Sterrenstelsels en Kosmos
Assignment 1 — Solutions

1. To give you an idea of the conditions in the center of the Sun, use

   (a) the equation of hydrostatic equilibrium to estimate the pressure at the center of the Sun, P_c, and

   Solution This is just Example 9.4 in Section 9.4 of Kutner. The equation of hydrostatic equilibrium is

   $$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

   Assume that the pressure at the outer edge of the Sun is zero. Then assume that $dP/dr = P_c/R$ and that the Sun has a uniform density of $\rho = \bar{\rho}$. Then we see that

   $$\frac{P_c}{R} \sim \frac{GM\bar{\rho}}{R^2}$$

   (the negative sign has gone away because of the direction of integration: the pressure is P_c at $r = 0$ and 0 at $r = R$, but you integrate along increasing r) so

   $$P_c \sim \frac{GM\bar{\rho}}{R} = 2.7 \times 10^{15}\,\text{dyn/cm}^2.$$

   Note that this is about 3 times less than the value in Kutner, because he ignored the factor of $4\pi/3$ in the conversion from density to mass and radius.

   (b) the equation of state for an ideal gas to estimate the temperature at the center of the Sun, T_c.

   Solution We can rewrite the ideal law at the center of the Sun,

   $$P_c = \frac{\bar{\rho}}{m} kT_c,$$

   where m is the average mass per particle, $m = 0.62 m_H$, in terms of T as

   $$T_c = \frac{mP_c}{k\bar{\rho}} = 1.4 \times 10^7\,\text{K}.$$

The average density of the Sun is $\bar{\rho} = 1.4\,\text{g/cm}^3$ and the typical mass of a particle in the Sun is $0.62 m_H$, where m_H is the mass of a hydrogen atom. You can find other constants, like the Boltzmann constant, the mass of the hydrogen atom, and the mass and radius of the Sun on page 553 of Kutner.

Compare the numbers you estimated above with the values inferred from accurate models: $P_c = 2.3 \times 10^{17}\,\text{dyne/cm}^2$ and $T_c = 1.6 \times 10^7\,\text{K}$. Do your estimates agree with these numbers? Examine your assumptions and try to figure out why they agree or disagree.

Solution Clearly the central pressure of the Sun disagrees with the accurate models by a factor of nearly 100. This is because the assumption of *constant density* is a very poor assumption for the Sun! Therefore the assumption of constant *mass* as a function of radius is also a very poor assumption, and since these terms each enter linearly into into the differential equation, they make a strong effect on the pressure structure of the Sun. The denisty gradient in the Sun is very, very strong in order to support the structure of the Sun.

However, the central temperature agrees *very* well! Is this suprising? Well, no, not if you actually plug the equation for P_c into the equation for T_c:

$$T_c \sim \frac{m}{k\bar{\rho}} \frac{GM\bar{\rho}}{R} = \frac{GMm}{kR}.$$

You see that the average density $\bar{\rho}$ has dropped out! This means that the central temperature of the Sun (to first order) *does not depend on its density structure.* This isn't quite true, but the effect of the structure on the central temperature is small.

2. Derive the relation between stellar mass and luminosity for stars on the main sequence. Below you will use this relation to estimate how long stars live at a given mass.

   First, write down the "order of magnitude equivalent" equations for mass conservation, hydrostatic equilibrium, and radiative transfer equations of stellar structure, as well as the equation of state for an ideal gas. *Make sure you keep terms in the opacity $\bar{\kappa}$ as you write down these equations!*

   To get you started, the equation of mass conservation,

$$\frac{dM}{dr} = 4\pi r^2 \rho(r), \tag{1}$$

   can be rewritten as

$$\frac{M}{R} \propto R^2 \rho, \tag{2}$$

   where M and R are the radius of the star and ρ is some measure of its density. We then have

$$\rho \propto MR^{-3}. \tag{3}$$

   Use this relation to eliminate ρ from the other equations.

   Solution Next is the equation of hydrostatic equilibrium,

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

or

$$P \propto R \times \frac{GM\rho}{R^2} \sim GM^2R^{-4}$$

after substituting in the relation for density derived above. This must equal the equation of state for an ideal gas,

$$P = \frac{\rho}{m} kT \propto \frac{k}{m} \frac{MT}{R^3},$$

so solving for T,

$$MR^{-4} \propto MTR^{-3}$$

and

$$T \propto MR^{-1}$$

(i.e., $T \propto M/R$).

The radiative transfer equation can be written as

$$L(r) \propto -\frac{r^2}{\bar{\kappa}\rho} T^3 \frac{dT}{dr}.$$

Next, use the approximation

$$\frac{dT}{dr} \approx \frac{T}{R} \tag{4}$$

at $L(r) = L(R)$ and your previous equations to write the radiative transfer equation in terms of luminosity L as a function of mass M and opacity $\bar{\kappa}$.

Solution Then using the equation for radiative transfer above, we can write

$$L = L(R) \propto \frac{R^2}{\bar{\kappa}\rho} T^3 \frac{T}{R}$$

Substituting in our scalings for density ρ and temperature T from above, we have

$$L \propto \frac{R^2 R^3}{\bar{\kappa} M} \frac{M^4}{R^4 R}$$

and therefore

$$L \propto \frac{M^3}{\bar{\kappa}}.$$

(Note that here I've surpressed terms in the mean mass of the particles, which is an important driver of the luminosity during the evolution of the stars but not on the zero-age main sequence.)

(a) For hot—that is, high-mass—stars, the opacity is a constant. What is the relation between luminosity and mass for these stars?

Solution If $\bar{\kappa} =$ constant, then

$$L \propto M^3.$$

(b) For cooler stars (like the Sun), the opacity goes as $\bar{\kappa} \propto \rho T^{-3.5}$. What is the relation between luminosity, mass, and radius for these stars?

Solution If $\bar{\kappa} \propto \rho T^{-3.5}$ (this is called *Kramer's law*), then

$$L \propto \frac{M^3 T^{3.5}}{\rho} \propto M^3 M^{3.5} R^{3.5} \frac{R^3}{M}$$

and

$$L \propto \frac{M^{5.5}}{R^{0.5}}$$

For solar mass stars, $R \propto M$ (roughly). What is the relation between luminosity and mass in this case?

Solution Then we have

$$L \propto \frac{M^{5.5}}{R^{0.5}} \frac{M^{5.5}}{M^{0.5}}$$

and

$$L \propto M^5.$$

Observationally, $L \propto M^{3.8}$, so $L \propto M^4$ is a reasonable approximation for the main sequence (except for the highest- and lowest-mass stars).

Now, the higher the luminosity, the more nuclear burning is required. The amount of fuel available is $\sim M$, and the timescale for nuclear burning is

$$\tau \propto \frac{M}{L}. \tag{5}$$

Then how does the timescale for nuclear burning vary with mass on the main sequence?

Solution If $L \propto M^4$, then

$$\tau \propto M^{-3}.$$

If the Sun will live 10^{10} years on the main sequence, how long will a star with $M = 10\,M_\odot$ live on the main sequence?

Solution A $M_{\rm big} = 10\,M_\odot$ star will live

$$\frac{\tau_{\rm big}}{\tau_\odot} = \left(\frac{M_{\rm big}}{M_\odot}\right)^{-3} = 10^{-3},$$

times the lifetime of the Sun or $\sim 10^7$ (10 million) years.

3. The Hyades open cluster has a radial velocity of $v_r = 35\,\mathrm{km/s}$, a proper motion of $\mu = 0.078\,\mathrm{arcsec/yr}$, and its convergent point is located at an angle of $A = 26°$ from the line-of-sight to the Sun.

   (a) How far away is the cluster from the Sun in parsecs (pc)?

   Solution Using equation (13.8) in Section 13.2 of Kutner, we have

   $$d = \frac{v_r \tan A}{4.74\mu},$$

   where d is in parsecs, v_r is in km/s, A is in degrees, μ is arcsec/year. Then

   $$d = \frac{35\tan(26)}{4.74 \times 0.078} = 46.2\,\mathrm{pc}$$

   What is its distance modulus $m - M$?

   Solution Recall that

   $$m - M = 5\log(d\,[pc]) - 5$$

   so

   $$m - M = 3.32\,\mathrm{mag}$$

   for the Hyades cluster.

   (b) The cluster's colour–*absolute* magnitude diagram (sometimes known as the "Hertzsprung-Russell diagram") is shown in Figure 1. Use the distance modulus you found above to write down the *apparent* magnitude scale on the right-hand side of the diagram.

   Solution Since $m - M = 3.32$, $m = M + 3.32$, so an absolute magnitude of $H_p = 0$ mag corresponds to an apparent magnitude of $m_{H_p} = 3.32$ mag.

   (c) What is the cluster's space motion (velocity)?

   Solution The space motion v of the Hyades cluster is, inverting equation (13.2) of Kutner,

   $$v = v_r/\cos A$$

   so $v = (35\,\mathrm{km/s})/\cos(26°) \approx 39\,\mathrm{km/s}$.