Cosmic Structure Formation – Assignment 1

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Cosmology is the study of the dynamics of our universe as a whole. The fundamental principles of relativistic cosmology are: **Cosmological Principle**, **Weyl's postulate** and **General Theory of Relativity**. In this assignment we will discuss them briefly and will derive some results of the standard model of cosmology.

The cosmological principle

This principle states that, at each epoch, universe presents same aspects at every point in space except the local small-scale irregularities such as galaxies and clusters. This implies that if we assume a cosmic time t and consider the spacelike hypersurfaces, then there is no privileged point on these slices. That means that the universe is ***homogeneous*** in space. According to the same principle, there should not be a privileged direction about any one of these points on the spacelike slides also. Of course, two observers at the same location but in relative motion with respect to each other will not see the surrounding universe as isotropic, but the strength of the cosmological principle resides in the possibility to define, at each location, a fundamental rest frame from which the universe will appear isotropic. This means that the universe is ***isotropic*** in space too.
A spacelike hypersurface is homogeneous if there is a group of isometries which map any point onto the other. If the space is isotropic also then it must be spherically symmetric.

Weyl's postulate

The galaxies lie in the spacetime on a congruence [1] of timelike geodesic and diverging from a point in the finite or infinite past. These geodesics don't intersect except at a singular point in past and possibly in a singular point in future. These geodesics are orthogonal to the family of spacelike hypersurfaces.

So, we may choose coordinates as (t, x^1, x^2, x^3) so that spacelike hypersurfaces are given by $t =$ constant and coordinates (x^1, x^2, x^3) are constant along the timelike geodesics. The coordinates (x^1, x^2, x^3) are called comoving coordinates. From the above arguments the most plausible choice of the line element is [2],

$$ds^2 = c^2dt^2 - h_{ij}dx^i dx^j, \tag{1}$$

where t plays the role of the cosmic time here. The world map is the distribution of the events on the surfaces of simultaneity. World picture is the events lying in the past light cone of the observer. The second part of the metric h_{ij}, must be independent of time except some time dependent overall scaling factor, as the cosmological principle states that, 3-space is homogeneous and isotropic *at each epoch*. Also the homogeneity and isotropy condition implies that, it must have same curvature in every points of space otherwise, the points would not be geometrically identical. Only time can enter as a space-independent overall scale factor which will act as a magnification factor, keeping the geometry same:

$$h_{ij} = [S(t)]^2 g_{ij} \tag{2}$$

[1] A congruence of curves is defined such that only one curve goes through each point on the manifold.
[2] Greek letters are used for spacetime coordinates in coordinate basis and Latin letters for spatial indices.

1 Friedmann-Robertson-Walker (FRW) metric:

As discussed above, cosmological principle tells us that 3-space must be of constant curvature which is characterized by:

$$R_{\mu\nu\gamma\delta} = K(g_{\mu\gamma}g_{\nu\delta} - g_{\mu\delta}g_{\nu\gamma}) \tag{3}$$

where $R_{\mu\nu\gamma\delta}$ is Riemann tensor and K is a constant which is called curvature. In 3-space by contraction we get:

$$R_{ij} = 2Kg_{ij} \tag{4}$$

where R_{ij} is now Ricci tensor in 3-space. As 3-space is isotropic, it must be spherically symmetric. So,

$$d\sigma^2 = e^{\lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{5}$$

where λ is a function of r only. Now using the above metric Eq.(5) and the condition of constant curvature Eq.(4) we get:

$$e^{-\lambda} = 1 - Kr^2 \tag{6}$$

So, the metric turns out to be,

$$d\sigma^2 = \frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{7}$$

1. Show that the above metric takes a conformally flat form

$$d\sigma^2 = (1 + \frac{1}{4}K\bar{r}^2)^{-2}[d\bar{r}^2 + \bar{r}^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{8}$$

   [Hint: use a radial coordinate transformation $r = \bar{r}/(1 + \frac{1}{4}K\bar{r}^2)$]

2. If we introduce $a(t) = [S(t)]^2/|K|$ show that the spacetime metric will have a form like:

$$ds^2 = c^2dt^2 - a(t)^2\Big(\frac{dx^2}{1 - kx^2/R_0^2} + x^2(d\theta^2 + \sin^2\theta d\phi^2)\Big) \tag{9}$$

   Find out the value of R_0 which is called the radius of curvature of the 3-space. k can take either of the values $+1, -1, 0$ corresponding to positively curved or negatively curved or flat space. [Hint: write $K = |K|^{\frac{1}{2}}k$ and use a rescaled radial coordinate $x = |K|^{\frac{1}{2}}r$]

3. If use the following transformations for different values of k

$$\begin{aligned} k = +1, \qquad & x = R_0\sin(r/R_0) \\ k = 0, \qquad & x = r \\ k = -1, \qquad & x = R_0\sinh(r/R_0) \end{aligned} \tag{10}$$

   show that in all the cases the metric becomes:

$$ds^2 = c^2dt^2 - a(t)^2\Big(dr^2 + S_k(r)^2d\Omega^2\Big) \tag{11}$$

   where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ and $x = S_k(r)$.

2 Proper distance, Hubble's law and Redshift

The proper distance $d_p(t)$ is obtained by integrating over the radial comoving coordinate r,

$$\begin{aligned} d_p(t) = a(t)\int_0^r dr = a(t)r &= a(t)R_0\mathtt{sin}^{-1}(x/R_0) \quad (k=+1) \\ &= a(t)x \qquad\qquad\qquad (k=0) \\ &= a(t)R_0\mathtt{sinh}^{-1}(x/R_0) \; (k=-1) \end{aligned} \tag{12}$$

The rate of change of the proper distance between two galaxies due to large scale cosmic expansion is,

$$\dot{d}_p = \dot{a}r = \frac{\dot{a}}{a}d_p \tag{13}$$

so, at present epoch, there is a linear relationship between proper distance and recessional velocity

$$v_p(t_0) = \dot{d}_p(t_0) = \left(\frac{\dot{a}}{a}\right)_{t=t_0} d_p(t_0) = H_0 d_p(t_0) \tag{14}$$

where H_0 is Hubble's parameter whose inverse, $1/H_0$ roughly gives us the age of our universe. Above equation is known as Hubble's law. Hubble distance is defined as

$$d_H(t_0) = c/H_0 \tag{15}$$

which implies that the points separated by a proper distance greater than d_H will have $v_p > c$.

1. Using the definition of redshift, $z = (\lambda_0 - \lambda_e)/\lambda_e$ show the following relation between z and $a(t)$:

$$1 + z = \frac{a(t_0)}{a(t_e)} = \frac{1}{a(t_e)} \tag{16}$$

with conventionally $a(t_0) = 1$.

Hence show that the proper distance between galaxies at time of emission $d_p(t_e)$ and observation of light $d_p(t_0)$ are related as,

$$d_p(t_e) = \frac{d_p(t_0)}{(1+z)} \tag{17}$$

assuming that the scale factor doesn't change significantly while emitting and observing the light.

3 Luminosity distance

Standard candles are objects whose luminosity L are known while the flux f is a measurable quantity. So, the luminosity distance d_L can be defined as,

$$d_L^2 = \frac{L}{4\pi f} \tag{18}$$

Suppose now, light is emanating from a galaxy at time t_e and observed by the observer at time t_0, $(t_e < t_0)$. At the present epoch $(t = t_0)$ the light will spread over a surface of sphere characterized by

$$ds^2 = -a(t_0)^2 S_k(r)^2 (d\theta^2 + \sin^2\theta d\phi^2) \tag{19}$$

which is the line element for a sphere of radius $a(t)S_k(r)$ and so the sphere has surface area $4\pi a(t_0)^2 S_k(r)^2$. Hence the observed flux will be,

$$f = \frac{L}{4\pi a(t_0)^2 S_k(r)^2 (1+z)^2} \tag{20}$$

where the two factors $(1+z)$ are coming one due to the reason that, the time interval of receiving of certain amount of energy is longer by a factor of $(1+z)$ than the interval of emission due to Doppler shift. And another factor is due to the fact that, the energy of each photon of light is reduced by $(1+z)$ due to expansion of universe.

1. From the above two equations show that for a spatially flat $(k=0)$ universe we get

$$d_L = r(1+z) = d_p(t_0)(1+z) \tag{21}$$

4 Angular diameter distance

Standard yardsticks are objects whose proper lengths l are known. If we observe a standard yardstick of constant proper length aligned perpendicular to our line of sight and measure an angular distance $\delta\theta$ between the ends of the yardstick then the angular diameter distance is defined as,

$$d_A = \frac{l}{\delta\theta} \tag{22}$$

The distance ds between the two ends of the yardstick measured at the time when the lights were emitted from the ends, t_e is

$$ds = a(t_e)S_k(r)\delta\theta \tag{23}$$

where $\delta\theta = \theta_2 - \theta_1$ is the angular size between two ends of the yardstick. Now as the standard yardsticks are tightly bound objects which are not expanding with the universe, we can write,

$$l = ds = a(t_e)S_k(r)\delta\theta = \frac{S_k(r)\delta\theta}{1+z} \tag{24}$$

So, the angular diameter distance is given by,

$$d_A = \frac{l}{\delta\theta} = \frac{S_k(r)}{1+z} \tag{25}$$

1. Show for a spatially flat $(k=0)$ universe we get,

$$d_A(1+z) = d_p(t_0) = \frac{d_L}{1+z} \tag{26}$$

5 Equations of cosmic dynamics

The equations of cosmic dynamics mostly consist of three equations, so called Friedmann's equation, acceleration equation and fluid equation. They are as follows:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}(\rho_m + \rho_r) - \frac{kc^2}{R_0^2 a^2} + \frac{\Lambda}{3} \tag{27}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\rho + 3P) + \frac{\Lambda}{3} \tag{28}$$

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0 \tag{29}$$

It can be shown that only two of the above equations are independent [*Optional homework*]. Matter, radiation and dark energy: each component of the universe obeys an equation of state between pressure (P) and energy density ρ through some parameter w as, $P = w\rho$.

1. Using the fluid equation show that for a flat ($k = 0$) universe with no cosmological constant, the energy density evolves with expansion parameter as (for $w \neq -1$),

$$\rho_w(a) = \rho_{w,0} a^{-3(1+w)} \tag{30}$$

2. From Friedmann equation show that the expansion parameter evolves with time as,

$$a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{(3+3w)}} \tag{31}$$

3. And the Hubble's parameter evolves as,

$$H_0 = \left(\frac{\dot{a}}{a}\right)_{t=t_0} = \frac{2}{3(1+w)} t_0^{-1} \tag{32}$$

which gives the age of the universe as,

$$t_0 = \frac{2}{3(1+w)} H_0^{-1} \tag{33}$$

4. The equation of state parameter $w = 0$ for non-relativistic matter, $w = 1/3$ for radiation. The critical density is followed from the Friedmann's equation by putting $k = 0$, which is $\rho_c = \rho_{w,0} = \frac{3c^2 H_0^2}{8\pi G}$. Rewrite the equation how energy density falls with time using the expression of critical density:

$$\rho_w(t) = \frac{c^2}{6\pi G(1+w)^2} t^{-2} \tag{34}$$

5. Now using the previous definition of redshift show that,

$$t_e = \frac{2}{3(1+w)H_0} \cdot \frac{1}{(1+z)^{3(1+w)/2}} \tag{35}$$

And the proper distance at present epoch is given by,

$$d_p(t_0) = \frac{2c}{H_0(1+3w)} \left[1 - (1+z)^{-(1+3w)/2}\right] \tag{36}$$

The horizon distance is defined in the limit $t_e \to 0$ ($z \to \infty$): $d_{hor}(t_0) = \frac{2c}{H_0(1+3w)}$

6. For Λ-only ($w = -1$) flat universe, the above treatment doesn't work. The energy density ρ, in this case remains constant. Show that in this case the scale factor varies with time as,

$$a(t) = e^{[(\frac{1}{3}\Lambda)^{\frac{1}{2}} t]} \tag{37}$$

And derive the following expression for proper distance,

$$d_p(t_0) = \frac{c}{H_0}\left[e^{H_0(t_0 - t_e)} - 1\right] = \frac{c}{H_0} z \tag{38}$$

References

1. Ray d'Inverno *Introducing Einstein's Relativity* Oxford University Press, 1992
2. Barbara Ryden *Introduction to Cosmology* Addison Wesley, 2007