Gaussian Random Fields

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1 Random Fields

A N dimensional random field is a set of random variables $Y(\mathbf{x})$, $\mathbf{x} \in \Re^N$, which has a collection of distribution functions

$$F(Y(\mathbf{x_1}) \leq y_1, ..., Y(\mathbf{x_n}) \leq y_n) \tag{1}$$

for any number of points n. The convention is that capital letters are random variables (like $Y(\mathbf{x_i})$), where as lower case letters denote a particular value or outcome of the random variable (for example y_i) and the bold faced letters denote spatial coordinates.

1.1 Probability Distributions

Furthermore let us stress that the *(Cumulative) Distribution Function* is the probability of a random variable X taking on the value x or smaller, i.e.;

$$P[X < x] = F(x) \tag{2}$$
$$F(-\infty) = 0 \tag{3}$$
$$F(\infty) = 1. \tag{4}$$

the distribution function F is a nonnegative increasing function. And its derivative is the probability density function.

$$f(y) = \frac{dF(y)}{dy} \tag{5}$$

Note that the names "distribution function" and "probability distribution function" are often confused. Strictly the distribution function is the integrated probability density function. For example for a gaussian random variable, the error function is the distribution function. Whilst the gaussian distribution is the probability density function for a gaussian variable.

1.2 A moment of appreciation

The *Expectation Value* of random variable Y is given by

$$E\{y\} = \int_{-\infty}^{\infty} y f(y) dy. \tag{6}$$

This is the also called the mean and often denoted also as $\mu, \langle y \rangle, \bar{y}$...etc. Higher order moments are computed in a similar way. The second order moment is the spread around the mean (the variance),

$$\sigma^2 = E\{(y-\mu)^2\} = \int_{-\infty}^{\infty} (y-\mu)^2 f(y) dy \tag{7}$$

The general expression for a kth order moment is

$$m_k = \int y^k f(y) dy \tag{8}$$

likewise the central kth order are

$$\mu_k = \int (y-\mu)^k f(y) dy \tag{9}$$

Of the higher order moments the third and fourth have special names, the *skewness* and *kurtosis*. The skewness is a measure of asymetry of the distribution around the mean, while the kurtosis measures the peakedness of the distribution.

2 A Random Field

A random field is homogeneous if all distribution functions (eq. 1) remain the same under a coordinate shift. By implication the moments of the field are not depend on the coordinate $\mathbf{x}$. For example the mean:

$$\mu(\mathbf{x}) = E\{Y(\mathbf{x})\} = m \tag{10}$$

(the capital E denotes the enseble average)

In the same manner one can define a second order moment as

$$C_2(\mathbf{x_1}, \mathbf{x_2}) = E\{Y(\mathbf{x_1})Y^\star(\mathbf{x_2})\} = C_2(r). \tag{11}$$

This function is called the two-point *noncentral* covariance function. More often we deal with the central covariance function, i.e. the moment with respect to the mean;

$$\xi(\mathbf{x_1}, \mathbf{x_2}) = E\{(Y(\mathbf{x_1}) - \mu(\mathbf{x_1}))\ (Y(\mathbf{x_2}) - \mu(\mathbf{x_2}))^\star\} \tag{12}$$

The $\star$ sign denotes complex conjugate, Y could also be a complex number.

How is this related to the expectation value $E(Y)$ (6)? If we were to ignore the spatial part of a random *homogeneous* field, and collect all the values of the field. From this collection we can construct a so called one-point distribution function $f_1(y)$. Then the mean of this distribution is:

$$m = \int y f_1(y) dy \tag{13}$$

This is only true if assume that field is *Ergodic*. The one-point distribution does not include all the information of the field! If we were to randomly reshuffle all points in the field we would still have the same one-point distribution function. But all the spatial correlations of the original field are lost. This data is contained in all the higher N-point distribution functions. They describe the joint distribution at multiple points in space, and therefore contain spatial dependancy .

For example the two-point distribution function the describes the probabilitly of having an outcome y_1 at location $\mathbf{x_1}$ and a value of y_2 at location $\mathbf{x_2}$:

$$f_2(Y(\mathbf{x_1}) = y_1, Y(\mathbf{x_2}) = y_2) \tag{14}$$

This can be generalized to any higher order N-point distribution function. In cosmology the two point distribution function has a very important role, because it is related to the auto-correlation function of the galaxy distribution. In the case of an isotropic process the two point distribution is only a function of distance. The mean of two point distribution is then the two point non-central covariance function:

$$C_2(r) = \int\int y_1^\star y_2\ f_2(y_1, y_2, r)\ dy_1 dy_2 \tag{15}$$

Any higher order covariance function is also defined as the expectation value of a N-point distribution function. The covariance function ξ has some important properties. Here we will treat two, the third property will be discussed in the next section.

The value of the covariance function a zero distance, i.e. $\xi(\mathbf{0})$ can be shown to be equal to the total variance of the field $\xi(0) = \sigma^2$. This can be seen if one takes $r = 0$ in equation 15.

The covariance function determines how strong values at a given distances are correlated. If the covariance tends to goto zero for certain distance. This implies that values seperated by this distance are not aware of each others presence. In the case that the correlation function is zero for every distance. The field values at all locations are completely independent resulting in a very wildly fluctuating field. If there are correlations at larger distances, this would give rise to a much smoother apperance of the field.

2.1 Power Spectrum

In a homogeneous field the Amplitudes of the Fourier components are statistically indepent distributed. These amplitudes are given by the Power Spectrum according to the following relationship

$$E\{\} \tag{16}$$

The power spectrum is related to the covariance function (assuming zero mean)

$$y \tag{17}$$

3 Gaussian Random Fields

The one-point Gaussian probability distribution function (pdf) is perhaps the most fundamental stochastic distribution function we know of. Many natural processes, as well as social processes, tend to have this distribution. For a Gaussian stochastic process Y_G with average y_c and dispersion σ, the probability for $Y_G = [y, y + dy]$ is given by

$$P(y)\,dy = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(y-y_c)^2}{2\sigma^2}\right\} dy\,. \tag{18}$$

to- When the stochastic process concerns an entire region of space we talk about a **Gaussian random field**. In a homogeneous Gaussian random field the one-point Gaussian distribution specifies the probability at any one location within that volume for having a value $Y_c = [y, y + dy]$ is given by the one-point Gaussian distribution given above.

In such a Gaussian random field one may also ask the probability for the field at location $\mathbf{x_1}$ having a value $Y(\mathbf{x_1}) = y_1$, at location $\mathbf{x_2}$ a value $Y(\mathbf{x_2}) = y_2$, and so on for N points, i.e. for $Y(\mathbf{x_N}) = y_N$ at location $\mathbf{x}_N$. Here we will concentrate on the two-point distribution function $P(y_1, y_2)$,

$$P(y_1, y_2)\, dy_1 dy_2 \;=\; \frac{1}{(2\pi)(\det M)^{1/2}}\; \exp\left\{ -\begin{pmatrix} y_1 & y_2 \end{pmatrix} \mathbf{M}^{-1} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} / \, 2 \right\} dy_1 dy_2$$

in which the $\mathbf{M}^{-1}$ is the inverse of the correlation matrix $\mathbf{M}$,

$$\mathbf{M} \;=\; \begin{pmatrix} \xi(0) & \xi(y_1, y_2) \\ \xi(y_2, y_1) & \xi(0) \end{pmatrix} \;=\; \begin{pmatrix} \sigma^2 & \xi(r) \\ \xi(r) & \sigma^2 \end{pmatrix} \tag{19}$$

A general random field (see above) demands the knowledge of all n-point probability distributions. For a Gaussian field these are

$$P(y_1, ..., y_n)\, dy_1..dy_n \;=\; \frac{1}{(2\pi)(\det M)^{1/2}}\; \exp\left\{ -\begin{pmatrix} y_1, & ..., & y_n \end{pmatrix} \mathbf{M}^{-1} \begin{pmatrix} y_1 \\ ... \\ y_n \end{pmatrix} / \, 2 \right\} dy_1, ...dy_n$$

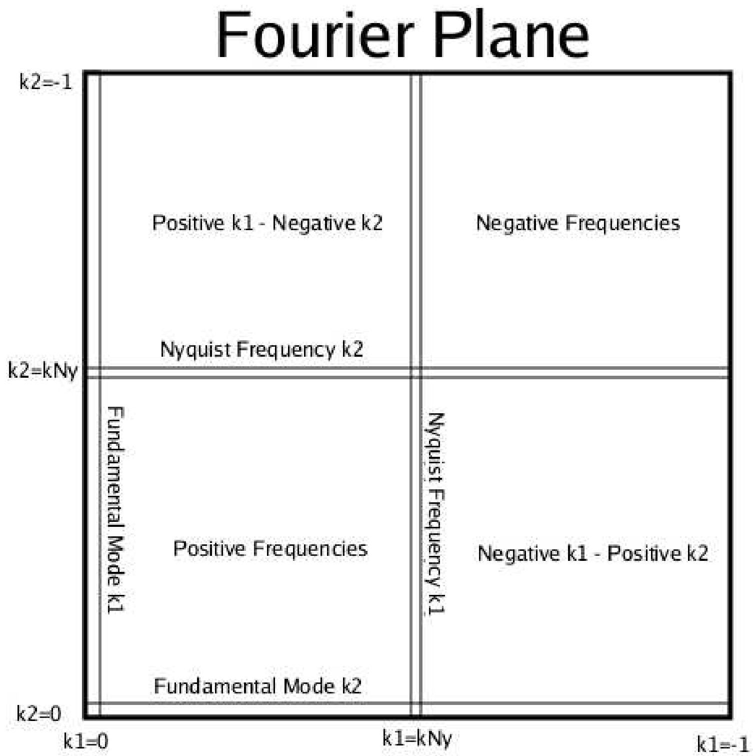
4 Generating Gaussian field in Fourier Space

Generating a Random Gaussian field is easiliest done in Fourier space. Then the complex Fourier amplitudes are $\tilde{Y} = |\tilde{Y}| \exp(i\phi)$. Where ϕ is a random phase and the modules are Rayleigh distributed

$$f(x) = \frac{2x}{\sigma^2} e^{-x^2/\sigma^2} \tag{20}$$

The dispersion is ofcourse related to the Power Spectrum as

$$\sigma^2 = (\delta k)^3 P(k) \tag{21}$$



5 Assignment

- **Fourier Plane** Your first task is to generate a 2D matrix (e.g. an image) with complex entries (the real and imaginary may just be uniform random numbers). This 2d-Matrix should have certain symetries that if one applies the inverse fourier transform its ouput are real numbers. (Hint this implies the following symmetry in Fourier Space $\tilde{Y}^(- k) = \tilde{Y}^\star(k)$!!!!!)
- **Gaussian Field** Generate a Gaussian fields with Power Spectrum; $P(k) \propto k^n$ Present this with contour plots (A filled contour with rainbow colorbar)