Quenched Connectivity Disorder: Spin Models on Random Lattices and Graphs

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Contents

The World a Jigsaw: Tessellations in the Sciences
Lorentz Center, Leiden University, 6 – 10 March 2006

Motivation

Non-perturbative quantum gravity – (at least . . .) two alternative functional integral approaches (analogous to path-integral quantization): $\longrightarrow$ Renate Loll

1. **Discretized Regge calculus:**
   - (dynamically) varying link lengths
   - fixed connectivities of simplicial lattices (regular *or* random)
   - is matter part influenced by (quenched) random lattices?
2. **Dynamical triangulations (DTRS):**
   - fixed link lengths
   - (dynamically) varying graph connectivities
   - how does matter part behave for frozen-in (quenched) connectivities?

Statistical physics point of view: Quenched vs annealed connectivity disorder

Random Graphs and Lattices

Locally varying connectivity of random graphs as special case of quenched (correlated) disorder applied to spin models.

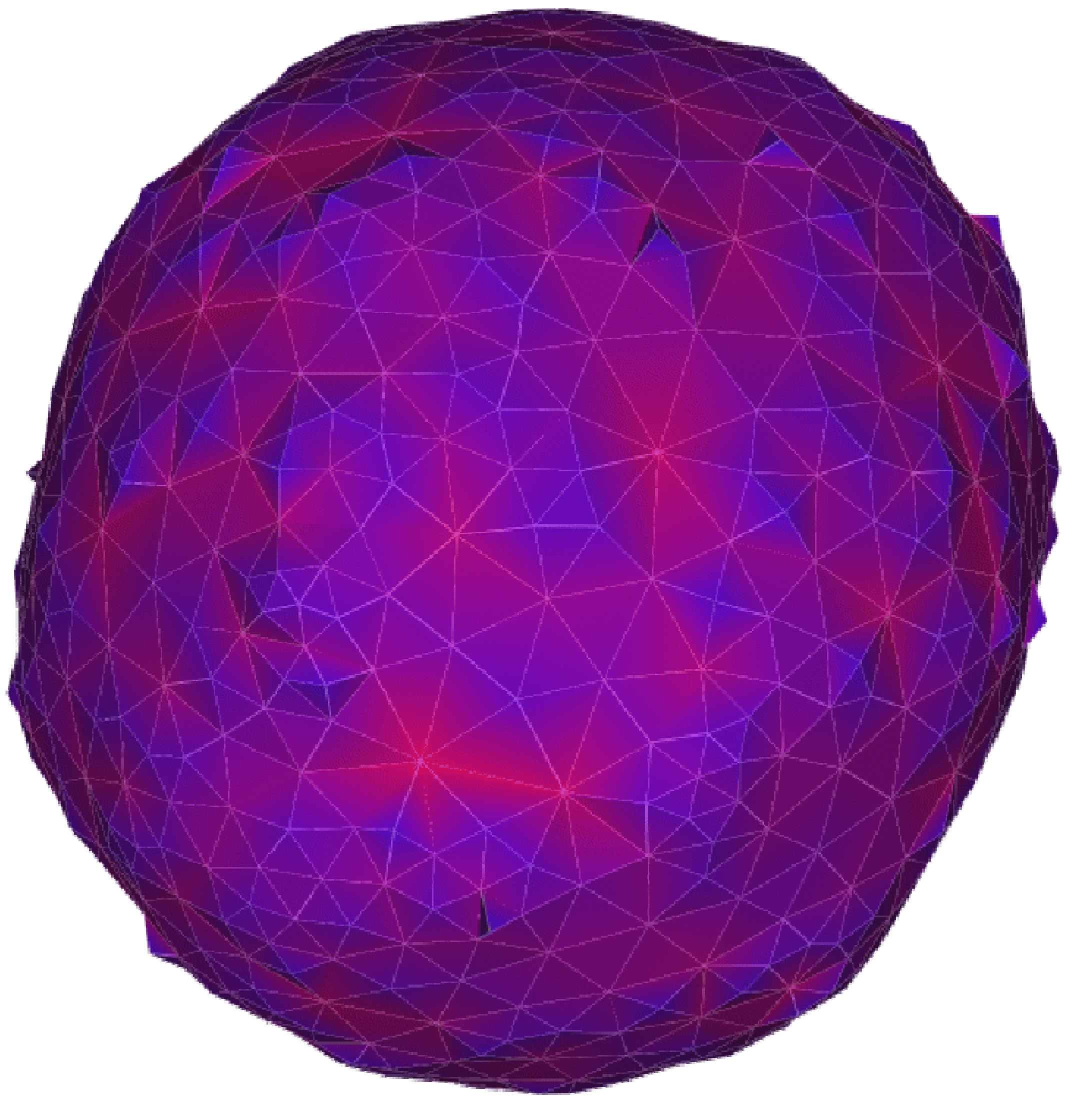
Voronoi-Delaunay triangulations:

- drop points randomly on the plane, construct Wigner-Voronoi cells and the corresponding dual bonds of the Delaunay triangulation
- Hausdorff dimension $d_h = 2$

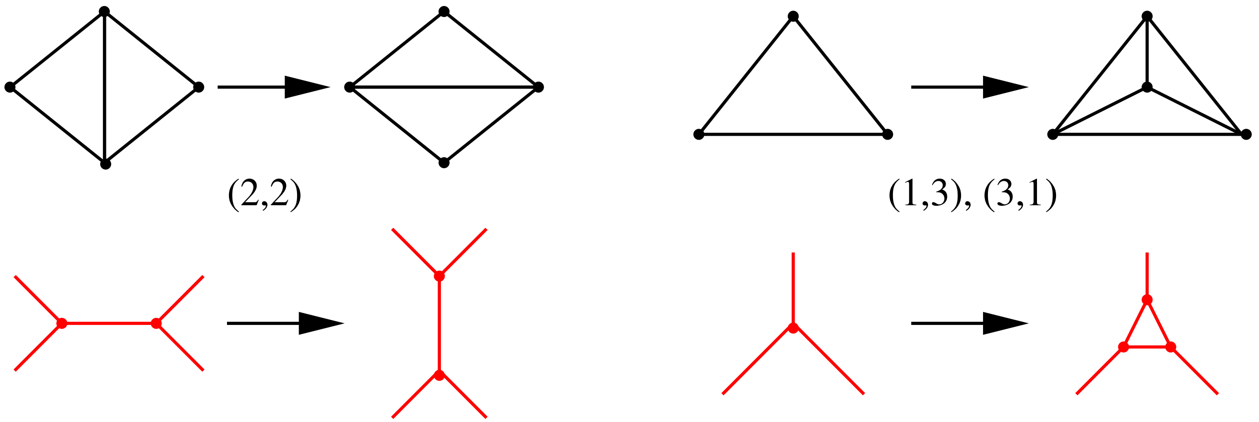
ϕ^3 **quantum gravity graphs:**

- dual graphs to *dynamical triangulations*
- graphs decompose into a tree of *baby universes*, Hausdorff dimension $d_h = 4$

Delaunay Triangulations/Dual Voronoi Graphs

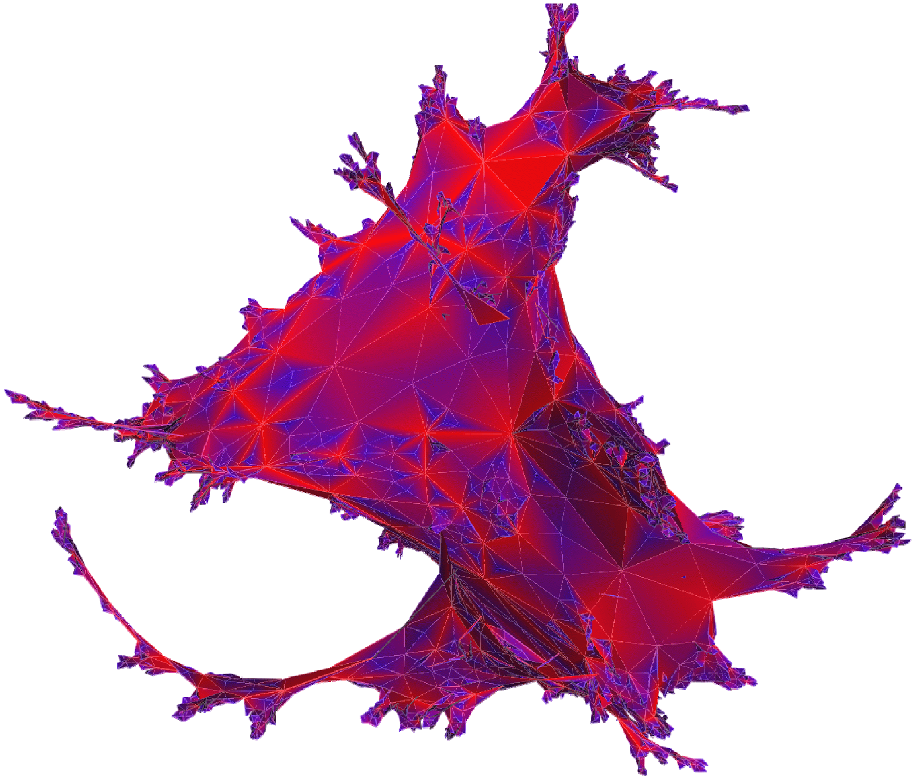


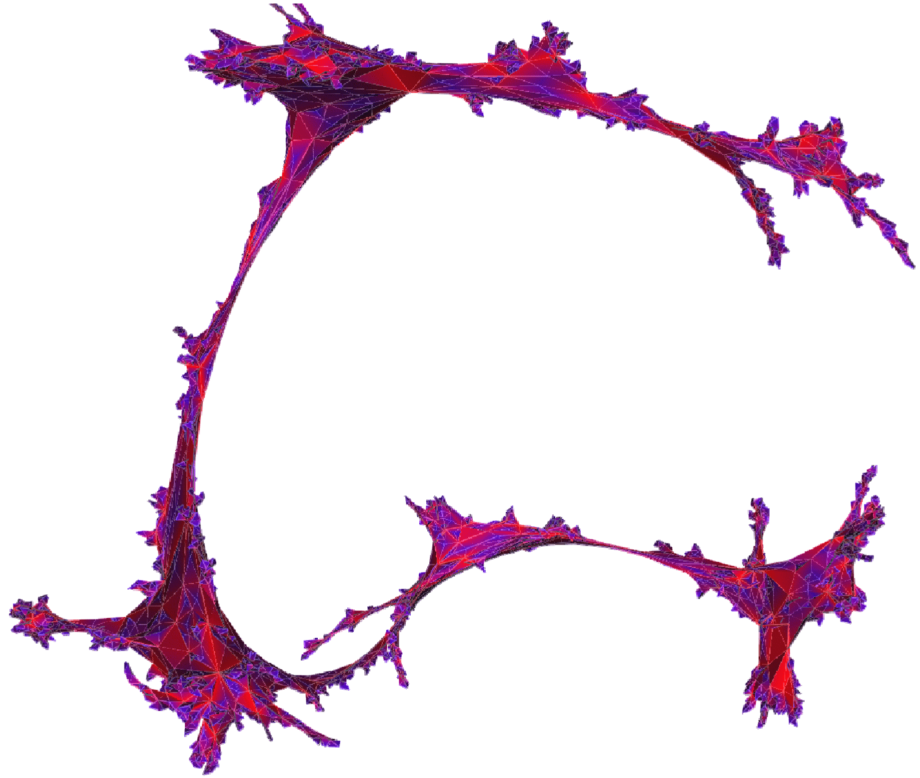
Link-Flip Moves for Dynamical Triangulations



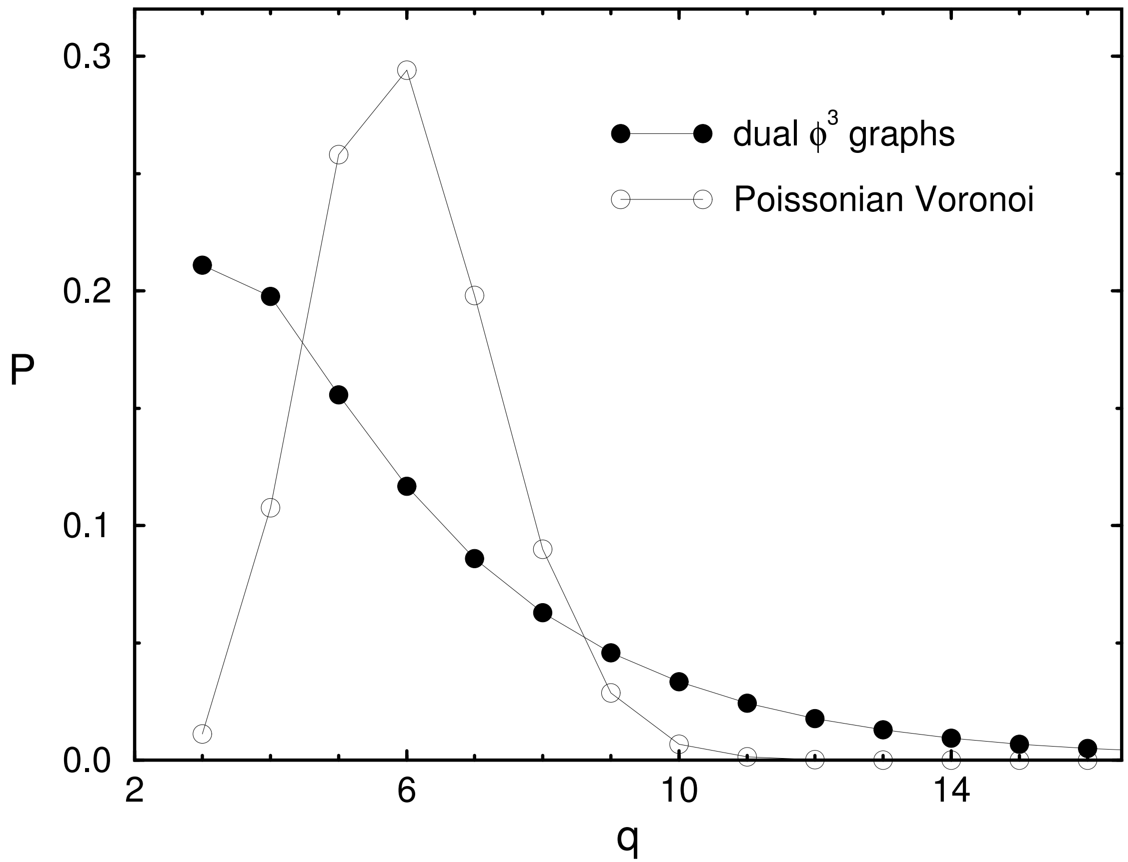
Or using the Tutte algorithm

Dynamical Triangulations/Dual Planar ϕ^3 Graphs





Connectivity Properties



Coordination-number distribution $P(q)$: Monotonic vs peaked; different tail behaviour

Spin Models and Phase Transitions

Ising model of ferromagnetism:

$$Z = \sum_{\{\sigma_i\}} \exp(-H_{\rm Is}/k_B T)$$

with Hamiltonian

$$H_{\rm Is} = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - h \sum_i \sigma_i \ , \qquad \sigma_i = \pm 1$$

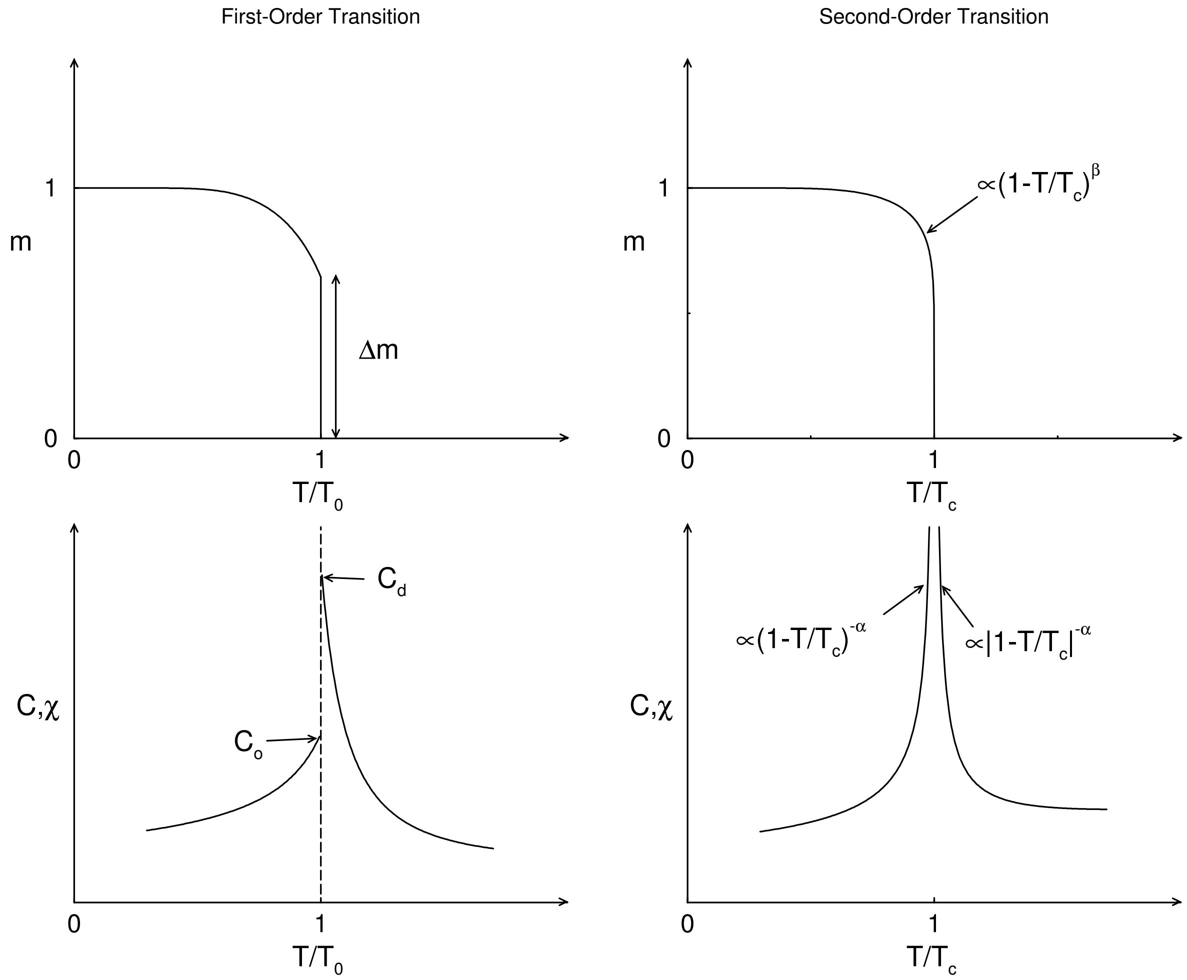
$T =$ temperature, $h =$ external magnetic field, $k_B =$ Boltzmann's constant

Potts models:

$$H_{\rm Potts} = -J \sum_{\langle ij \rangle} \delta_{\sigma_i \sigma_j} \ , \qquad \sigma_i \in 1, \dots, q$$

$\delta_{\sigma_i \sigma_j} =$ Kronecker symbol ($q = 2 \leftrightarrow$ Ising)

Phase Transitions



jumps, finite correlation length   singularities, diverging correlation length

Critical Phenomena

Correlation length:

$$\xi = \xi_0 |1 - T/T_c|^{-\nu} + \ldots$$

Magnetisation:

$$m = m_0 (1 - T/T_c)^{\beta} + \ldots \qquad (T \leq T_c)$$

Susceptibility:

$$\chi = \chi_0 |1 - T/T_c|^{-\gamma} + \ldots$$

Specific heat:

$$C = C_{\rm reg} + C_0 |1 - T/T_c|^{-\alpha} + \ldots$$

ν, β, γ, α: universal critical exponents

Table of critical exponents:

model	ν	α	β	γ
2D Ising	1	0 $(\log)$	1/8	7/4
3D Ising*)	0.630 05(18)	0.109 85	0.326 48	1.237 17(28)
2D $q = 3$ Potts	5/6	1/3	1/9	13/9
2D $q = 4$ Potts	2/3	2/3	1/12	7/6

*) "world average" [M. Weigel, WJ, Phys. Rev. **B62** (2000) 6343]

2D Potts with $q \geq 5$: first-order phase transition of increasing strength (measured by latent heat or interface tension)

Quenched Connectivity Disorder

Static random lattices $\leftrightarrow$ quenched disorder in coordination numbers ???

If YES, one would expect:

1. Pure system (regular lattice) with a 2nd order transition. Uncorrelated quenched disorder is for

$$\alpha \begin{cases} < 0 & \text{irrelevant} \\ = 0 & \text{marginal} \\ > 0 & \text{relevant} \end{cases}$$

   perturbation (Harris criterion), governed for $\alpha > 0$ by a “random” fixed point characterized by a new set of critical exponents.

2. Pure system (regular lattice) exhibits 1st order transition. Uncorrelated quenched disorder induces a

   softening to 2nd order transition

Typical case verifying this scenario: random-bond Ising and Potts models

A Possible Refinement: Disorder Correlations

Disorder

Annealed

Quenched

Uncorrelated

Harris criterion:

$\alpha > 0$

– **random bond model**

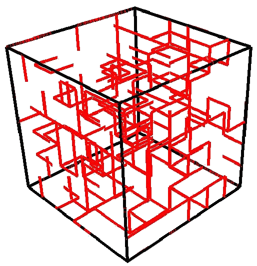
Correlated

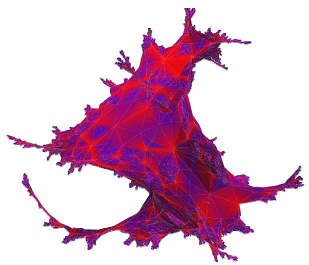
Luck criterion:

$(1-\alpha)/(2-\alpha) < \beta$

– **QG graphs**

– **Voronoi/Delaunay**





The Harris-Luck Criterion

Uncorrelated disorder:

Spin model with weak quenched bond disorder: $J_{i,j} = J_0(1+\epsilon_{i,j})$. The fluctuation of the mean coupling induces a fluctuation of effective critical temperatures:

$$\begin{aligned}\sigma(J) &\equiv (J-J_0)/J_0 \sim \xi^{-d/2} \sim L^{-d/2} \\ \sigma(t) &\equiv (t-t_0)/t_0 \sim t^{d\nu/2}\end{aligned}$$

Disorder is relevant if (Harris, 1974):

$$d\nu/2 < 1 \Leftrightarrow \alpha = 2 - d\nu > 0$$

Generalization for correlated disorder (B "ball" of radius R):

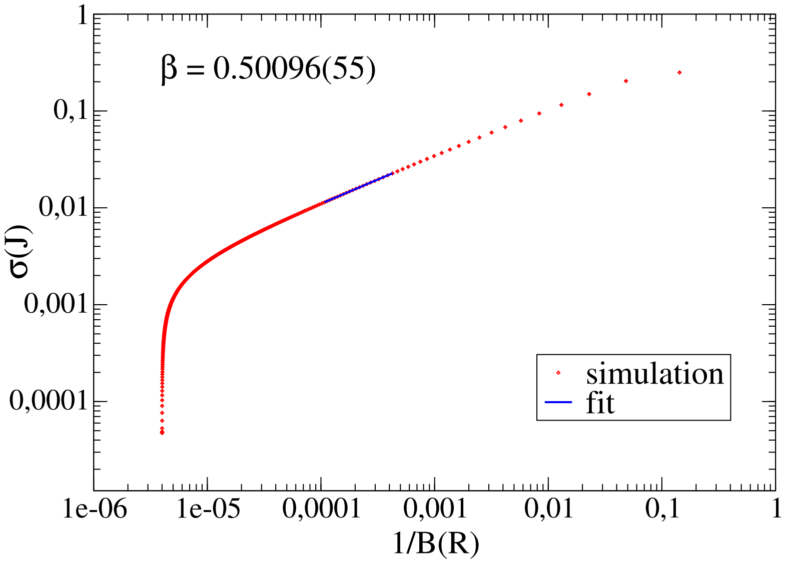
$$\sigma(J) \equiv \frac{J(R)-J_0}{J_0} \sim B(R)^{\beta-1} \sim L^{-d(1-\beta)}$$

with the wandering exponent β. Disorder is relevant if (Luck, 1993)

$$\beta > \beta_c \equiv 1 - 1/d\nu = (1-\alpha)/(2-\alpha)$$

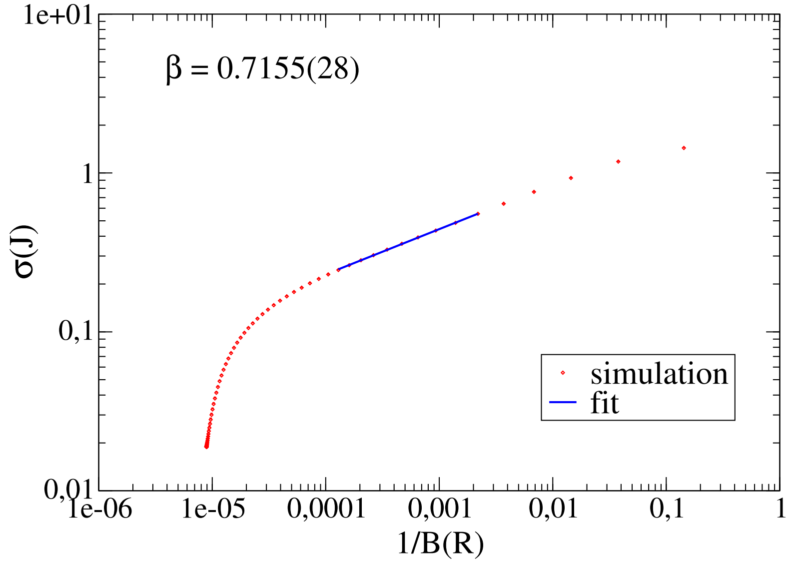
Wandering Exponents β

Delaunay graphs ($500\,000$ sites)



- correlations decay faster than $1/R^2$ (presumably exponentially fast)
- $q = 2$: $\alpha = 0 \Rightarrow$ marginal case
  $q = 3$: $\alpha = \frac{1}{3} \Rightarrow$ should be relevant!

ϕ^3 graphs ($250\,000$ sites)



- strong, algebraically decaying correlations between co-ordination numbers
- If $\alpha > \alpha_c = (1 - 2\beta)/(1 - \beta) \approx -1.5149 \Rightarrow$ should be always relevant

M. Weigel, WJ, Phys. Rev. **B69** (2004) 144208

Potts Models on ϕ^3 Graphs

Annealed gravity graphs:

- exact large-N matrix model solutions
- continuum CFT predictions via KPZ formula (c: central charge)

$$\tilde{\Delta} = \frac{\sqrt{1-c+24\Delta} - \sqrt{1-c}}{\sqrt{25-c} - \sqrt{1-c}}$$

Δ: bare conformal weight, $\tilde{\Delta}$: gravitationally dressed conformal weight

$$C \sim t^{-\alpha}, \qquad m \sim t^{\beta}, \quad t = |1 - T/T_c|$$
$$\alpha = \frac{1-2\Delta_\epsilon}{1-\Delta_\epsilon}, \qquad \beta = \frac{\Delta_\sigma}{1-\Delta_\epsilon}$$

Ising ($c = 1/2$)	Δ_ϵ	Δ_σ	α	β	γ	δ
Onsager	1/2	1/16	0	1/8	7/4	15
KPZ	2/3	1/6	-1	1/2	2	5

Quenched gravity graphs (replica trick):

$$[F]_{\rm av} = -[\ln Z]_{\rm av} = [\lim_{n\to 0}(Z^n - 1)/n]_{\rm av} = \lim_{n\to 0}([Z^n]_{\rm av} - 1)/n$$

with

$$[Z^n]_{\rm av} = \left[\left(\sum_{\{s\}} e^{\sum_{\langle ij\rangle} C_{ij} s_i s_j}\right)^n\right]_{\rm av} = \sum_{\rm geometries}\sum_{\{s^{(1)}\}}\cdots\sum_{\{s^{(n)}\}} e^{\sum_{k=1}^{n}\sum_{\langle ij\rangle} C_{ij} s_i^{(k)} s_j^{(k)}}$$

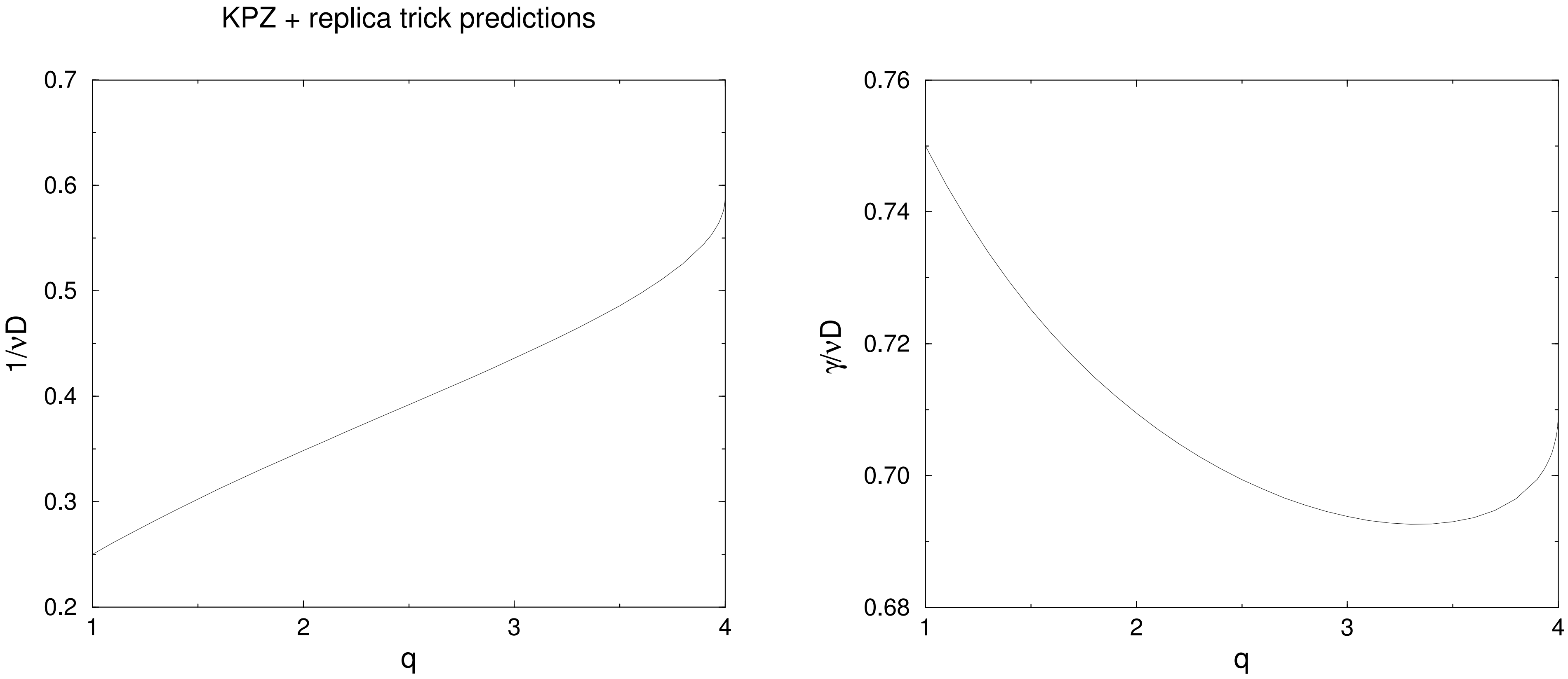
$\Rightarrow$ annealed ensemble of n matter copies with total central charge $c \longrightarrow nc$ in KPZ formula. Replica limit $n \to 0$:

$$\tilde{\Delta} = \frac{\sqrt{1+24\Delta} - 1}{4}$$

Effective "dressing" due to quenched connectivity disorder

D.A. Johnston, WJ, Phys. Lett. **B460** (1999) 271

Theoretical Predictions: q-State Potts Model on Quenched ϕ^3 Graphs



Monte Carlo (MC) Simulations

Example: Finite-size scaling study of the 3-state Potts model on 256 random graph realizations with $N = 500$, 1000, 2000, 5000, and 10000 lattice sites.

Compute, e.g., average susceptibility $[\chi]_{\rm av}$ and perform fits to

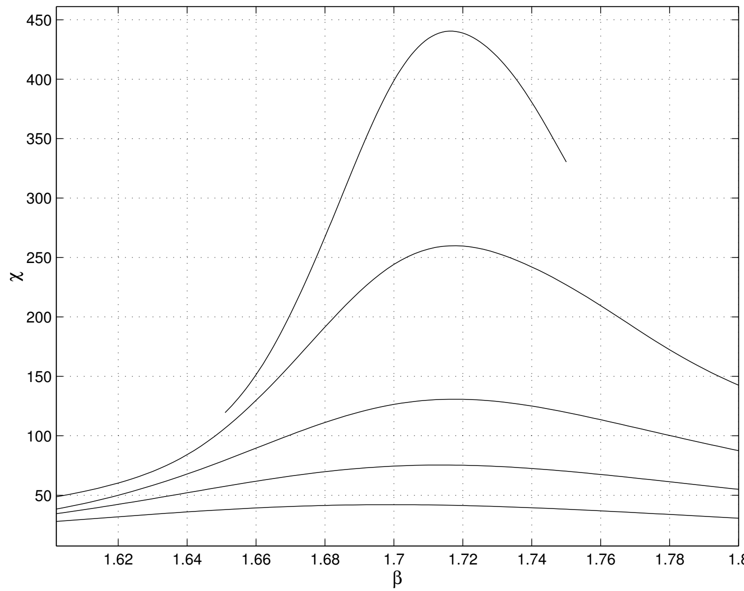
$$[\chi]_{\rm av} = N^{\gamma/\nu d_h} f_\chi(x)[1 + \ldots]$$

with fractal dimension $d_h = 4$ and ($\beta \equiv 1/T$)

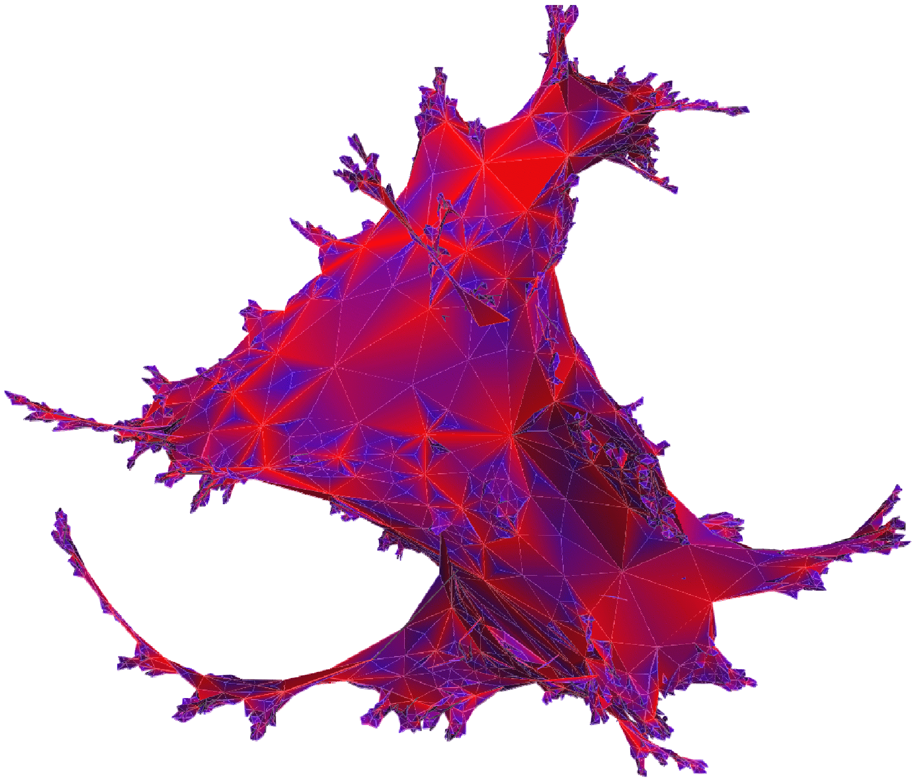
$$x = (\beta - \beta_c)N^{1/\nu d_h}$$

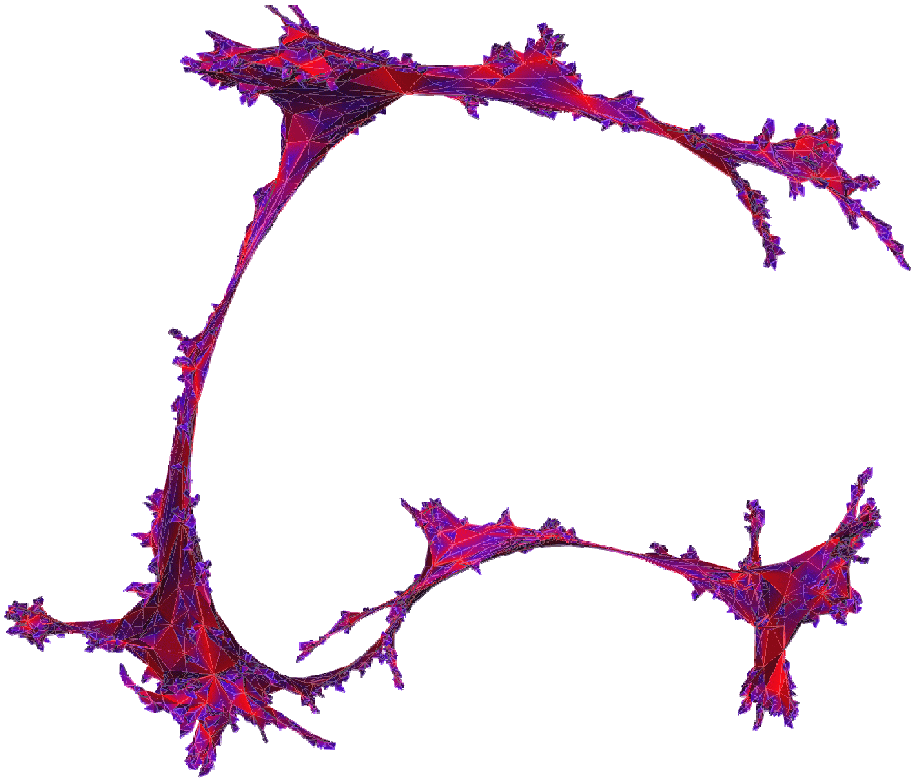
At maxima locations $x = \text{const.}$, i.e.,

$$\beta_{\max}(N) = \beta_c + aN^{-1/\nu d_h}$$



Recall the dynamical triangulations with $d_h = 4$





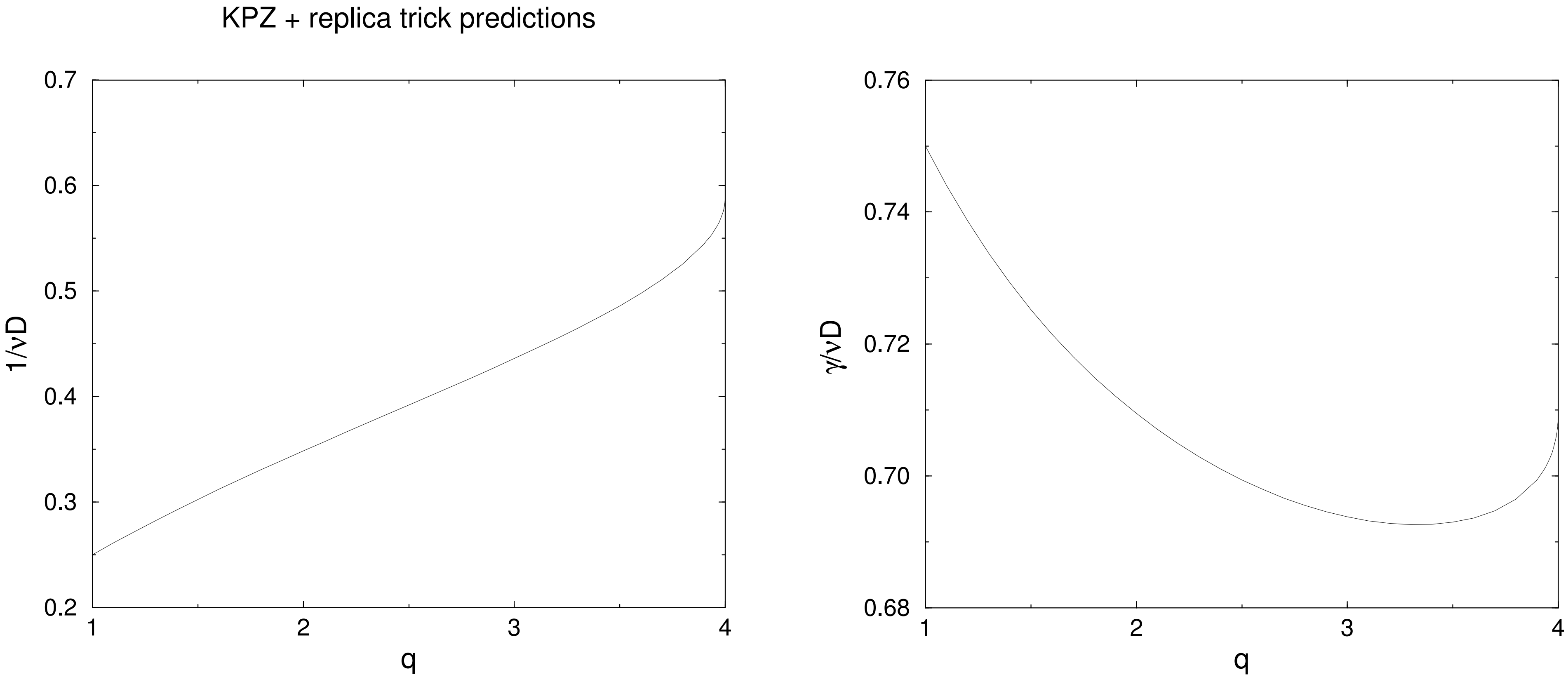
Results for $q = 3$ [A. Wernecke, WJ, in preparation]

	$1/\nu d_h$	$\gamma/\nu d_h$	$(1-\beta)/\nu d_h$
regular	0.6	0.8666. . .	0.5333. . .
replica theory	0.4360. . .	0.6937. . .	0.2829. . .
MC max $[\mathcal{O}]_{\rm av}$	0.4027(25)	0.7395(53)	0.2861(42)
MC $[\max \mathcal{O}]_{\rm av}$	0.407(12)	0.7536(76)	0.3022(82)
MC std	0.439(39)	0.724(78)	0.387(38)

- clear effect of quenched connectivity disorder
- but MC results do not quite fit KPZ + replica trick
- last two lines show results of alternative procedure: Find for each realization the maxima and then average these maxima $\Rightarrow$ distribution of maxima, i.e., also their standard deviation (std) $\Rightarrow$ check of non-self-averaging properties!
- qualitatively similar to $q = 2$ and 4 results [D.A. Johnston, WJ, Nucl. Phys. **B578 [FS]** (2000) 681]

	w/o disorder: 2nd order			1st order
	$q = 2$	$q = 3$	$q = 4$	$q = 10$
$1/\nu d_h$	0.34(3)	0.4027(25)	0.42(2)	0.58(2)
$\gamma/\nu d_h$	0.79(1)	0.7395(53)	0.75(1)	0.71(1)

Theoretical Predictions: q-State Potts Model on Quenched ϕ^3 Graphs



3-State Potts Model on Voronoi Spheres

Previous studies:

No qualitative effects of connectivity disorder seen for:

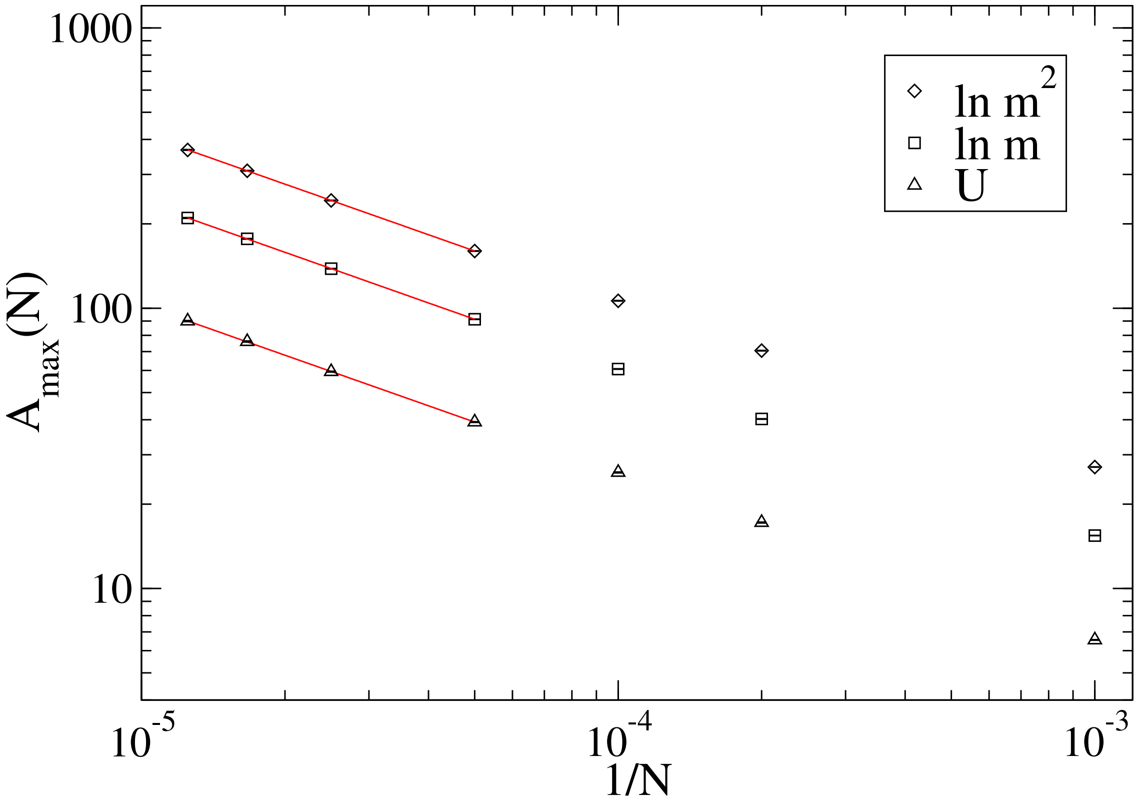
- 2D Ising ($\alpha = 0$, i.e. marginal)
  [M. Katoot, R. Villanova, WJ, Phys. Lett. **B315** (1993) 412; Phys. Rev. **B49** (1994) 9644]
- 2D 8-state Potts (first-order transition)
  [R. Villanova, WJ, Phys. Lett. **A209** (1995) 179]

Recent finite-size scaling study for 2D 3-state Potts:

- spins living on trivalent vertices of Voronoi tessellation
- $N = 1\text{k}, 5\text{k}, 10\text{k}, 20\text{k}, 40\text{k}, 60\text{k}$, and 80k triangles
- 100 realizations per lattice size
- $T = 5 \times 10^4$ independent measurements each
- state-of-the-art histogram scaling analysis

Results

FSS determination of ν ($A_{\max} \propto N^{1/2\nu}$)



Fits yield $\nu = 0.8335(26)$, in perfect agreement with the exact regular lattice value of $\nu = 5/6 = 0.833\bar{3}$ $\Rightarrow$ no influence of quenched connectivity disorder detectable (. . . at least up to size $N = 80\,000$).

Similarly:

$$C_{\max} \propto N^{\alpha/2\nu}, \quad m_{\inf} \propto N^{-\beta/2\nu}, \quad \chi_{\max} \propto N^{\gamma/2\nu}$$

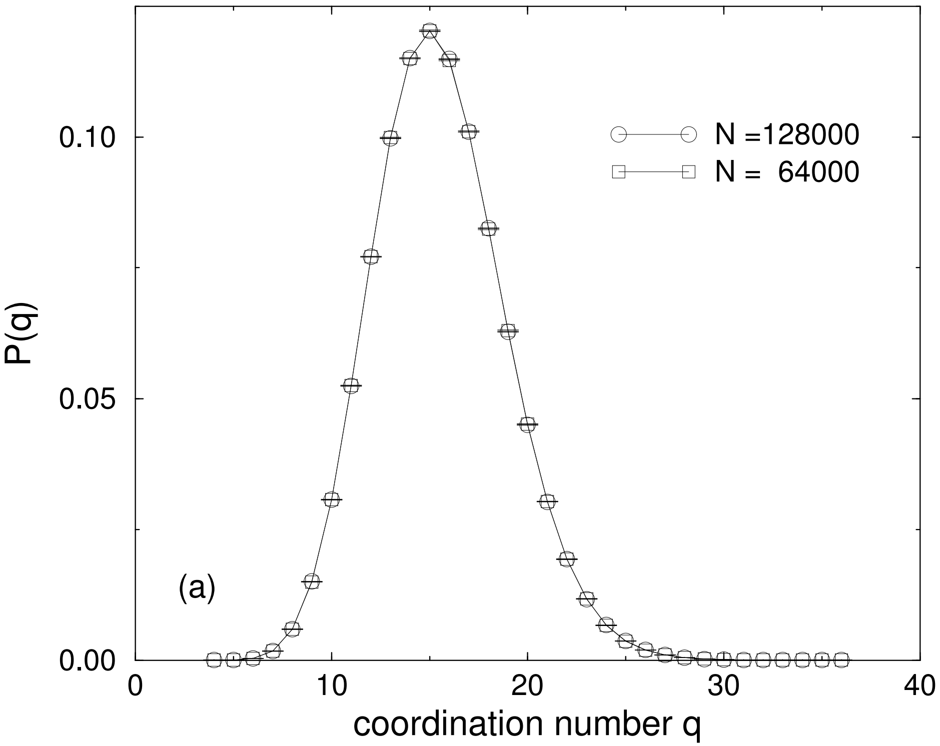
Fits yield again agreement with the exact regular lattice values

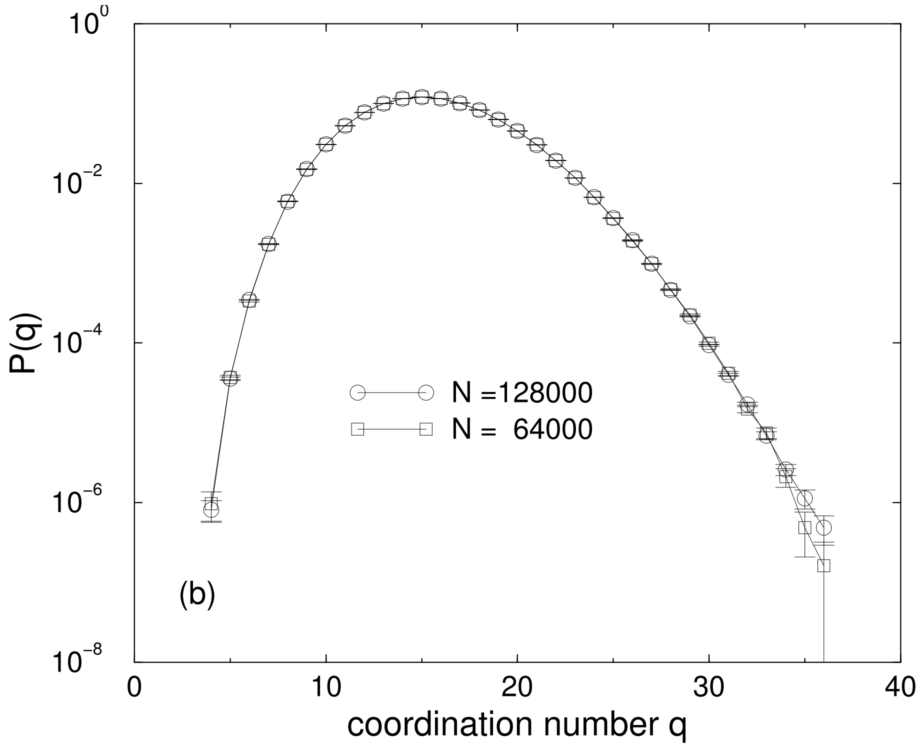
$$\begin{aligned} \alpha/2\nu &= 0.2201(27) \approx 1/5 = 0.2, \\ \beta/2\nu &= 0.0617(14) \approx 1/15 = 0.066\bar{6}, \\ \gamma/2\nu &= 0.8718(12) \approx 13/15 = 0.866\bar{6} \end{aligned}$$

M. Weigel, WJ, Acta Physica Polonica **B34** (2003) 4891; and in preparation

Ising Model on 3D Voronoi/Delaunay Tessellations

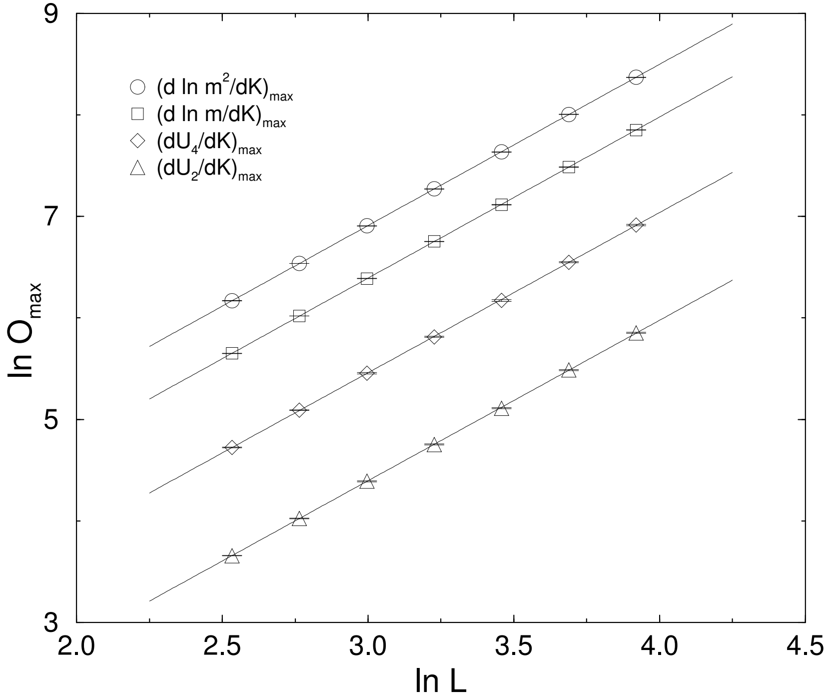
Delaunay random lattices with $N = 2000$ up to $128\,000$ sites, 96 realisations

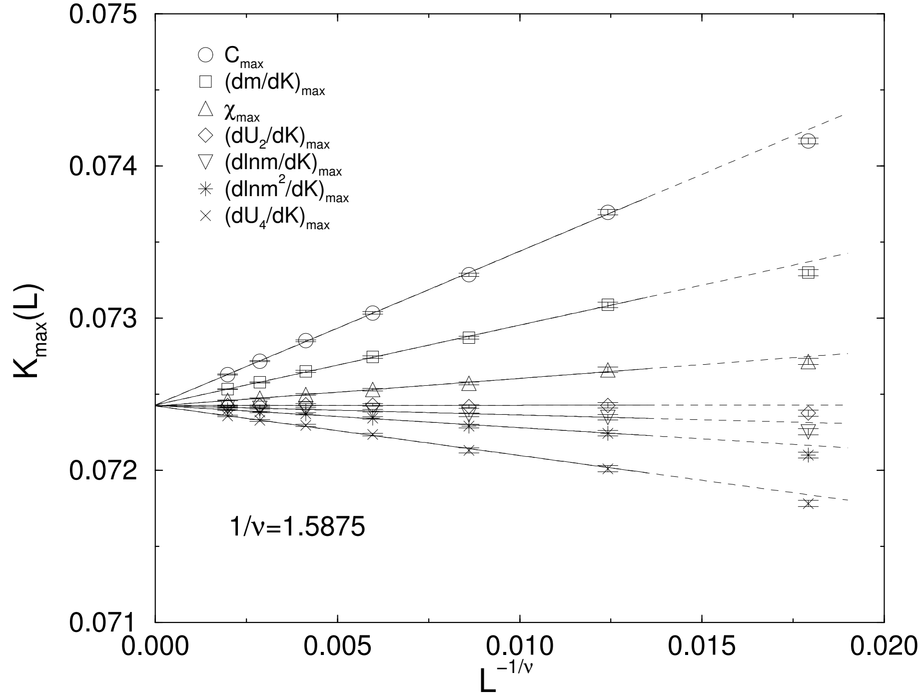




$N = 128\,000$: $\overline{q} = 15.5349(5) \approx 2 + 48\pi^2/35 = 15.5354\ldots$

Critical Exponent ν





$4 \times 7 = 28$ fits (goodness $Q > 0.15$):

$$\begin{aligned} 1/\nu &= 1.5875(12) \\ \nu &= 0.6299(5) \end{aligned}$$

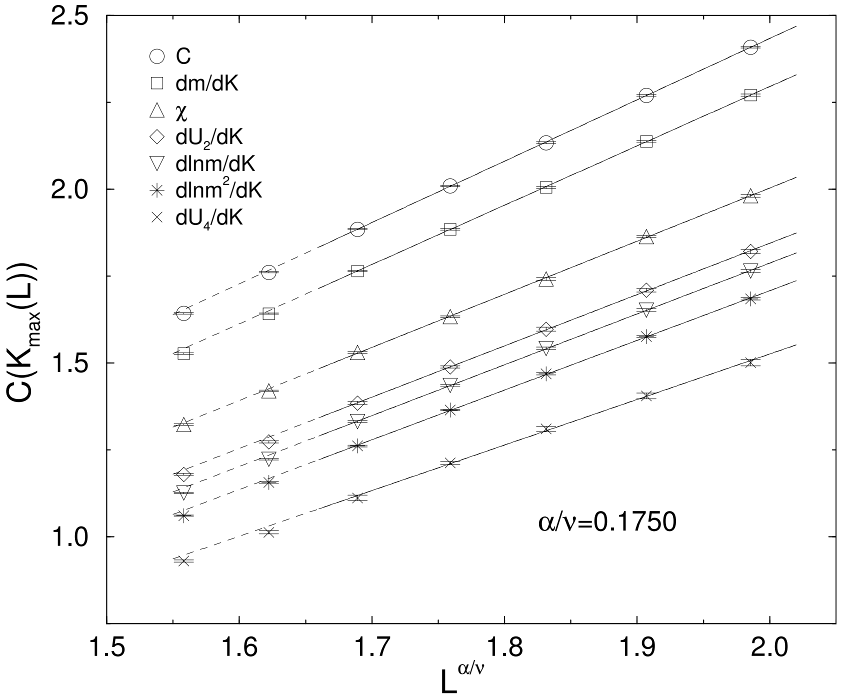
"World average" for regular lattices:

$$\nu = 0.63005(18)$$

Combined fit results:

$$K_c = 0.0724\,249(40)$$

Critical Exponents α/ν and γ/ν



$$C = \text{const.} + aN^{\alpha/3\nu} + \dots$$

Assuming hyperscaling: $\alpha/\nu = 2/\nu - d$

Similar FSS analyses of the susceptibility ($\chi \propto N^{\gamma/3\nu}$) give:

$$\gamma/\nu = 1.9576(13)$$

"World average" for regular lattices:

$$\gamma/\nu = 1.9636(10)$$

Also here, no indication of relevance of quenched connectivity disorder

R. Villanova, WJ, Phys. Rev. **B66** (2002) 134208

Summary and Outlook

- Quantitative analysis of correlations in random graphs and lattices (wandering exponent)
- Quenched connectivity disorder is relevant for planar ϕ^3 gravity graphs but apparently not for Voronoi-Delaunay (Poissonian) random lattices
- Analytical predictions for ϕ^3 graphs based on KPZ formula + replica trick match only approximately

Todo list:

- Further analytical work for ϕ^3 gravity graphs (CFT, matrix models,. . .)
- Generalize Voronoi-Delaunay case to link-length dependent interactions ($\longrightarrow$ Goetz Kähler)
- Study of simpler (and tunable) correlated disorder

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Summary and Outlook