Today, we are going to study the evolution of a spherical overdensity (or spherical underdensity) embedded in an expanding Friedmann-Robertson Walker Universe.

Starting as a tiny primordial density fluctuation, the spherical perturbation will at first evolve linearly. It would start to expand along with the Hubble expansion, but as the tiny perturbation grows its velocity would start to deviate more and more from the Hubble rate: its (negative) peculiar velocity would become larger and larger.

At some stage the spherical overdensity would become so substantial, i.e. $\delta \approx 1$, that you cannot follow it anymore with linear theory. Interestingly, in the case of a spherical configuration we can follow the complete evolution up to the collapse of the object.

As the density excess of the spherical overdensity increases, the object will slow down its expansion so far that it comes to a standstill: its peculiar velocity compensates for the Hubble expansion. At that *turnaround* point, the fate of the object will be inescapable: the object will start to contract, also physically (ie. not only in comoving space). After some time the contraction leads to the climax of its evolution, the collapse of the entire object to one point. There is life after death: following this the energy and momentum of the mass within the sphere will get distributed to an equilibrium configuration, which it reaches when it attains virial equilibrium. The latter we will not analyze today.

The beauty of the spherical object's evolution is that we can describe and understand its evolution in detail, and by analytical means. It requires some good understanding of your basic dynamics knowledge, in combination with that of linear evolution. The *spherical model* forms a key reference point for our understanding of cosmic structure formation. An analytically tractable idealizations such as the spherical model helps to understand various aspects of clumps and void evolution. It therefore represents the key reference model against which we may assess the evolution of more complex configurations.

The structure of a spherical void or peak can be treated in terms of mass shells. In the "spherical model" concentric shells remain concentric and are assumed to be perfectly uniform, without any substructure. The shells are supposed never to cross until the final singularity, a condition whose validity is determined by the initial density profile.

The resulting solution of the equation of motion for each shell may cover the full nonlinear evolution of the perturbation, as long as shell crossing does not occur. As a result, each shell can be characterized by the mass M contained within its interior.

Part I: spherical object, linear evolution.

In the lectures, we have learned that in the linear regime, the peculiar velocity is directly proportional to the accompanying peculiar gravitational acceleration,

$$\boldsymbol{v} \;=\; \frac{2f(\Omega)}{3H\Omega}\,\boldsymbol{g}\,. \tag{1}$$

For a spherical overdensity, we may find a relation for this in terms of the mean overdensity Δ of the spherical peak. To this end we use the well-known fact that for a spherical object the dynamics is fully and solely dependent on the mass $M(r)$ that is contianed within the radius r.

a) If a spherical object at time t has a radial density profile $\rho(r,t)$, demonstrate that the mean density excess (or deficit) within radius r is given

$$\Delta(r,t) \quad = \quad \frac{3}{r^3}\int_0^r \delta(y,t)\,y^2\,dy\,, \tag{2}$$

where $\delta(r,t)$ is the radial density perturbation profile $\delta(r,t)$.

b) Infer an expression for the peculiar gravitational acceleration $g(r,t)$ at radius r, in terms of the mean density excess $\Delta(r,t)$. For this, use the fact that the entire gravitational acceleration (including the background contribution) is given by

$$\frac{d^2r}{dt^2} \;=\; -\frac{GM(r,t)}{r^2} \tag{3}$$

Do not forget to remove the background term, ie. the term corresponding to the unifomr density $\rho_u(t)$.

c) With the finding of the previous question, infer that the peculiar velocity of a shell with radius r of the spherical perturbation in the linear regime, restricting oneself to the growing mode, is given by

$$v_{lin}(r,t) \;=\; -\frac{H(t)r}{3}\,f(\Omega)\,\Delta(r,t) \tag{4}$$

It clearly demonstrates the growth of the linear peculiar velocity as the density excess $\Delta(r,t)$ is growing.

Part II: Spherical shell, equations of motion.

In this exercise we study the full nonlinear evolution of a shell in a spherical mass concentration.

a) As we pointed out above, the dynamics of a spherical shell is completely determined by the mass M interior to the shell. Look at a shell that at time t has reached a radius $r(t)$. Show that its equations of motion are given by:

$$\begin{aligned} \frac{d^2 r}{dt^2} &= -\frac{GM(r,t)}{r^2} \\ \frac{1}{2}\left(\frac{dr}{dt}\right)^2 - \frac{GM(r,t)}{r} &= E \end{aligned} \tag{5}$$

where $E = cst$ is the energy of the shell. A bound shell has (total) energy $E < 0$, an open shell has $E > 0$ and a critical shell has $E = 0$.

b) If $E > 0$, show that the solution to the equation above may be given in parameterized form

$$\begin{aligned} r(\Phi) &= \frac{GM}{2E}\left(\cosh\Phi_r - 1\right) \\ t(\Phi) - t_i &= \frac{GM}{2E\sqrt{2E}}\left(\sinh\Phi_r - \Phi_r\right) \end{aligned} \tag{6}$$

where Φ_r is the socalled "development angle" parameter, and has a different behaviour for each individual shell. You may try to demonstrate this by filling in this solution in the equations of motion (eqn. 6), but it is a lot more revealing to try to derive this solution from first principle.

HINT: To be able to solve the second order differential equation (for an open shell) from first principle, transform the energy equation into the following second order differential equation

$$r\frac{d}{dt}\left(r\frac{dr}{dt}\right) = GM + 2Er \tag{7}$$

and introduce the parameter Φ_r via the tranformation,

$$\frac{d}{d\Phi_r} = \frac{1}{\sqrt{2E}}\, r\frac{d}{dt}\,. \tag{8}$$

On the basis of the latter, it is not difficult to infer the following set of 2 solvable differential equations,

$$\begin{aligned} \frac{d^2 r}{d\Phi_r^2} &= \frac{GM}{2E} + r \\ \frac{dt}{d\Phi_r} &= \frac{r}{\sqrt{2E}}\,. \end{aligned} \tag{9}$$

Note that the first equation should be easily solvable, as it simply concerns harmonic functions. The transformation is almost the same for a closed shell, with the exception of a minus sign.

c) Also infer the solution for a closed shell, ie. for $E < 0$. Show that then

$$\begin{aligned} r(\Phi) &= \frac{GM}{2E}\left(\cos\Phi_r - 1\right) \\ t(\Phi_r) - t_i &= \frac{GM}{2E\sqrt{-2E}}\left(\sin\Phi_r - \Phi_r\right) \end{aligned} \tag{10}$$

d) Finally, infer that for a critical shell with $E = 0$

$$r(t) = \left(\frac{9}{2}GM\right)^{1/3}(t - t_i)^{2/3} \tag{11}$$

As an aside, also note that the Friedmann equations for a FRW Universe, for $\Lambda = 0$, have exactly the same form. The solutions that you will find for this problem, is therefore entirely applicable to such a mass-dominated FRW universe. In this case, you find the solution for the expansion factor $a(t)$ by using the transformation,

$$E \rightarrow \frac{1}{2}H_0^2\left(1 - \Omega_0\right) \tag{12}$$

$$\tag{13}$$

$$GM \rightarrow \frac{1}{2}\Omega_0 H_0^2 \tag{14}$$

Part III: Spherical shell evolution, parameterization.

In task I, we already argued that the solution for each shell depends on the mean density contrast within the radius of the shell (ie., implicitly, on the entire mass within the shell), $\Delta(r,t)$. We inferred that the mean density excess of such a shell is given by

$$\begin{aligned}\Delta(r,t) &= \frac{3}{r^3}\int_0^r \left[\frac{\rho(y,t)}{\rho_u(t)} - 1\right] y^2\,dy \\ &= \frac{3}{r^3}\int_0^r \delta(y,t)\,y^2\,dy\,,\end{aligned} \tag{15}$$

In this problem we will look at the parameterization of the shell at radius r in terms of its density excess/deficit, corresponding linear peculiar velocity and related energy of the shell. This to be able to transform the quantities GM and E in the shell's equation of motion (problem set II) into relevant perturbation quantities.

a) The shell may be embedded in a $\Omega \neq 1$ Universe. The density perturbation profile $\delta(r,t)$ is with respect to the corresponding cosmic density $\rho_u(t)$. However, to be able to judge whether a shell is bound or unbound, we should ideally refer to a critical (Einstein-de Sitter) universe. Only when it is bound with respect to and EdS universe, a shell will be really gravitationally bound and ultimately collapse. In other words, in an open Universe a perturbation might be an overdensity, yet still represent an gravitationally unbound entity. To this end, we introduce a mean over/underdensity $\Delta_c(r,t)$ of a shell with respec to an EdS Universe.

   Argue and demonstrate that the mean density excess $\Delta_c(r,t)$ with respect to an Einstein-Universe (ie. the critical Universe) is given by

$$1 + \Delta_c(r,t) \,=\, \Omega(t)\,\left(1 + \Delta(r,t)\right)\,. \tag{16}$$

   [. b)] Infer that for a shell that started at initial time t_i with a raidus r_i, the mass M can be written as

$$GM \,=\, \frac{1}{2} H_i^2 r_i^3 \left(1 + \Delta_{ci}\right)\,, \tag{17}$$

   In this expression, Δ_{ci} is the initial critical interior overdensity, $\Delta_{ci} = \Delta_c(r_i, t_i)$. Note that the interior mass $M(r)$ of a shell remains constant during its evolution: its radius changes, but its interior mass is assumed to remain constant (ie. there is not shell-crossing).

c) To quantify the corresponding total energy (E) of a shell, we also have to take into account its kinetic energy. To this end, we introduce a parameter

α that quantifies the relative contribution of the peculiar velocity v with respect to the Hubble velocity Hr. Its definition is

$$\alpha(r,t) \;=\; \left(\frac{v}{Hr}\right)^2 , \tag{18}$$

for a shell with radius $r(t)$ and peculiar velocity $r(t)$.

Show that for a shell with energy E and interior mass M, starting from an initial radius r_i, the energy E of the shell can be written as

$$\begin{aligned} E &= \frac{1}{2}\left(\frac{dr}{dt}\right)^2 - \frac{GM}{r} \\ &= \frac{1}{2}H_i^2 r_i^2 \,\left(\alpha_i - \Delta_{ci}\right) . \end{aligned} \tag{19}$$

In this expression, Δ_{ci} is the initial critical interior overdensity, $\Delta_{ci} = \Delta_c(r_i,t_i)$, and α_i the corresponding initial velocity perturbation parameter, $\alpha_i = \alpha(r_i,t_i)$.

d) It is revealing to infer the initial value of the kinetic energy parameter $\alpha(t_i) = \alpha_i$, ie. the value in the linear regime. In question I, we inferred that for a spherical shell, the peculiar velocity in the linear regime is given by

$$v_i \;=\; -\frac{H_i r_i}{3}\, f(\Omega_i)\,\Delta(r_i,t_i)\,. \tag{20}$$

On the basis of this, show that in the linear regime, restricting oneself to the growing mode solution, at initial time t_i the parameter α_i has the value

$$\alpha_i \;=\; -\frac{2}{3} f(\Omega_i)\Delta(r_i,t_i)\,, \tag{21}$$

where $f(\Omega_i)$ is the well-known Peebles factor at time t_i, when $\Omega(t) = \Omega_i$.

e) In the case of an Einstein-de Sitter Universe, $\Omega(t) = \Omega_i = 1$ and $f(\Omega) = 1$. Show that in that case in the linear regime the energy of a shell is given by

$$E \;=\; -\frac{5}{3}\,\frac{1}{2} H_i^2 r_i^2\, \Delta_i\,, \tag{22}$$

where we use the fact that $\Delta_{ci} = \Delta_i$ in an Einstein-de Sitter Universe. The factor $\frac{5}{3}$ is a very important factor that comes back in many evaluations of linear velocity perturbations when these are presumed to be growing mode only.

Part IV: Spherical shell evolution: radius, density and velocity.

In this problem, we are going to infer the solutions for shells in and around a spherical density excess. We will infer the evolution of its radius $r(t)$, the corresponding interior density excess/deficit, and the corresponding peculiar velocity.

We are going to look at the evolution of the spherical shell radius for a shell that has an initial radius r_i at initial time t_i (and comoving radius x_i, with $r_i = a(t_i)x_i$)

Referring to the general solutions for a spherical mass M, with energy E, we will distinguish the following situations:

$$\begin{aligned} \text{openshell}: &\qquad E > 0, \quad \alpha_i > \Delta_{ci} \\ \text{criticalshell}: &\qquad E = o, \quad \alpha_i = \Delta_{ci} \\ \text{closedshell}: &\qquad E < 0, \quad \alpha_i < \Delta_{ci} \end{aligned} \tag{23}$$

Given the equations of motion for the shell (see question II), and the similarity to the dynamics for an FRW universe, we may actually describe the expansion/contraction of the shell's radius in terms of a dimensionless expansion factor $\mathcal{R}(t, r_i)$. This factor describes the relative expansion/contraction of a shell's radius with respect to its initial radius r_i,

$$r(t, r_i) \,=\, \mathcal{R}(t, r_i) r_i \,. \tag{24}$$

a) Combining the general spherical evolution solutions derived in question II with the equations for the mass M and energy E of a shell, ie. equations 17 and 22, show that the expansion factor $\mathcal{R}(t, r_i)$ of a shell is described by the following (partially parameterized) solutions:

$$\mathcal{R}(t, r_i) \,=\, \begin{cases} \frac{1}{2}\,\dfrac{1+\Delta_{ci}}{(\alpha_i - \Delta_{ci})}\,\,(\cosh\Phi_r - 1) & \Delta_{ci} < \alpha_i, \\ \\ \frac{1}{2}\,\dfrac{1+\Delta_{ci}}{(\Delta_{ci} - \alpha_i)}\,\,(1 - \cos\Phi_r) & \Delta_{ci} > \alpha_i\,, \end{cases} \tag{25}$$

in which the development angle Θ_r is related to time t via

$$H_i(t - t_0) \,=\, \begin{cases} \frac{1}{2}\,\dfrac{1+\Delta_{ci}}{(\alpha_i - \Delta_{ci})^{3/2}}\,\,(\sinh\Phi_r - \Phi_r) & \Delta_{ci} < \alpha_i \\ \\ \frac{1}{2}\,\dfrac{1+\Delta_{ci}}{(\Delta_{ci} - \alpha_i)^{3/2}}\,\,(\Phi_r - \sin\Phi_r) & \Delta_{ci} > \alpha_i \end{cases} \tag{26}$$

while for a critical shell the solution is given by the direct relation

$$\mathcal{R}(t) \,=\, \left\{\frac{3}{2} H_i (1+\Delta_{ci})^{1/2} t\right\}^{2/3} \qquad\qquad \Delta_{ci} = \alpha_i\,. \tag{27}$$

b Sketch the solution for the radius $r(r_i, t)$ of the a range of different shells of an initially tophat overdensity, as well as the expansion factor $\mathcal{R}(r_i, t)$ of these shells. A tophat overdensity has a density profile $\delta(r)$,

$$\delta(r,t) = \begin{cases} \delta_s & r < R_s \\ \\ 0 & r > R_s \end{cases} \tag{28}$$

i.e. it has an overdensity within radius R_s and no density pertubation outside this radius.

c) Show that for a shell with an initial density contrast $\Delta(r_i)$, the density contrast $\Delta(r, t)$ at any subsequent time t is given by,

$$1 + \Delta(r,t) = \frac{1 + \Delta_i(r_i)}{\mathcal{R}^3} \frac{a(t)^3}{a_i^3} . \tag{29}$$

d) Subsequently, show that in an Einstein-de Sitter universe the density contrast of the shell, at development angle Φ_r, is given by

$$1 + \Delta(r,t) = \frac{9}{2} f(\Phi_r) \tag{30}$$

with the function $f(\Phi_r)$ given by

$$f(\Phi) = \begin{cases} \dfrac{(\sinh \Phi_r - \Phi_r)^2}{(\cosh \Phi_r - 1)^3} & \text{open}\,, \\ \\ 2/9 & \text{critical}\,, \\ \\ \dfrac{(\Phi_r - \sin \Phi_r)^2}{(1 - \cos \Phi_r)^3} & \text{closed}\,. \end{cases} \tag{31}$$

e) Show that the shell's peculiar velocity (at development angle Φ_r), ie. the velocity with respect to the global Hubble velocity,

$$\begin{aligned} v_{pec}(r,t) &= \frac{dr}{dt} - H(t)r(t) \\ &= \frac{d\mathcal{R}}{dt} r_i - H(t)r(t) \end{aligned} \tag{32}$$

can be inferred from the expression

$$v_{pec}(r,t) = \frac{3}{2} H(t) r(t) \left\{ g(\Phi_r) - \frac{2}{3} \right\} , \tag{33}$$

where $H(t)$ is the Hubble parameter at time t and the cosmic "velocity" function is

$$g(\Phi_r) = \begin{cases} \dfrac{\sinh \Phi_r \left(\sinh \Phi_r - \Phi_r\right)}{\left(\cosh \Phi_r - 1\right)^2} & \text{open}\,, \\[2ex] \dfrac{2}{3} & \text{critical}\,, \\[2ex] \dfrac{\sin \Phi_r \left(\Phi_r - \sin \Phi_r\right)}{\left(1 - \cos \Phi_r\right)^2} & \text{closed} \end{cases} \tag{34}$$

f) Sketch the mean density profiles $\Delta(r,t)$ and velocity profiles of the spherical object with an initial tophat density profile (see eqn. b). Do this for a range of shells (in and around the object). Do this both for the total velocity as well as the peculiar velocity.

g) For an evolving overdense spherical shell you may infer a few interesting stages. To this end, we restrict ourselves to the evolution in an Einstein-de Sitter Universe. At which point does the initial overdense shell get to a halt ? Ie., at which development angle Φ_r does it attain a peculiar velocity $v(r,t) = 0$.

h) Show that the density excess at this *turnaround* point is given by

$$1 + \Delta(r, t_{ta}) = (3\pi/4)^2 \approx 5.55 \tag{35}$$

This is an important finding: if you take a radius around an (approximately spherical) density excess in which the density excess is in the order of 4.55, then the matter inside that sphere is contracting and on the way towards collapse. We see this happening around rich clusters of galaxies.

i) By what factor has the comoving (!!!!) radius of the spherical overdensity shrunk at this turnaround point ?

j) Had the perturbation evolved according to linear theory, then turnaroudn would happen at the redshift z_{ta} when the linear theory prediction for the density excess would reach the value $\Delta_{lin} \approx 1.062$? Infer this number from the equations above, and using linear theory. It is called the *linearly extrapolated turnaround density*.

k) Full collapse is corresponding to a development angle $\Phi_r = 2\pi$ (see eqn. (a)). What is the physical density at full collapse ? Much more important, show that the linearly extrapolated collapse density is

$$\Delta_{lin}(z_c) \approx 1.686\,. \tag{36}$$

This will turn out to be one of the most important numbers in our study of cosmic structure formation !!! You will find it back in much of the analyses that will follow.

l) Why so important ? You know that in an Einstein-de Sitter Universe the linear density growth factor is equal to $D(t) = a(t) = 1/(1+z)$. Given an initial (linear) density field $\Delta(t_i)$. Identify the peaks with a spherically averaged overdensity $\Delta(t_i)$. At which redshift would they collapse ? What do you learn from this ? NOTE: try to realize how important this consequence is !!!!