Chapter 3:
The Friedmann-Robertson-Walker-Lemaître Universe

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1. Newtonian Cosmology

Gravity is ruling the Universe. It was Newton who was the first to establish this. Following in the footsteps of Copernicus, Galilei and Kepler, giants upon whose "shoulders he was standing", he formulated his theory of gravity. On the basis of his theory the first true scientific cosmological description and comprehensive model of the world could be formulated. It forms an interesting and illustrative contrast to the 20^{th} century theories based on Einstein's General Theory of Relativity. **Newtonian cosmology** involved:

- Absolute and uniform time
- Space & Time independent of matter
- Space is absolute, static & infinite
- Universe does not have a boundary and no center
- Dynamics:
  action at a distance
  instantaneous
- Newtonian Cosmological Principle:
  Universe looks the same
  at every location in space
  at every moment in time

2. Relativistic Cosmology

In 1915, Albert Einstein had just completed his General Theory of Relativity, which explained gravity in a different way from Newton's law. Gravity was a manifestation of the geometry, curvature, of spacetime and his theory of gravity was a **"metric theory"**. The force of gravity became a *metric force*, resulting from the local curvature of spacetime.

Einstein's General Theory of Relativity revolutionized our thinking about the nature of space and time: no longer Newton's static and rigid background. It had turned into a dynamic medium, intimately couple to the universe's content of matter and energy. It was all encrypted into that impressive framework of the Einstein Field Equations. They form the description of the mutual interaction between the matter-energy content of the Universe and its geometry, responsible for the gravitational interactions acting on all matter and energy,

... Spacetime becomes a dynamic medium
integral part of the structure of the cosmos ...
curved spacetime becomes force of gravity

$$R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R = -\frac{8\pi G}{c^4} T^{\alpha\beta}$$

... its geometry rules the world,
the world rules its geometry ...

One of the very first applications of General Relativity concerned the Universe itself. Only with the availability of the new metric theory which had transformed spacetime into a dynamic medium was it possible to infer a meaningful description of the Cosmos, one that bears any resemblance to reality.
The first attempts towards applying General Relativity to cosmology were made by Einstein himself, in 1917. To this end he made the simplifying assumptions that the Universe is **homogeneous** and **isotropic** (see later, sect: 3). Subsequently, he looked for a static solution of the Einstein field equations. To his surprise, his equations implied the Universe to be unstable, either it should expand or collapse ! This prodded him to introduce the concept of the "cosmological constant" (see chapter 2) in order to balance the books. While rather ad-hoc, we have also seen that it is certainly permitted by the equations. His 1917 Universe model, including cosmological constant Λ, is known by the name **Einstein Universe**. It is well-known that after Hubble's discovery in 1929 of the systematic recession velocities of galaxies, Einstein discarded his introduction of Λ as his "greatest blunder". However, set in proper perspective one should realize at the time Einstein proposed his model our conception of the Universe did not reach further than our own Galaxy. The 1920 "Great Debate" between Curtis and Shapley on the nature of the galaxies was still 3 years away. For sure, observational cosmology in 1917 was far from a mature branch of astronomy ! In around the same year Willem de Sitter, director of Leiden Observatory, assessed another solution of the Einstein field equations, that of an empty Universe. While perhaps slightly "unrealistic" given our own existence, the **de Sitter** Universe is still a theoretically important model. He showed it to be in a *steady state*, having a constant Hubble parameter and a constant *deceleration term q*.

Soon, more systematic cosmological assessments of the Einstein field equations allowed Aleksandr Friedman (Russia) in 1922-1924 and George Lemaítre (1927) to find the general solutions to the Einstein field equations for a homogeneous and isotropic medium. These solutions, the **Friedman-Robertson-Walker-Lemaítre equations** still form the basis of our modern cosmological worldview. For sure, that Friedman and Lemaítre testified of great vision may be appreciated from the fact that Hubble's discovery still was a few years in the waiting.

What then about the names Robertson and Walker ? Recall that Einstein's General Relativity theory is a **metric** theory. One needs to understand the geometry of the Universe to be able to describe its dynamics. In their general form they effectively involve 10 potentials, or independent elements of the Ricci tensor $R_{\alpha\beta}$. In other words, they form a notoriously complex and generally insolvable set of equations. Any hope for progress in cosmology relied upon identifying **symmetries** which might severely curtail its geometry and by implication its dynamics. These **symmetries** are embodied by the **Cosmological**

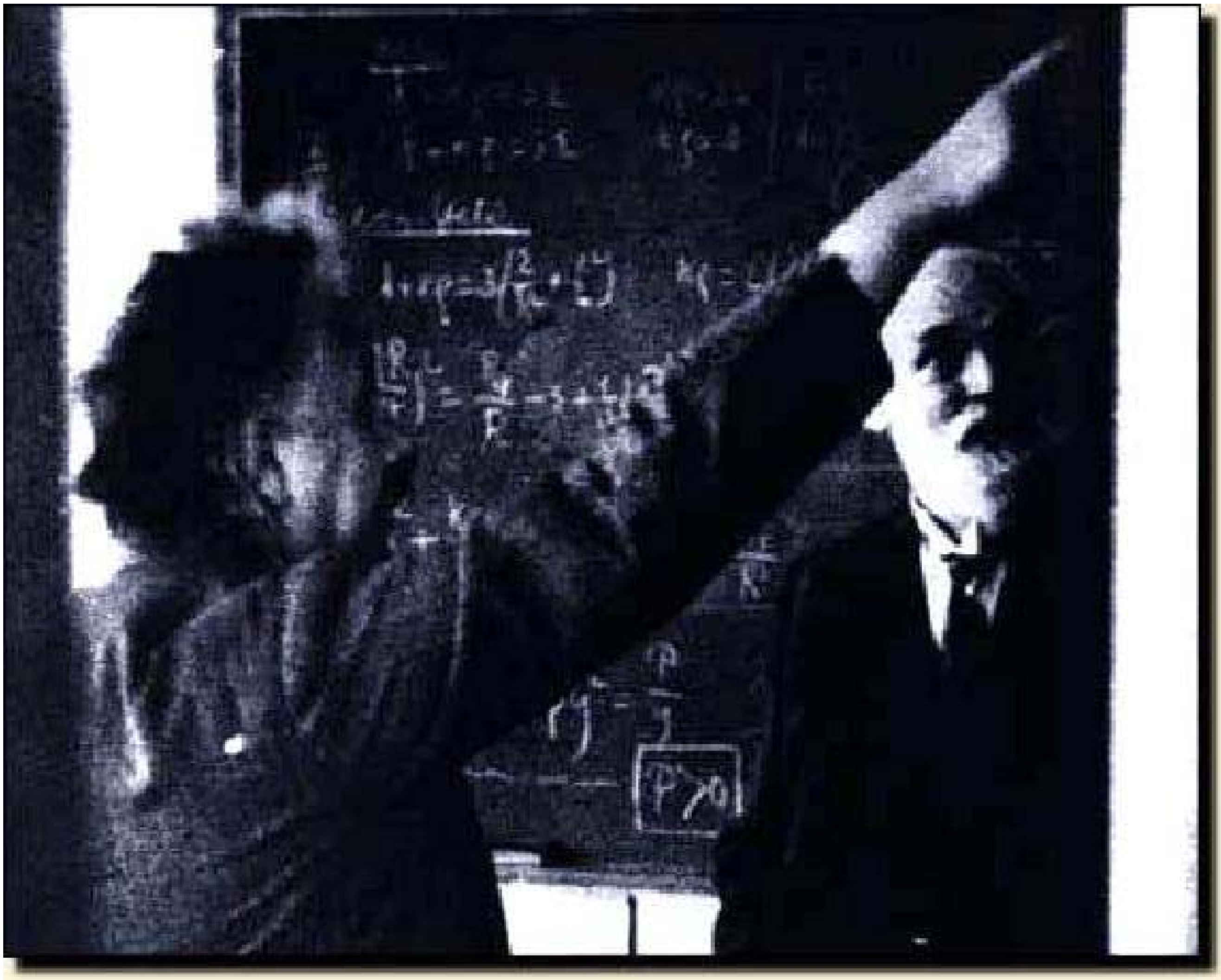


Figure 1. "Claves Lugduni, Claves Caeli": Einstein and de Sitter discussing the Universe in Leiden.

Principle. Their translation into the geometry of the Universe, in terms of a proper metric $g_{\alpha\beta}$, is what has become known as the **Robertston-Walker metric**. It were **Robertson** and **Walker** who around 1935 independently solved the implied form of spacetime. They provided a solid mathematical derivation of the **metric of spacetime** for **all isotropic, homogeneous, uniformly expanding** models of the Universe. Interestingly, their formulation dates from years after the formulation by Friedmann and Lemaítre of their relativistic cosmological models. Nonetheless, the work by Robertson and Walker was essential in putting this on a firm mathematical basis.

Despite the rather erratic historical path along which our modern worldview of the Big Bang Universe got traced out, we will follow the at hindsight more systematic path of reasoning.

3. the Cosmological Principle

> "all places in the Universe are alike"

In 1935 Einstein aptly characterized the structure of our Universe in these words. We have the privilige (and luck) to live in rather simple Universe. A Universe beautiful because of its simplicity. And beauty and simplicity here stand for highly symmetric. Naturally, this is for someone indifferent to the richness and variety of objects and structures in the Universe. This ultimately will form our main source of attention and interest. However, once we smooth out over all (infra)structure on scales of a few hundred Megaparsecs and less, the global Universe indeed has all appearance of being a highly symmetric entity.

Humanity has always considered the simplicity of the World and the Universe as a great source of beauty. This is underlined by statements of two reputable philosophers from antiquity,

> *"... God is an infinite sphere whose centre is everywhere and its circumference nowhere"*
>
> *Empedocles, 5^{th} century BC*

> *"Whatever spot anyone may occupy, the universe stretches away from him just the same in all directions without limit"*
>
> *Lucretius, "De Rerum Natura", 1^{st} century BC*

On Gigaparsec scales the appearance of the Universe is encrypted by the **Cosmological Principle**. In essence it states that we do not occupy any priviliged position in the Universe. There are a few versions of the cosmological principle. Here we distinguish between the *essential cosmological principle* and the *extended cosmological principle*. The top two items in the following table are the essential points of the *Cosmological Principle*. They are essential for sensitbly constraining the geometry of the Universe. The two last items are often implicitly assumed, but in fact should be stated explicitly. The **Homogeneity** of the Universe states that the physical conditions are the same everywhere in the visible Universe. Conditions like the temperature T and density ρ are the same throughout the Universe. In other words, the Universe looks the same at every point in space. **Isotropy** reassures that there is no preferential direction in the Universe, in every direction it looks the same. Note that **homogeneity** and **isotropy** are complementary concepts. A homogeneous medium is not necessarily isotropic nor is the impression of isotropy at one location evidence for the overall homogeneity of the Universe !!!

The *extended cosmological principle* includes two additional issues. The **uniform Hubble expansion** is in a sense included in the first two items. If the Hubble expansion were not uniform, it would be in conflict with the statement that the Universe's conditions would be everywhere and in each direction the same. Still it may be seen as an extra steement to assure that if at any one cosmic time *homogeneity* and *isotropy* holds up, it will remain like that. The assumption of **universality** may seem implicit, but has been the subject of some highly interesting recent studies. It is clear that if the physical laws are not the same throughout the Universe, or throughout cosmic history, we have a problem in formulating a sensible cosmological theory. How can one ever be sure of what happens elsewhere if the same physics is not valid. Even while physical laws may have the same form, the constants characterizing them and the interactions between matter and energy in the Universe might in fact vary. While at first one may think

Cosmological Principle

• **Homogeneity**	$\Longrightarrow$	The Universe is the *same* everywhere, *at every location* - physical quantities (ρ, T, ...)
• **Isotropy**	$\Longrightarrow$	The Universe the *same* *in every direction*
• **Uniformly Expanding**	$\Longrightarrow$	Universe *grows with same rate* - at every location - in every direction
• **Universality**	$\Longrightarrow$	*Physical Laws & Constants the same everywhere*

of this as pure imagination, recent work on high-resolution absorption spectra of quasar light may indeed suggest a systematic trend of the fine structure constant $\alpha = e/hc$. Because this would affect atomic physics, atomic line transitions get to change as a function of cosmic time. While it is not unfeasible this may turn out to be due to systematics of the measurements, or interpretation, the fact that indeed there might be more between "heaven and earth" than meets the eye is intriguing. Nonetheless, as it would complicate our analysis substantially, and the effect if existent at all would be rather small, we keep to the essential tenet that *physical laws* and *physical constants* are indeed universally valid.

4. Cosmic Time: Weyl's Postulate

In the former section we addressed the issue of space in the cosmos. Equally pressing is the issue of time in the Universe.

It is clear that to Newton the issue of a proper definition of a "cosmic time" did not pose any problem. By default there was a rigid, well-defined absolute time throughout the whole of the Universe. In principle even existing independently of the world itself.

For General Relativity the story is quite different. In principle one is free to describe physical processes in any inertial or free-falling frame, there is no sense of absolute space and absolute time. This causes a complication with respect to our attempt to describe the structure and dynamics of the Universe. What do we mean with measuring distances between two objects in the Universe. What is the meaning of '*simultaneity* ? In principle there is no fundamental answer to this question, simultaneity being entirely dependent on the choice of reference frame.

To proceed further in a meaningful fashion, we therefore follow **Weyl** in defining a **Universal Time** as follows. Interestingly, **Weyl's Postulate** (1923) upon which it is based was introduced by Weyl in 1923, quite well before Hubble's discovery of the recession of the nebulae:

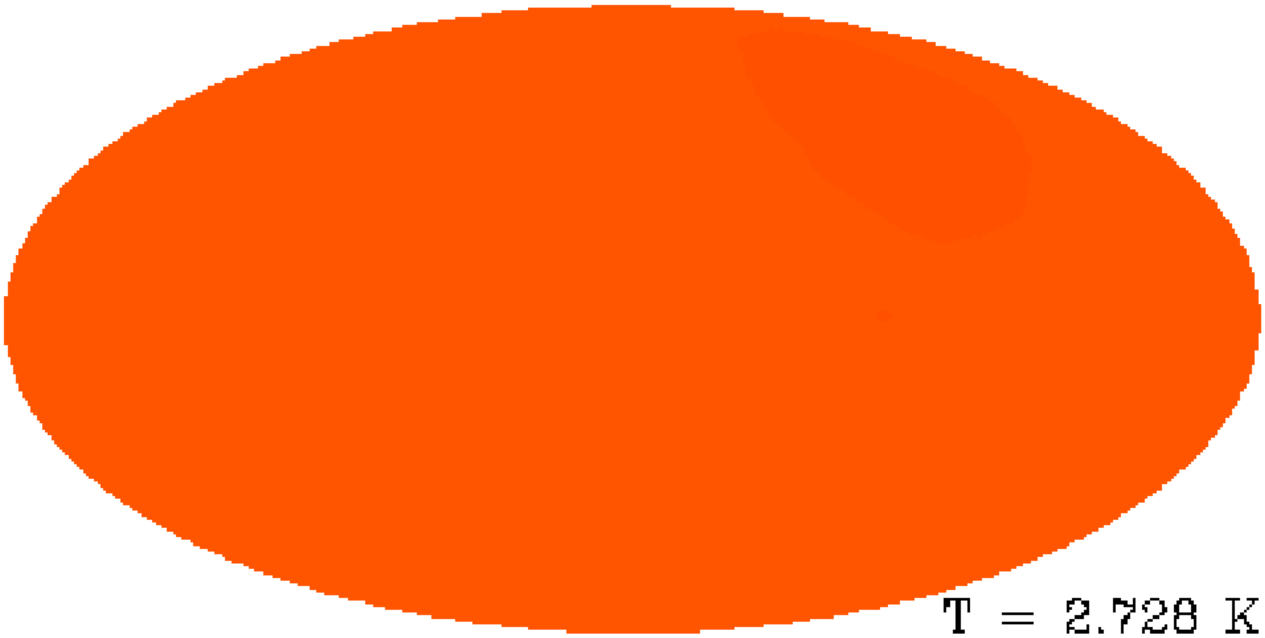
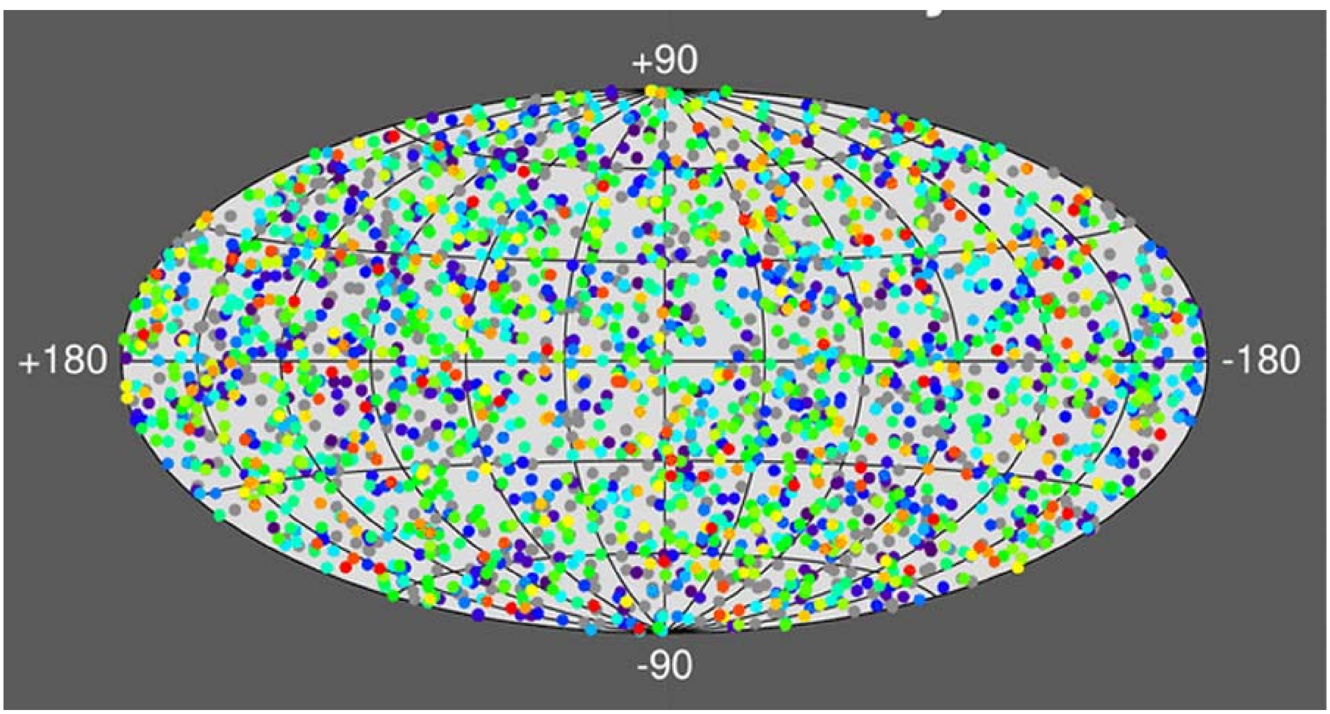


Figure 2. Isotropy of the Universe. Top: The CMB temperature sky distribution, T=2.725 K (COBE); Bottom: Sky distribution 2704 BATSE GRB sources.



> "The particles of the substratum (representing the nebulae) lie in spacetime on a bundle of geodesics diverging from a point in the (finite or infinite) past."

Perhaps the word "substratum" needs some additional explanation. It is the underlying "fluid" defining the overall kinematics of a system of galaxies. In this view, the galaxies act as the occasional "discrete" beacons, flowing along their geodesics (i.e., they are "freely falling"). An immediate repercussion of Weyl's postulate is that the worldlines of galaxies do not intersect, except at a*singular point in the finite/infinite past*. Moreover, only one geodesic is passing through each point in spacetime, except at the origin. This allows one to define the concept of **Fundamental Observer**, one for each worldline. Each of these is carrying a *standard clock*, for which they can synchronize and fix a **Cosmic Time** by agreeing on the initial time $t = t_0$ to couple a time t to some density value. This guarantees a homogenous Universe at each instant of cosmic **Universal Time**, and fixes its definition.

While for homogeneous Universe it is indeed feasible to use Weyl's postulate to define a universal time, this is no longer a trivial exercise for a Universe with inhomogeneities. The worldlines will no longer only diverge, as structures contract and collapse worldlines may cross. Also, if we were to tie a *cosmic time* to a particular density value we would end up with reference frames that would occur rather contrived to us. Also, we would end up with the problem of how to define a density perturbation. We would have a freedom of choice for the reference frame with respect to which we would define it. As usually stated, the density perturbation is dependent upon the chosen gauge, i.e. the chosen metric. This issue came prominently to the fore when Lifschitz tried to solve the perturbed Einstein field equations. The solution was a proper gauge choice, which has become known as 'synchronous gauge". In essence, it involves a choice for the time and spatial coordinates based upon a homogeneous background Universe.

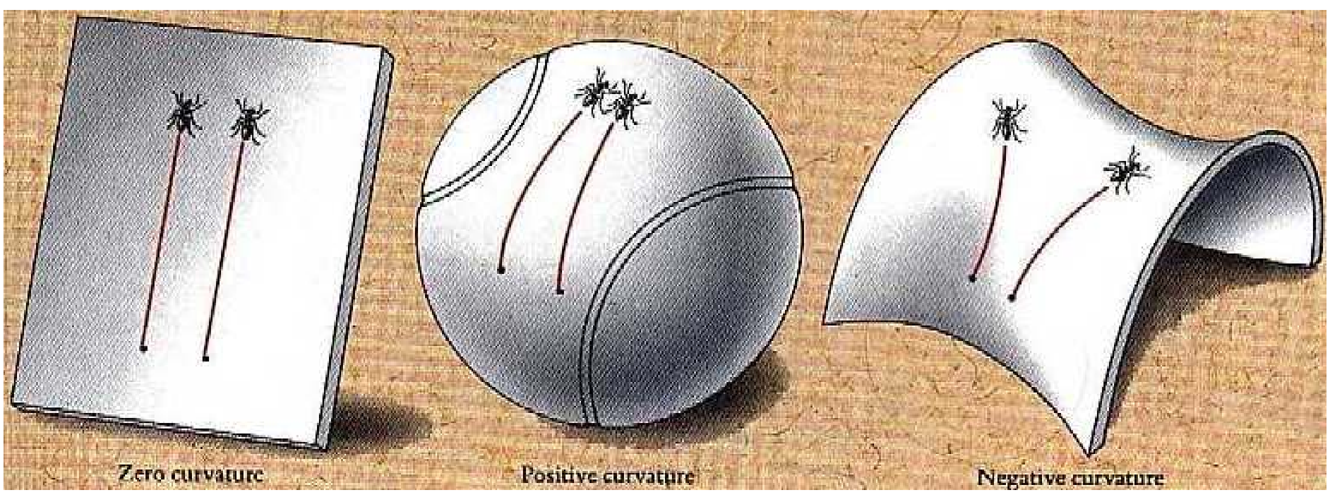


Figure 3. The three homogeneous and isotropic spaces: flat, spherical and hyperbolic.

Homogeneous and Isotropic Spaces

The constraints imposed by the cosmological principle on the geometry of the Universe are strong. Amongst an infinity of academically possible geometries, only **three uniform geometries**, geometries that are indeed fully *homogenous* and *isotropic*.

The Three Uniform Spaces

• **Uniform Hyperbolic Space**	Negatively Curved	Gauß
• **Euclidian Flat Space**	Zero curvature	Euclides
• **Uniform Spherical Space**	Positively Curved	Riemann Lobachevski Bolyai

Known since millennia is the *flat geometry*, commonly known by the name of *Euclides*, the scholar from Alexandria who in the first century A.D. in his book *Elements* (the most published book in the world after the Bible !), inventorised the full body of geometrical knowledge attained since Thales. Five basic axioms form the foundation of *Euclidian Geometry*, and for nearly 18 centuries they were considered to be absolutely fundamental. Still even Kant thought that the Euclidian axioms were "a priori" true and *"an inevitable necessity of thought"*. One of the five, the **Euclidian Parallel Postulate**, turned out **NOT** to be universally true:

> "Two straight lines drawn in some plane are
> and remain parallel if they do not intersect"
>
> Euclides, "The Elements"

Finally, the "parallel postulate" turned out to be a less universal principle than what had been presumed for centuries to be perfectly trivial and plausible. While true for flat spaces, with the discovery of generally curved spaces dawned the realization that parallel lines may at some point intersect. In fact, it is the norm, and Euclidian space is the one exception where it holds true.

With the gradual development of the description of curved spaces, the subject of *differential geometry*, came the *"discovery"* that in addition to *Euclidian flat space* there were two other *uniform spaces*. One is a negatively curved space with a uniform *Hyperbolic Geometry*, discovered by Gauß, Lobachevski and Bolyai. An equivalent positively curved space with a uniform *Spherical Geometry*. The latter was discovered by Riemann, the mathematician who lay the foundations for differential geometry, also called Riemannian geometry, and thus paved the way for the formulation of the General Theory of Relativity by Einstein. Most of this material was presented in one of the most classical lectures in the history of mathematics, "Über die Hypothesen welche der Geometrie zu Grunde liegen", delivered on 10 June 1854 at the University of Göttingen (in the presence of Gauß). Of crucial important is the ability to distinguish on the basis of **local characteristics** between the three different uniform spaces. Such characteristics al low us the ability to decide in which space we ourselves live. Four local properties in which the different are clearly different are **1)** the behaviour of parallel lines, **2)** the interior angles (of e.g. triangles), **3)** the ratio between circumference S of a circle and its radius r and **4)** the curvature k. In table 4 we list the values for each of these characteristics for the three spaces, along with an extra post on their **5)** extent and **6)** boundary.

	Parallel Lines	Triangular Angles	Circumference Circle	Curvature	Extent	Boundary
		$\alpha + \beta + \gamma$	$x \equiv \frac{S}{2r}$	k		
Flat Space	parallels: 1 never intersects	π	π	0	open: infinite	unbounded
Spherical Space	parallels: ∞ along great circles, all intersect	$> \pi$	$< \pi$	$1/R^2$ > 0	closed: finite	unbounded
Hyperbolic Space	parallels: ∞ diverge & never intersect	$< \pi$	$> \pi$	$-1/R^2$ < 0	open: infinite	unbounded

Table 1. Characteristics Uniform Geometries

Robertson-Walker metric

Formally, the implications of the **Cosmological Principle** should be worked out in terms of a metric $g_{\alpha\beta}$. These metric solutions for homogeneous, isotropic spaces are named after the persons who worked out their explicit form, the **Robertson-Walker metric**. They were the mathematicians who proved towards the later thirties that

For a coordinate system with cosmic time t and spatial coordinates (r, θ, ϕ), in which r is the comoving radial distance of a body, we find that distances ds^2 are specified by

$$ds^2 \,=\, c^2\,dt^2 \,-\, R_c(t)^2(t)\,\left\{ d\left(\frac{r}{R_c}\right)^2 \,+\, S_k^2\left(\frac{r}{R_c}\right)\,\left(d\theta^2 \,+ \sin^2\theta\,d\phi^2\right) \right\} \tag{1}$$

in which $R_c(t)$ is the **radius of curvature** at time t and, by convention, R_c the radius of the present universe.

The function $S_k(x)$ depends on which geometry our Universe has, specified through the index k. The latter is the normalized curvature constant, which can attain only three values

- $k \,=\, \begin{cases} 1 & \text{spherical} \\ 0 & \text{flat} \\ -1 & \text{hyperbolic} \end{cases}$

Dependent on whether we live in a positively curved, flat or negatively curved space, S_k has the following expressions:

$$\begin{array}{rrcl} k=1 & S_1(x) & = & \sin(x) \\ k=0 & S_0(x) & = & x \\ k=-1 & S_{-1}(x) & = & \sinh(x) \end{array}$$

(2)

A useful quantity is the dimensionless expansion factor $a(t)$. It specifies the growth of the Universe in time, and is defined such that at the present epoch $a_0 = 1$, while at the moment of the Big Bang $a = 0$. In other words, it specifies how the distance $s(t)$ between two *fundamental observers*, having a present-day distance s_0, changes with time:

$$s(t) \,=\, a(t)\,s_0 \tag{3}$$

One can easily show that for a uniformly expanding universe the radius of curvature $R_c(t)$ also evolves along with the expansion factor $a(t)$:

$$R_c(t) \,=\, a(t)\,R_c\,. \tag{4}$$

A convenient alternative expression for the Robertson-Walker metric is therefore given by

$$ds^2 \,=\, c^2\,dt^2 \,-\, a(t)^2(t)\,\left\{ dr^2 \,+\, R_c^2\,S_k^2\left(\frac{r}{R_c}\right)\,\left(d\theta^2 \,+ \sin^2\theta\,d\phi^2\right) \right\} \tag{5}$$

For

$$ds^2 \,=\, c^2\,dt^2 \,-\, R^2(t)\,\left\{ \frac{dr^2}{1-kr^2} \,+\, r^2\,\left(d\theta^2 \,+ \sin^2\theta\,d\phi^2\right) \right\} \tag{6}$$

in which (r, θ, ϕ) are the comoving coordinates of a body, while the curvature of space is specified by the radius of curvature:

$$R(t) :\ \text{radius of curvature}\,. \tag{7}$$

Note that the simple homogeneous and isotropic space-times we are discussing can be characterized by only one *radius of curvature*, instead of by the 10 components of the general *Ricci tensor*. The curvature is usually specified via a (normalized) curvature constant k which can only attain three different values,

Friedmann-Robertson-Walker-Lemaıtre Equations

$$\ddot{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) a + \frac{\Lambda}{3} a$$

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - \frac{kc^2}{R_0^2} + \frac{\Lambda}{3} a^2$$

At the lecture we have derived the evolution of the Hubble parameter in a generic Universe, filled with radiation, matter, dark energy and with a curvature k term,

$$H^2(a) = H_0^2 \left\{ \frac{\Omega_{rad,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{vac,0} + \frac{(1-\Omega_0)}{a^2} \right\} \tag{8}$$

in which the total Ω_0 is the sum of the contributions by radiation, matter and dark energy,

$$\Omega_0 = \Omega_{rad,0} + \Omega_{m,0} + \Omega_{vac,0} \,. \tag{9}$$

From the above relation we can find directly the solution for the general evolution of a FRW Universe,

$$H_0 t = \int_0^a \frac{da}{\sqrt{\Omega_{rad,0}a^{-2} + \Omega_{m,0}a^{-1} + \Omega_{vac,0}a^2 + (1-\Omega_0)}} \,. \tag{10}$$

5. Observables FRW Universes

The **angular diameter distance** of a source at redshift z is given by

$$d_A = \frac{2c}{H_0\Omega_0^2}\frac{1}{(1+z)^2}\left\{\Omega_0 z + (\Omega_0 - 2)\left(\sqrt{1+\Omega_0 z} - 1\right)\right\}$$

and you are advised to show that for $z \gg 1$ you may approximate this by

$$d_A = \frac{2c}{H_0\Omega_0}\frac{1}{z} \,.$$

6. Cosmic Horizons

The (particle) horizon of our Universe $R_H(t)$ at a cosmic time t is defined by the expression

$$R_H(t) = a(t) \int_0^t \frac{c\, dt'}{a(t')}$$