Degeneracy pressure: an alternate derivation

An addendum to “Stellar Structure and Evolution,” part of “Sterrenstelsels en Kosmos”

Degeneracy pressure for non-degenerate electrons

- To begin, we need to discuss the concept of "dynamical phase space"
- In dynamics, our interest is in the *positions* and *momenta* of particles
- We can imagine a 6-dimensional "hyperspace" that tells us the position and momentum of each particle we're following: (x, y, z, p_x, p_y, p_z)

- Now, because of the Pauli exclusion principle, only *two* electrons can occupy some volume of phase space
  - two because each electron has the possibility of having spin=1/2 or spin=-1/2
- The size of this volume in phase space is given by the Heisenberg uncertainty principle: $\Delta x \Delta p_x \geq h/2\pi$
  - so the minimum volume per particle must be $\Delta x \Delta y \Delta z \Delta p_x \Delta p_y \Delta p_z \sim h^3$

- Now, the *number density of electrons* in real space is $n_e = 1/(\Delta x \Delta y \Delta z)$

- If all the electrons have some *characteristic momentum*, which we'll call p_0, then we can say that there are two electrons per allowable volume of *phase space*,

$$p_0^3/n_e \sim h^3/2$$

- or

$$n_e/p_0^3 \sim 2/h^3$$

- Of course, the number density of electrons is the total density divided by the mass of available electrons, $n_e = \rho/(\mu_e m_p)$
  - here μ_e is the number of electrons per nucleon and m_p is the proton mass
- so the momentum can be found from (ignoring factors of order unity)

$$p_0^3 = h^3 \rho/(\mu_e m_p)$$

- and therefore

$$p_0 = h[\rho/(\mu_e m_p)]^{1/3}$$

- Now, the *pressure* is the just the energy density (again ignoring term of order unity)

$$P = \rho v^2$$

- but remember that the velocity of the electrons is just $v = p_0/m_e$

- On substituting, we find $P \propto \rho p_0^2$

- and therefore, substituting in the momentum, $P \propto \rho \rho^{2/3}$

- So finally, we have the desired relation:

$$P \propto \rho^{5/3}$$