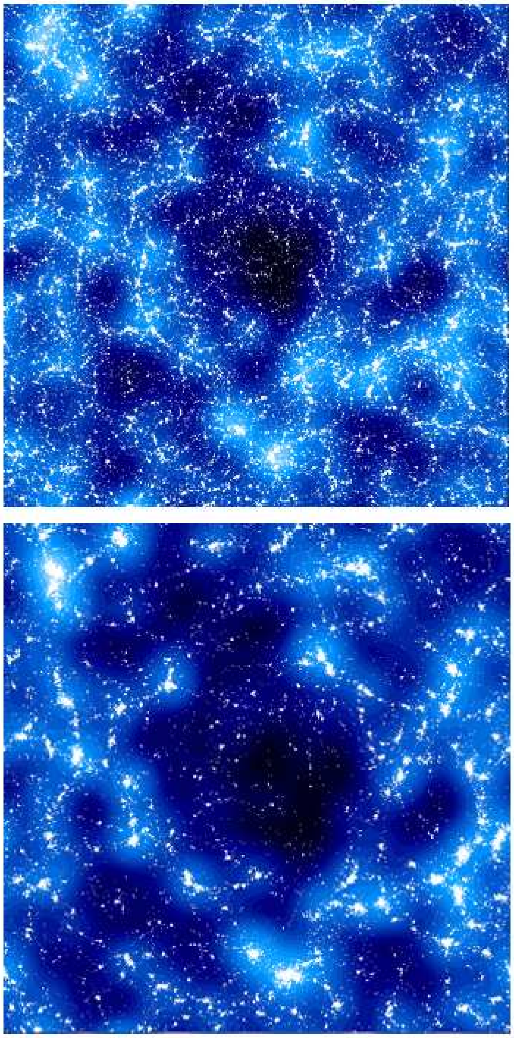

Figure 3. Void evolution: two timesteps in the evolution of a void region in a N-body computer simulation of structure formation (in a SCDM model). Courtesy: Erwin Platen.

jects. Recent work by [10] managed to describe the corresponding *"Void Sociology"* through an analytical description within the context of the excursion set formalism [3]. Fig. 3 contains an illustration of the evolving Cosmic Web, marked by the surge and development of large cosmic voids. The evolution of voids is dictated by two processes:

• Voids merge with surrounding voids into a larger void.

• Voids may get destroyed when embedded within a larger collapsing overdensity.

When working out the implication of the precise interplay between these two processes, one finds that voids tend to define a population marked by a self-similar peaked size distribution centered around a characteristic void size: at any one cosmic instant the void population is dominated by voids of nearly the same size (fig. 3). This indeed forms a telling confirmation of the observed foamlike distribution of galaxies with the characteristic pattern punctuated by voids whose size is $\approx 10 - 30h^{-1}$ Mpc.

3.2. From Voids to Voronoi

A bold leap leads to a geometrically interesting situation. Taking the voids as the dominant dynamical component of the Universe, we may think of the Megaparsec scale structure as a close packing of spherically expanding regions. The asymptotic configuration for a "peaked" void distribution is one of a single characteristic void size. Pursuing this situation we would find a cosmic matter distribution which would be organized by a population of *equally sized* and *equally fast expanding*, *spherically shaped* voids.

Matter will collect at the interstices between the expanding voids. In this asymptotic limit of the outflow being the same in all voids, these interstices are the *bisecting planes* between two neighbouring expansion centres: the *walls* and *filaments* would be found precisely at the (bisecting) midplanes between expanding voids. For any given set of expansion centres, or *nuclei*, the arrangement of these planes defines a unique process for the partitioning of space: the resulting skeleton of the matter distribution would be nothing else than a *Voronoi tessellation* [17, 7] (see fig. 2). In other words, the *ASYMPTOTIC* description of the cosmic clustering process leads to a geometrical configuration that is one of the main concepts in the field of stochastic geometry:

VORONOI TESSELLATIONS

A cosmological realisation of this process is called a *Voronoi foam* [6].

4. Voronoi Clustering Models

Voronoi Clustering Models are a class of heuristic models for cellular distributions of matter which use the Voronoi tessellation as the skeleton of the cosmic matter distribution [16, 13, 14].

It is the stochastic yet non-Poissonian geometrical distribution of the *walls*, *filaments* and *clusters* embedded in the cosmic web which generates the large-scale clustering properties of matter and the related galaxy populations. It is precisely this aspect which is modelled in great detail by Voronoi tessellation. The small-scale distribution of galaxies, i.e. the distribution within the various components of the cosmic skeleton, will involve the complicated details of highly nonlinear small-scale interactions of the gravitating matter. Ideally, well-defined and elaborate physical models and/or N-body computer simulations would fill in this

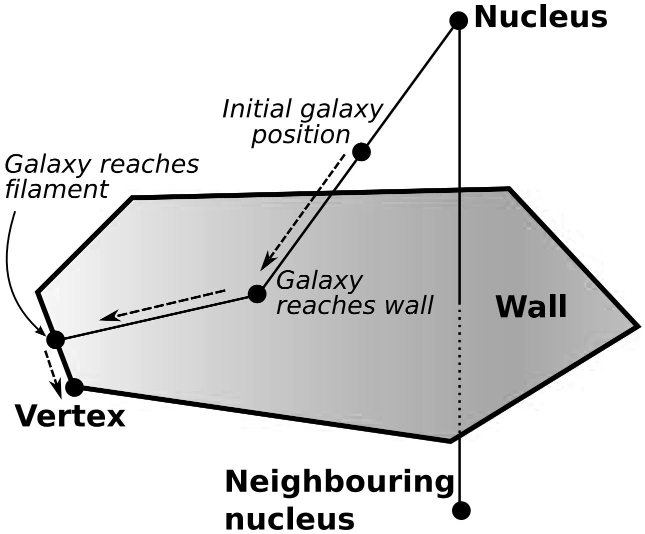

Figure 4. Schematic illustration of the Voronoi kinematic model. Courtesy: Miguel Aragón-Calvo.

aspect. In the Voronoi models described here we complement the geometrically fixed configuration of the Voronoi tessellations with a heuristic prescription for the placing of particles/model galaxies within the tessellation.

The *Voronoi Kinematic Voronoi Model* is based upon the notion that voids play a key organizational role in the development of structure and make the Universe resemble a soapsud of expanding bubbles [5]. It forms an idealized and asymptotic description of the outcome of the cosmic structure formation process within gravitational instability scenarios with voids forming around a dip in the primordial density field. This is translated into a scheme for the displacement of initially randomly distributed galaxies within the Voronoi skeleton. Within a void, the mean distance between galaxies increases uniformly in the course of time. When a galaxy tries to enter an adjacent cell, the velocity component perpendicular to the cell wall disappears. Thereafter, the galaxy continues to move within the wall, until it tries to enter the next cell; it then loses its velocity component towards that cell, so that the galaxy continues along a filament. Finally, it comes to rest in a node, as soon as it tries to enter a fourth neighbouring void.

4.1. Voronoi Kinematic Model

All different Voronoi models are based upon the displacement of a sample of N *Voronoi galaxies*. The initial spatial distribution of these N galaxies within the sample volume V is purely random, their initial locations $\mathbf{x}_{n0}$ defined by a homogeneous Poisson process. A set of M nuclei or *expansion centres* within the volume V corresponds to the cell centres, or *expansion centres* driving the evolving matter distribution.

The first stage of the procedure consists of the generation of initial conditions of the Voronoi galaxy distribution, along with the specification of the properties (width) of the structural elements:

- Distribution of M nuclei, *expansion centres*, within the simulation volume V. The location of nucleus m is $\mathbf{y}_m$.
- Generate N model galaxies whose initial locations, $\mathbf{x}_{n0}$ $(n = 1, \ldots, N)$ are randomly distributed throughout the sample volume V.
- Of each model galaxy n determine the Voronoi cell $\mathcal{V}_\alpha$ in which it is located, ie. determine the closest nucleus j_α.
- The width of the walls, filaments and vertices is set by assuming them to have a Gaussian radial density distribution, with wall width R_{W}, filament width R_{F} and vertex width R_{V} input parameters of the Voronoi model.

The second stage of the procedure consists of the calculation of the complete *Voronoi track* for each galaxy $n = 1, \ldots, N$. Figure 4 contains a sketch of a typical Voronoi galaxy track:

- The first step is the calculation of the galaxy tracks is the dtermination for each galaxy n the Voronoi cell $\mathcal{V}_\alpha$ in which it is initially located, ie. finding the nucleus j_α which is closest to the galaxies' initial position $\mathbf{x}_{n0}$.
- Subsequently, the galaxy follows a radial path within Voronoi cell $\mathcal{V}_\alpha$. It moves along the path emanating from its expansion centre j_α, ie. along the direction marked by the unity vector $\hat{\mathbf{e}}_{n\alpha}$ defined by the line starting at $\mathbf{y}_\alpha$ and pointing radially outward from j_α, $\mathbf{r}_{n\alpha} \equiv \mathbf{y}_\alpha + R_n\,\hat{\mathbf{e}}_{n\alpha}$. This is pursued until the galaxy intersects the Voronoi wall $\Sigma_{\alpha\beta}$.
- After this the galaxy's displacement is restricted to the wall $\Sigma_{\alpha\beta}$ as the displacement component perpendicular to the wall is damped. The galaxy moves towards the one edge $\Lambda_{\alpha\beta\gamma}$ defining the nearest intersection with wall path $r_{n\alpha\beta}$. This continues until it intersects Voronoi edge $\Lambda_{\alpha\beta\gamma}$.
- Subsequently, it pursues its path along the spine of the edge $\Lambda_{\alpha\beta\gamma}$ as the velocities' component perpendicular to the edge has been suppressed ("the self-gravity of the filament damps its corresponding velocity"). This continues till the galaxy finally arrives at vertex $\Xi_{\alpha\beta\gamma\delta}$. The vertex is the final destination of the galaxy, the deep potential well of the cluster at its location will keep the galaxy within its reach.

By determining the total *Voronoi track* for each galaxy n it is rather straightforward to compute the location of the galaxy at any cosmic epoch t by determining the *displacement* $\mathbf{x}_n$ that each galaxy has traversed along its path. For each cosmic epoch we may therefore easily compute the corresponding mass distribution.

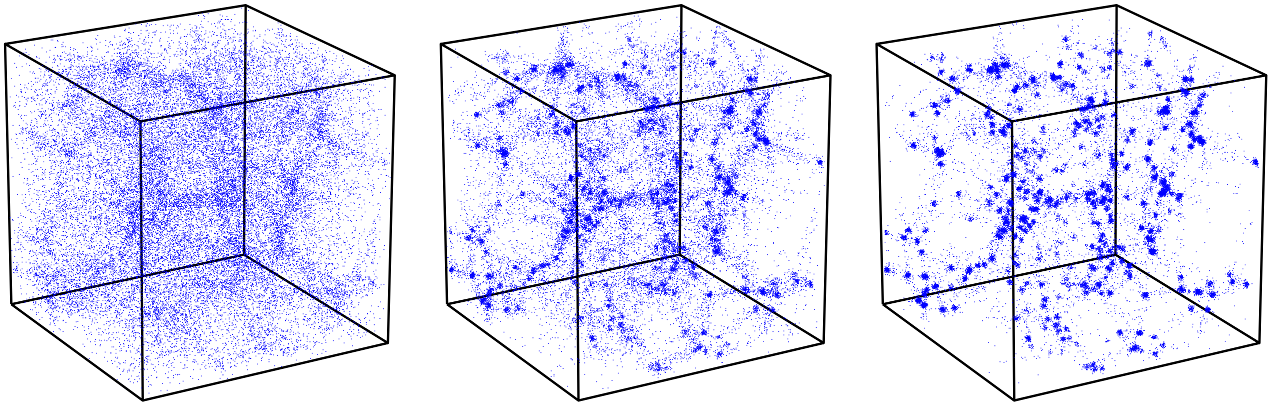

Figure 5. A sequel of three consecutive timesteps within the kinematic Voronoi cell formation process. The depicted boxes have a size of $100h^{-1}$Mpc. Within these cubic volumes some 64 Voronoi cells with a typical size of $25h^{-1}$Mpc delineate the cosmic framework with 32000 galaxies.

4.2. Kinematic Evolution

The evolutionary progression within the Voronoi kinematic scheme, from an almost featureless random distribution, via a wall-like and filamentary morphology towards a distribution in which matter ultimately aggregates into conspicuous compact cluster-like clumps can be readily appreciated from the sequence of 6 cubic 3-D particle distributions in Figure 5 [14].

The steadily increasing contrast of the various structural features is accompanied by a gradual shift in topological nature of the distribution. The virtually uniform particle distribution at the beginning (upper lefthand frame) ultimately unfolds into the highly clumped distribution in the lower righthand frame. At first only a faint imprint of density enhancements and depressions can be discerned. In the subsequent first stage of nonlinear evolution we see a development of the matter distribution towards a wall-dominated foam. The contrast of the walls with respect to the general field population is rather moderate (see e.g. second frame), and most obviously discernable by tracing the sites where the walls intersect and the galaxy density is slightly enhanced. The ensuing frames depict the gradual progression via a wall-like through a filamentary towards an ultimate cluster-dominated matter distribution. By then nearly all matter has streamed into the nodal sites of the cellular network. The initially almost hesitant rise of the clusters quickly turns into a strong and incessant growth towards their appearance as dense and compact features which ultimately stand out as the sole dominating element in the cosmic matter distribution (bottom righthand frame).

5. Superclustering vs. Vertex Clustering

Upon quantitatively testing the spatial distribution of galaxies various interesting properties of clustering within Voronoi foams are revealed. The most salient and outstanding properties concern the distribution of the Voronoi vertices. A first inspection of the spatial distribution of Voronoi vertices (Fig. 6, bottom righthand frame) immediately reveals that it is not a simple random Poisson distribution.

The impression of strong clustering, on scales smaller than or of the order of the cellsize λ_c[3], is most evidently expressed by the corresponding two-point correlation function $\xi(r)$ (Fig. 6). Not only one finds a clear positive signal out to a distance of at least $r \approx 1/4\,\lambda_c$ but also – surprising at the time of its finding on the basis of similar computer experiments [16] – the correlation function appears to be an almost perfect power-law,

$$\begin{aligned} \xi_{vv}(r) &\equiv \frac{\langle n_v(\mathbf{x}+\mathbf{r})\, n_v(\mathbf{x})\rangle}{\bar{n}_v^2} - 1 \approx \left(\frac{r_o}{r}\right)^{\gamma}, \\ \gamma &= 1.95; \quad r_o \approx 0.3\,\lambda_c\,. \end{aligned} \tag{1}$$

The solid line in the log-log diagram in Fig. 6 represents the power-law with these parameters, $\gamma \approx 1.95$ and $r_o \approx 0.3\,\lambda_c$.

Beyond this range, the power-law behaviour breaks down and, following a gradual decline, the correlation function rapidly falls off to a zero value once distances are of the order of (half) the cellsize. Assessing the behaviour of $\xi(r)$ in a linear-linear plot, we get a better idea of its behaviour around the zeropoint "correlation length" $r_a \approx 0.5\lambda_c$ (bottom righthand frame fig. 6). Beyond r_a the distribution of Voronoi vertices is practically uniform. Its only noteworthy behaviour is the gradually declining and alternating quasi-periodic ringing between positive and negative values similar to that we also recognized in the "galaxy" distribution, a vague echo of the cellular patterns which the vertices trace out. Finally, beyond $r \approx 2\lambda_c$ any noticeable correlation seems to be absent.

[3] in the following we use the cellsize λ_c, ie. the intranucleus distance, as unit of distance for the generated Voronoi foams.

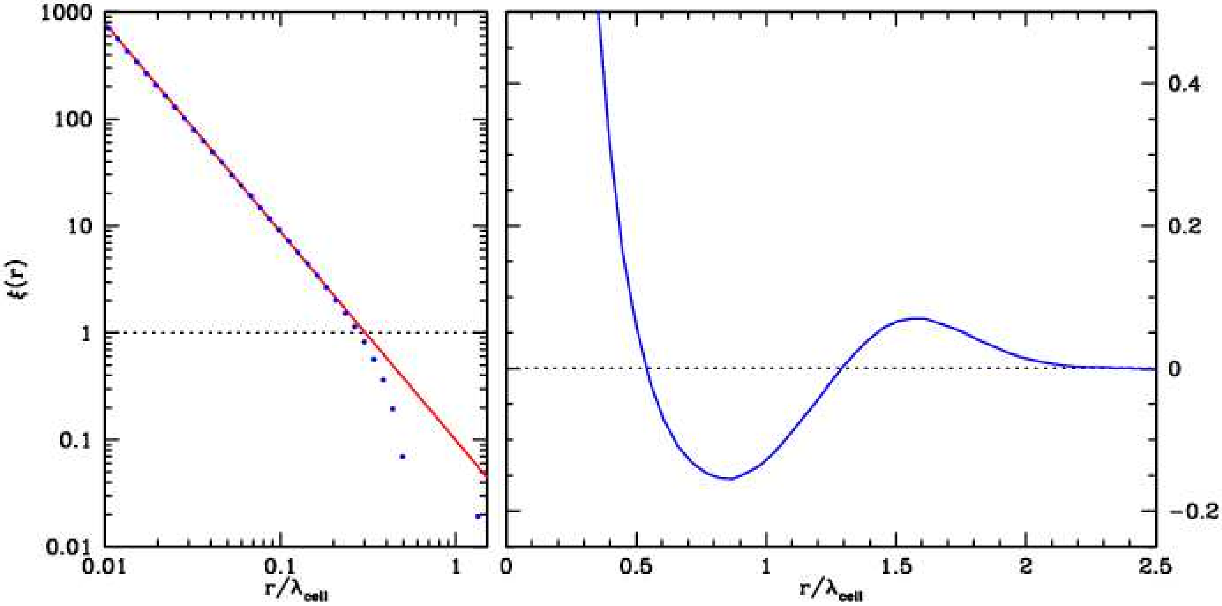

Figure 6. Two-point correlation function of (Poisson) Voronoi vertices. Distance in units of inter-nucleus distance. Left: log-log plot of $\xi(r)$. Right: lin-lin plot of $\xi(r)$.

The correlation function of Voronoi vertices is a surprisingly good and solid match to the observed world. Maps of the spatial distribution of clusters of galaxies show that clusters themsvelves are not Poissonian distributed. On the contrary, they are highly clustered and aggregrate into huge supercluster complexes [8, 1]. Such superclusters represent moderate density enhancements on a scale of tens of Megaparsec, still co-expanding with Hubble flow, be it at a slightly decelerated rate.

It sheds an alternative view on the power-law clustering with power law $\gamma \approx 2$ found in the cluster distribution. Also, the observed cluster clustering length $r_o \approx 20h^{-1}$ Mpc can be explained within the context of a cellular model, suggesting a cellsize of $\lambda_c \approx 70h^{-1}$ Mpc as the basic scale of the cosmic foam. On the other hand, this also reveals a complication. The suggested cell scale seems to be well in excess of the $25h^{-1}$–$35h^{-1}$ Mpc size of the voids in the galaxy distribution. Also, it does not correspond to the clustering of objects in the walls and filaments of the same tessellation network when tying it to the observed galaxy-galaxy correlation. The solution to this dilemma leads to an intriguing finding for the scaling of vertex clustering.

6. Geometric Scaling

A *prominent* characteristic of superclustering in the observational world is that clustering of clusters is considerably more pronounced than that of galaxies. This characteristic ties in with a generally observed property of clustering to be dependent on the nature of objects. Amongst clusters we observe a trend for more luminous/massive clusters to be more strongly clustered. While the two-point correlation function $\xi_{cc}(r)$ of clusters is consistent with it being a scaled version of the power-law galaxy-galaxy correlation function, with almost the same slope $\gamma \approx 1.8$, the correlation amplitude and clustering length r_0 increase as the characteristic "mass" of the clusters in the sample gets larger.

6.1. Vertex Selections

The kinematic Voronoi model allows a geometric modelling of mass-dependent clustering properties by assigning a "mass" to each vertex. Brushing crudely over the details of the temporal evolution, we may assign each Voronoi vertex a "mass" by equating this to the total amount of matter which according to the "Voronoi streaming" description will ultimately flow towards that vertex. The related nuclei are the ones that supply the Voronoi vertex with inflowing matter. Evidently, vertices surrounding large cells are expected to be more massive. It is reasonably straightforward if cumbersome to calculate the "vertex mass" $\mathcal{M}_V$ by pure geometric means. The details, turn out to be challengingly complex, as it concerns the (purely geometric) calculation of the volume of a non-convex polyhedron centered on the Voronoi vertex.

In our computer experiments [14, 15] we set up realizations of a (Poisson) Voronoi foam comprising 1000 cells with an average size of $25h^{-1}$Mpc. From the full vertex distribution we selected subsets of vertices, each subset comprising vertices with a progressively higher lower mass limit.

A telling illustration of the significant stronger cluster-

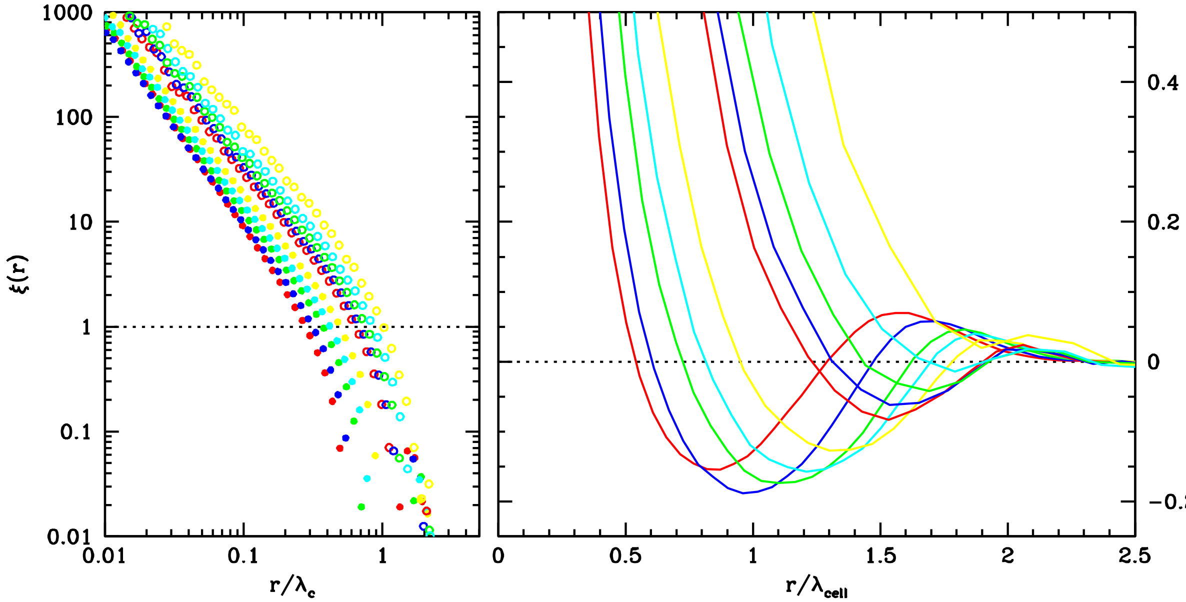

Figure 7. The two-point correlation for a variety of vertex subsamples, selected on the basis of "richness/mass". The largest subsample (with weakest correlation function) contains all vertices, the riches only the 2.5% most massive ones. Left: log-log plot of $\xi(r)$. Right: lin-lin plot of $\xi(r)$.

ing for more massive vertices is shown in Fig 6. It shows the vertex distribution in a central slice through the full 3-D cubic distribution. While the top lefthand panel shows the distribution of the full Voronoi vertex distribution, the distribution of the 12.5% richest vertices (top righthand panel) and the 2.5% richest vertices (lower lefthand panel) is obviously significantly different. Not only do we observe a marked increase in clustering strength as the sample includes more massive vertices, also the spatial extent of the clustering patterns appears to grow as a function of vertex richness. When correcting for the possibly confusing influence of the sampling dilution, by sampling an equal number of vertices from each "selected" sample, the effect is even more prominent. When lifting the central $1/8^{th}$ region out of the 20% vertex subsample in the (top righthand) frame and sizing it up to the same scale as the full box, we observe the similarity in point process between the resulting (bottom righthand) distribution and that of the 2.5% subsample (bottom lefthand). It is spatial self-similarity in its purest form !

6.2 Vertex Correlation Scaling

To quantify the impression given by the distribution of the biased vertex selections, we analyzed the two-point correlation function for each vertex sample. We computed $\xi(r)$ for samples ranging from the complete sample down to the ones merely containing the 2.5% most massive ones[4]

The surprising finding is that all subsamples of Voronoi vertices do retain a two-point correlation function (Fig. 7) displaying the same qualitative behaviour as the $\xi_{vv}(r)$ for the full unbiased vertex sample (Fig. 6). Out to a certain range it invariably behaves like a power-law (lefthand frame), while beyond that range the correlation functions all show the decaying oscillatory behaviour that already has been encountered in the case of the full sample. Nonetheless, we can immediately infer significant systematic trends.

- *Clustering Strength.* The *first* observation is that the amplitude of the correlation functions increases monotonously with rising vertex sample richness. Expressing the amplitude in terms of the "clustering length" r_o we find a striking and almost perfect linear relation (Fig. 9, lefthand frame). The "fractal" clustering scaling description of [11], according to which the clustering scale r_0 scales with the average object distance $L(r)$, turns out to be implicit for weblike geometries:

$$\xi_{cc}(r) = \beta\,(L(r)/r)^{\gamma}\,; \qquad L(R) = n^{-1/3}\,. \qquad (2)$$

This finding is in line with studies based on cluster samples of rich clusters or selected on the basis of their X-ray emission [9, 2]. They do indeed show a trend of an increasing clustering strength as the clusters are richer (≈ massive).

[4] We use the average distance $\lambda_v(R) = n(R)^{-1/3}$ between the sample vertices for characterizing the richness of the sample. This is based on λ_v increasing monotonously with subsample richness.

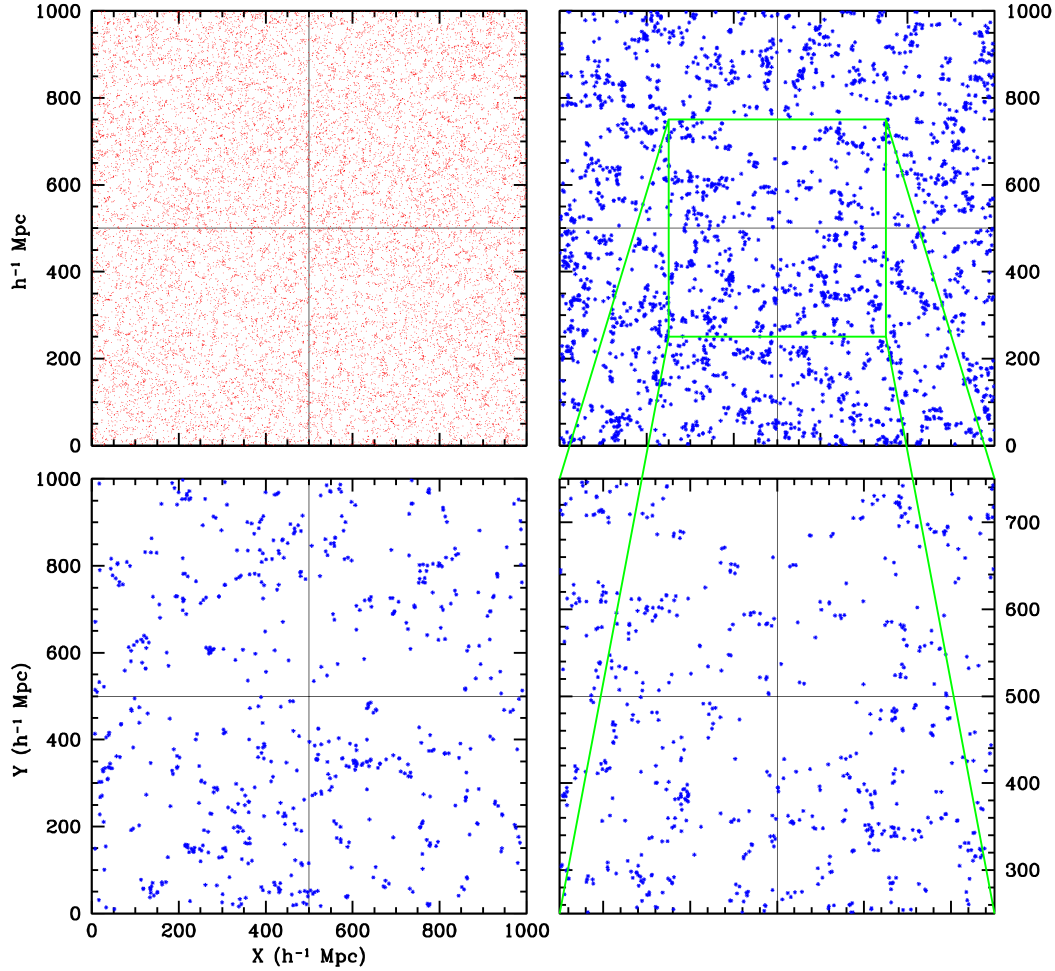

Figure 8. Illustration of the 'self-similar' vertex distribution. The more massive vertex samples are (1) more strongly clustered and (2) define clustering patterns on a larger spatial scale. When properly scaled a pure self-similar pattern emerges. See text for explanation.

• *Spatial Coherence.* A *second* significant observation, from the large scale behaviour inferred from the lin-lin plot, is that $\xi_{\rm vv}$ extends out to larger and larger distances as the sample richness is increasing. The oscillatory behaviour is systematically shifting outward for the richer vertex samples, reflecting their more extended clustering patterns. Even though the basic cellular pattern has a characteristic scale of only λ_c, the sample of the 5% richest nodes sets up coherent patterns at least 2 to 3 times larger (also cf. fig. 6). Pursuing the systematics of this geometric clustering, Fig. 9 also plots the *coherence scale* $r_{\rm a}$. It scales almost perfectly linear with average vertex distance λ_v !!

These finding are in line with observational evidence that $\xi_{cc}(r)$ extends out considerably larger scales than the galaxy-galaxy correlation ξ_{gg}, possibly out to $50-100 h h^{-1}{\rm Mpc}$, even though less evident than the increased strength of clustering. The inescapable conclusion is that *weblike geometries manage to define coherent structures significantly larger than their basic size. They induce clustering in which richer objects not only cluster more strongly, but also over an ever larger spatial range.* !!!

• *Self-Similarity.* Combining the behaviour of clustering scale scale $r_{\rm o}$ and coherence scale $r_{\rm a}$ a remarkable *self-similar* scaling behaviour is revealed. The ratio of correlation versus clustering length is virtually constant for all vertex samples, $r_{\rm a}/r_{\rm o} \approx 1.86$. *Weblike geometries may involve intriguing self-similar patterns of clustering*.

• *Clustering Slope.* A *final* interesting detail on the vertex clustering scaling behaviour is that a slight and interesting trend in the behaviour of power-law slope. The richer samples correspond to a tilting of the slope. Fig. 9 shows a gradual change from a slope $\gamma \approx 1.95$ for the full sample, to a robust $\gamma \approx 1.8$ for the selected samples. Note than this suggestive value is found in the majority of observed galaxy

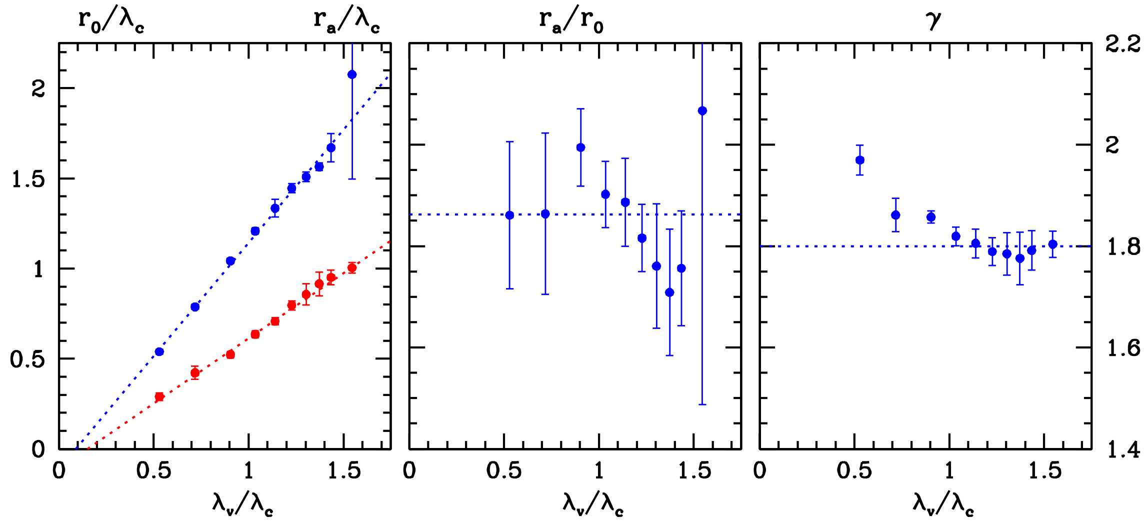

Figure 9. Scaling of Voronoi vertex two-point correlation function parameters, as function of average vertex separation in (mass) selected subsample, λ_v/λ_c. Left: clustering length r_0 (lower, $\xi(r_0) \equiv 1.0$); coherence length r_a (top, $\xi(r_a) \equiv 0.0$). Centre: ratio clustering-coherence length, r_a/r_0. Right: power-law slope γ.

and cluster samples !

7 Summary: Self-Similar Cosmic Geometry

The revealed systematic trends of vertex clustering have uncovered a hidden *self-similar* clustering of vertices. It forms a tantalizing indication for the existence of self-similar clustering behaviour in spatial patterns with a cellular or foamlike morphology. It may hint at an intriguing and intimate relationship between the cosmic foamlike geometry and various measures of clustering in the Universe.

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