Part I.

In this exercise we learn how to deal with the comoving coordinate transform. After completing it, you will be able to derive the Newtonian 'dust' comoving equations from basic equations.

a. In general, suppose we have some function $f(\boldsymbol{x}, t)$ and we move with some velocity with respect to the coordinate system of $\boldsymbol{x}$, say $\boldsymbol{x} = g(\boldsymbol{y}, t)$. Give an expression for $\partial_t f|_y$.

Now we define

$$\boldsymbol{r} \equiv a\boldsymbol{x} \qquad \boldsymbol{v} \equiv a\dot{\boldsymbol{x}} \qquad \phi \equiv \Phi + \frac{1}{2}a\ddot{a}x^2$$

$$\begin{aligned} \partial_t \rho + \boldsymbol{\nabla}_r \cdot (\rho \boldsymbol{u}) &= 0 && \text{Continuity} \\ \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \boldsymbol{\nabla}_r)\boldsymbol{u} &= -\boldsymbol{\nabla}_r \Phi && \text{Euler} \\ \nabla_r^2 \Phi &= 4\pi G \rho && \text{Poisson} \end{aligned}$$

b. For a function $f(r, t)$, write down $\partial_t f|_r$ in terms of $\partial_t f|_x$.

d. Rewrite the Continuity equation in comoving coordinates and in terms of the density perturbation $\delta \equiv \rho/\rho_u - 1$.

c. Rewrite the Euler equation in comoving coordinates.

e. The potential $\phi_u = -a\ddot{a}x^2/2$ reflects the background dynamics of the Universe of some density ρ_u. Compute $\nabla_r^2 \phi_u$; combining with the Poisson equation, do you recognise the result?

f. Assume that we can split the dynamics of LSS from the background Universe (this is not trivial!). Rewrite the Poisson equation in comoving coordinates and in terms of the density and potential perturbations.

Part II: Cosmology and the Linear Density Growth Factor.

The equation for linear structure formation is

$$\ddot{D} + 2H\dot{D} = \frac{3}{2}\Omega_0 H_0^2 a^{-3} D.$$

We will learn how to compute the *growing mode* solution for a universe with matter, curvature and a cosmological constant. The Friedman equation for such a universe can be written as

$$H^2 = H_0^2 \left[\Omega_m a^{-3} + (1-\Omega_c)a^{-2} + \Omega_\Lambda\right],$$

where $H = H(t)$ is the Hubble parameter, and Ω_i are values at $t = t_0$.

a. Write an expression for $\ddot{H} + 2H\dot{H}$ (try differentiating the Friedman equation). Does this result seem familiar?

b. Using both equations, work your way to

$$\frac{d}{dt}\left[a^2 H^2 \frac{d}{dt}\left(\frac{D}{H}\right)\right] = 0$$

This is solved by integrating

$$D_+ \propto H \int \frac{dt}{a^2 H^2}$$

Most of the time the result will be shown as a function of redshift $z + 1 = 1/a$. This is normalised to the Einstein-de Sitter Universe in the limit $a \to 0$, where $D_+ = a$.

c. Show that

$$D_+(z) = \frac{5}{2}\Omega_m H_0^2 H(z) \int_z^\infty dz' \frac{1+z'}{H^3(z')}.$$

d. Solve it for the cases of an Einstein-de Sitter, free expanding, and a de Sitter universe (Λ-only).

Part III Coupled Growth of a Matter-Radiation medium.

Up to this point, we have limited the study of linearly evolving density fluctuations to media of a single nature (either matter or radiation). Here we are considering the case of a fluidum of matter and radiation. In other words, we look at the evolution of perturbations in the early universe.

To keep it simple, you are allowed to ignore the pressure force of the radiation. However, not the contribution of pressure to the inertia of the matter-radiation fluidum.

a. Show that the linearized form of the continuity equation for matter and radiation are:

$$\begin{aligned} \frac{\partial \delta_m}{\partial t} + \frac{1}{a} \boldsymbol{\nabla}_x \cdot \boldsymbol{v} &= 0 \\ \frac{\partial \delta_{rad}}{\partial t} + \frac{4}{3} \frac{1}{a} \boldsymbol{\nabla}_x \cdot \boldsymbol{v} &= 0 \end{aligned} \tag{1}$$

b. While the Euler equation for both radiation and matter remains the same (discarding, unjustifiably, the pressure force),

$$\frac{\partial \boldsymbol{v}}{\partial t} + \frac{\dot{a}}{a} \boldsymbol{v} = -\frac{1}{a} \boldsymbol{\nabla} \phi$$

the Poisson equation establishes the coupling between the radiation and matter component. Show that for the Poisson equation in comoving coordinates,

$$\nabla_x^2 \phi = 4\pi G\, a^2 \left[\rho_{m,u}\, \delta_m + 2\, \rho_{rad,u}\, \delta_{rad}\right]. \tag{2}$$

c. Show that the combination of the continuity equation, Euler equation and Poisson equation for both matter and radiation, leads to the following system of two coupled second order differential equations,

$$\begin{aligned} \frac{\partial^2 \delta_m}{\partial t^2} + 2\, \frac{\dot{a}}{a} \frac{\partial \delta_m}{\partial t} &= 4\pi G\, a^2 \left[\rho_{m,u}\, \delta_m + 2\, \rho_{rad,u}\, \delta_{rad}\right] \\ \frac{\partial^2 \delta_{rad}}{\partial t^2} + 2\, \frac{\dot{a}}{a} \frac{\partial \delta_{rad}}{\partial t} &= 4\pi G\, a^2 \left[\frac{4}{3}\, \rho_{m,u}\, \delta_m + \frac{8}{3}\, \rho_{rad,u}\, \delta_{rad}\right]. \end{aligned} \tag{3}$$

d. Subsequently, show that this can be condensed in an insightful linear matrix equation,

$$L \begin{pmatrix} \delta_m \\ \delta_{rad} \end{pmatrix} = 4\pi G \begin{pmatrix} \rho_{m,u} & 2\rho_{rad,u} \\ \frac{4}{3}\rho_{m,u} & \frac{8}{3}\rho_{rad,u} \end{pmatrix} \begin{pmatrix} \delta_m \\ \delta_{rad} \end{pmatrix} \tag{4}$$

in which the linear operator L is defined by

$$L \equiv \frac{\partial^2}{\partial t^2} + 2\,\frac{\dot{a}}{a}\,\frac{\partial}{\partial t}\,. \tag{5}$$

e. Given the expression above, reason under which circumstances the perturbations in radiation and matter are full coupled. What does this imply for the corresponding perturbation in the entropy density,

$$\frac{\delta\mathcal{S}}{\mathcal{S}} = \frac{3}{4}\,\delta_{rad} - \delta_m. \tag{6}$$

Part IV Perturbation Growth of potential, gravitational acceleration and velocity.

We are going to look into the growth, in tersm of cosmic expansion factor $a(t)$, of the perturbations in the gravitational potential, gravitational acceleration and velocity. In other words, the growth of ϕ, peculiar gravity $\mathbf{g}$ and peculiar velocity $\mathbf{v}$.

a. Given the linear factor of growth of density perturbations, $D(t)$, what is the growth factor D_ϕ of the potential perturbation ϕ ? (You need to infer this quantitatively using the equations for structure growth). **b.** In addition, infer the linear growth factor D_g for gravity perturbations. **c.** Given that we know that the velocity perturbations $\mathbf{v}$ are directl proportional to the peculiar gravity $\mathbf{g}$, via the relation

$$\mathbf{v} \;=\; \frac{2\,f}{3H\Omega}\,\mathbf{g}, \tag{7}$$

with $f(\Omega)$ the Peebles factor,

$$f \;\equiv\; \frac{a}{D}\frac{dD}{da} \;=\; \frac{d\log D}{d\log a} \propto \Omega^{\gamma} \tag{8}$$

show that the linear velocity perturbation growth factor is given by

$$D_v(t) \;=\; aD\,H f(\Omega_m) \tag{9}$$

d. Infer that in an Einstein-de Sitter Universe the peculiar velocity increases as

$$v(t) \propto a^{1/2} \tag{10}$$

e. On the other hand, show that in a low-Omega universe the peculiar velocity decreases in time as

$$v(t) \propto a^{-0.6} \tag{11}$$

To infer this, you should use the asymptotic results that for $a \to \infty$, $\Omega(t) \propto \Omega_0/a$ and $H \propto 1/a$.

f. Finally, show that in a perfectly smooth Universe, ie. one with no potential perturbations and $\phi = 0$, that

$$v(t) \propto 1/a(t)\,. \tag{12}$$

To reach the last result, infer from the Euler equation that for a particle moving in a smooth universe,

$$\frac{da\mathbf{v}}{dt} \;=\; 0. \tag{13}$$