Cosmic Structure Formation – Assignment 3

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Until the Assignment 2 we have limited our study of linearly evolving density fluctuations to media of a single nature (either matter or radiation or Λ). Here we will consider the case of a fluidum of matter and radiation. In other words, we look at the evolution of perturbations in the early universe. In the second problem we consider the Lagrangian approach to the perturbation theory and derive Raychaudhuri's equation.

1 Coupled growth of matter-radiation medium:

We have seen that the linearized form of the continuity equation for pure matter dominated universe leads to the following equation:

$$\frac{\delta_m}{\delta t} + \frac{1}{a}\boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \boldsymbol{v} = 0 \tag{1}$$

1. For a pure radiation-dominated universe if we take into account a pressure inertia term in the continuity and Euler equation:
$$\rho \to \rho + \frac{P}{c^2} \tag{2}$$
along with a pressure force term $-\nabla_r P$, show that in first order of fluctuations Continuity equation leads to:
$$\frac{\delta_{rad}}{\delta t} + \frac{4}{3}\frac{1}{a}\boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \boldsymbol{v} = 0 \tag{3}$$

2. While the Euler equation for both radiation and matter remains the same (discarding, unjustifiably, the pressure force):
$$\frac{\delta \boldsymbol{v}}{\delta t} + \frac{\dot{a}}{a}\boldsymbol{v} = -\frac{1}{a}\boldsymbol{\nabla}\delta\Phi \tag{4}$$
the Poisson equation establishes the coupling between the radiation and matter component. Show that for the Poisson equation in comoving coordinates:
$$\boldsymbol{\nabla}^2\delta\Phi = 4\pi G a^2[\bar{\rho}_m\delta_m + 2\bar{\rho}_{rad}\delta_{rad}] \tag{5}$$

3. Show that the combination of the continuity equation, Euler equation and Poisson equation for both matter and radiation, leads to the following system of two coupled second order differential equations:
$$\ddot{\delta}_m + 2\frac{\dot{a}}{a}\dot{\delta}_m = 4\pi G a^2[\bar{\rho}_m\delta_m + 2\bar{\rho}_{rad}\delta_{rad}] \tag{6}$$
$$\ddot{\delta}_{rad} + 2\frac{\dot{a}}{a}\dot{\delta}_{rad} = 4\pi G a^2[\frac{4}{3}\bar{\rho}_m\delta_m + \frac{8}{3}\bar{\rho}_{rad}\delta_{rad}] \tag{7}$$

4. Subsequently show that this can be condensed in a linear matrix equation:

$$L \begin{pmatrix} \delta_m \\ \delta_{rad} \end{pmatrix} = 4\pi G \begin{pmatrix} \bar{\rho}_m & 2\bar{\rho}_{rad} \\ \frac{4}{3}\bar{\rho}_m & \frac{8}{3}\bar{\rho}_{rad} \end{pmatrix} \begin{pmatrix} \delta_m \\ \delta_{rad} \end{pmatrix} \tag{8}$$

where the linear operator L is defined by

$$L = \frac{\partial^2}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial}{\partial t} \tag{9}$$

5. Given the expression above, reason under which circumstances the perturbations in radiation and matter are fully coupled. What does this imply for the corresponding perturbation in the entropy density,

$$\frac{\delta \mathcal{S}}{\mathcal{S}} = \frac{3}{4}\delta_{rad} - \delta_m \tag{10}$$

2 Lagrangian perturbation theory:

In the Eulerian approach we describe the evolution of a system at a fixed location. The change in the physical state of a system at a particular location is the result of two factors. First there are the intrinsic local changes of the system. Secondly, the system changes locally as a result of the energy flows in the system, i.e. as a result of energy flowing in and out of the local volume. The alternative approach to describing the physical evolution of the system is to follow the matter (energy) elements on their path through the evolving cosmic matter field. Mathematically, the conversion involves a transformation from an Eulerian to a Lagrangian time derivative. For the comoving coordinate system, the Lagrangian time derivative is defined by the transformation

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{1}{a}\boldsymbol{v} \cdot \boldsymbol{\nabla} \tag{11}$$

with $\boldsymbol{v}$ the peculiar velocity of the fluid element.

1. In the Lagrangian description the deformation of a fluid element is described as the gradient of its velocity field (the rate-of-strain tensor) as it is straightforward to see that the deformations are closely linked to the flow field in and around the fluid element. We can distinguish three different modes, the **expansion/compression** θ, the **shear** σ_{ij} and the **vorticity** tensor elements ω_{ij} which contribute to the velocity deformation tensor as follows:

$$\frac{1}{a}\frac{\partial v_i}{\partial x_j} = \frac{1}{3}\theta\delta_{ij} + \sigma_{ij} + \omega_{ij} \tag{12}$$

in which the expansion θ is the trace of the velocity field gradient, $\sigma_{ij} = \sigma_{ji}$ the traceless symmetric part and $\omega_{ij} = -\omega_{ji}$ the antisymmetric part. In component notation it becomes:

$$\theta = \frac{1}{a}\sum_i \frac{\partial v_i}{\partial x_i} \tag{13}$$

$$\sigma_{ij} = \frac{1}{2a}\Big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\Big) - \frac{1}{3}(\boldsymbol{\nabla} \cdot \boldsymbol{v})\delta_{ij} \tag{14}$$

$$\omega_{ij} = \frac{1}{2a}\Big(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\Big) \tag{15}$$

Using the above definitions derive the following equations governing comoving Lagrangian formulation of perturbation theory:

$$\frac{d\delta}{dt} + a(1+\delta)\theta = 0 \tag{16}$$
$$\frac{d\boldsymbol{v}}{dt} = -\frac{\dot{a}}{a}\boldsymbol{v} - \frac{1}{a}\boldsymbol{\nabla}\delta\Phi \tag{17}$$
$$\boldsymbol{\nabla}^2\delta\Phi = 4\pi G a^2 \bar{\rho}\delta \tag{18}$$
$$\frac{d\theta}{dt} + 2\frac{\dot{a}}{a}\theta + \frac{1}{3}\theta^2 + \sigma^{ij}\sigma_{ij} - 2\vec{\omega}^2 = 4\pi G\bar{\rho}\delta \tag{19}$$
$$\frac{d\omega^i}{dt} + 2\frac{\dot{a}}{a}\omega^i + \frac{2}{3}\theta\omega^i - \sigma_{ij}\omega^j = 0 \tag{20}$$
$$\frac{d\sigma_{ij}}{dt} + 2\frac{\dot{a}}{a}\sigma_{ij} + \frac{2}{3}\theta\sigma_{ij} + \sigma_{ik}\sigma^k_j - \frac{1}{3}\delta_{ij}(\sigma^{kl}\sigma_{kl}) = -T_{ij} \tag{21}$$

where T_{ij} is the gravitational tidal field working on the fluid element

$$T_{ij}(x) = \frac{1}{a^2}\Big(\frac{\partial^2\delta\Phi}{\partial x_i \partial x_j} - \frac{1}{3}\nabla^2\delta\Phi\,\delta_{ij}\Big) \tag{22}$$

And vorticity tensor ω_{ij} is related to the vorticity $\vec{\omega}$ of the fluid through the relations:

$$\omega_{ij} = \epsilon_{ijk}\omega^k \tag{23}$$
$$\vec{\omega} = \frac{1}{2a}\boldsymbol{\nabla} \times \boldsymbol{v} \tag{24}$$

References

1. Lecture notes: Gravitational Instability & Linear perturbation Theory, Rien van de Weygaert
2. Lecture notes: Going nonlinear: Progressing complexity, Rien van de Weygaert