Topological Persistence

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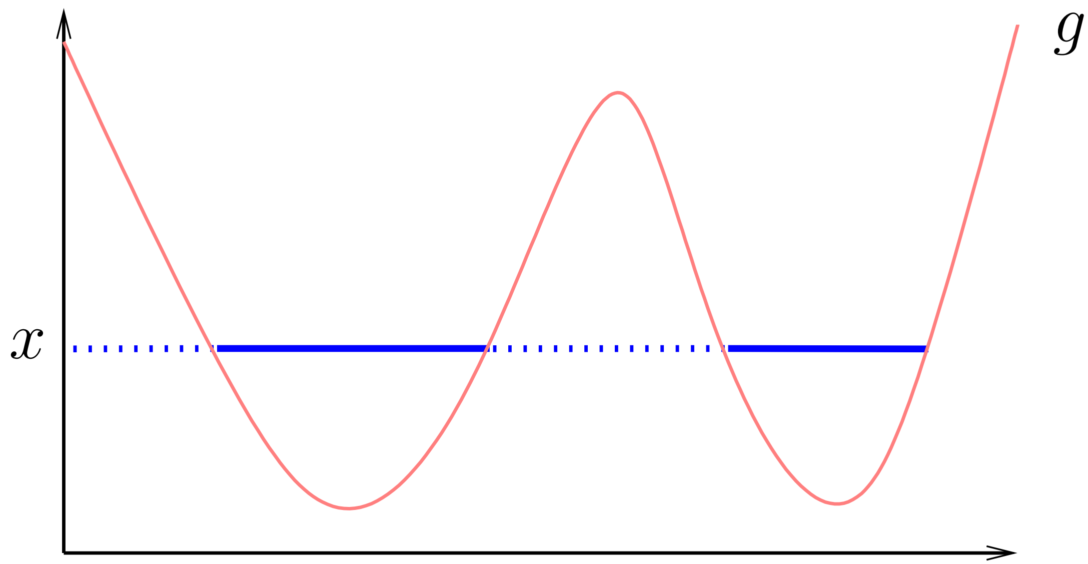
SOPHIA ANTIPOLIS

Introduction

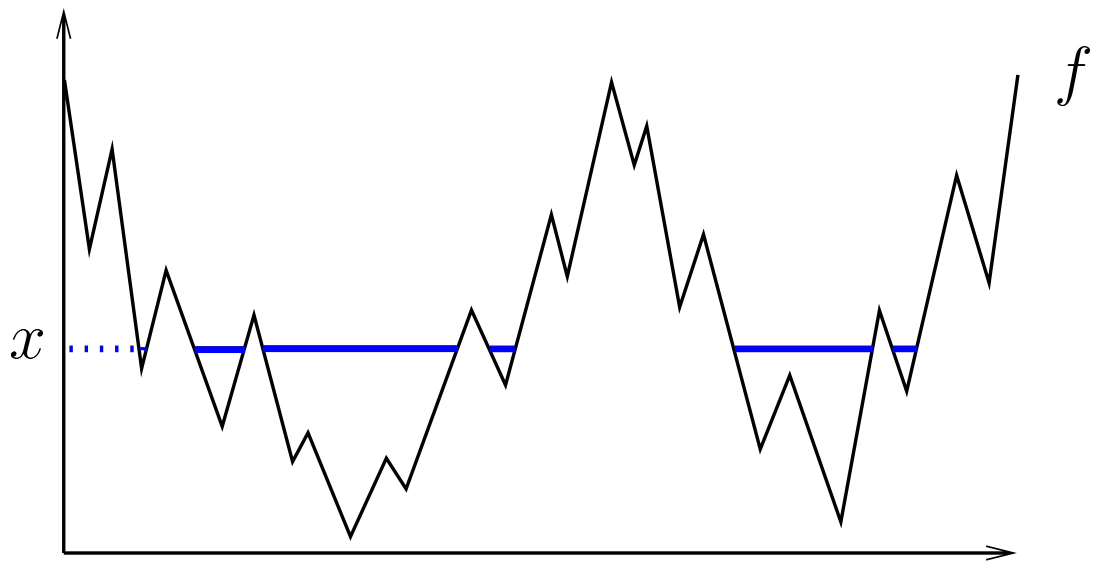
- *Topological persistence and simplification* by H. Edelsbrunner et. al. (2000)
- Topological approach for separating signal from noise.
- Data = real function over a topological space.

The idea of persistence



- How many components in $g^{-1}(-\infty, x]$?

The idea of persistence



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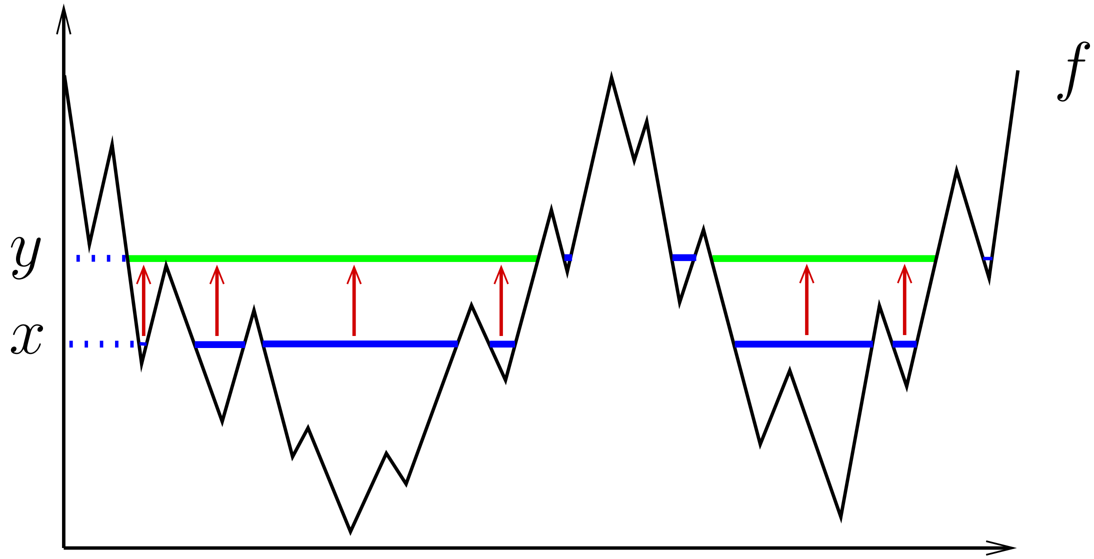
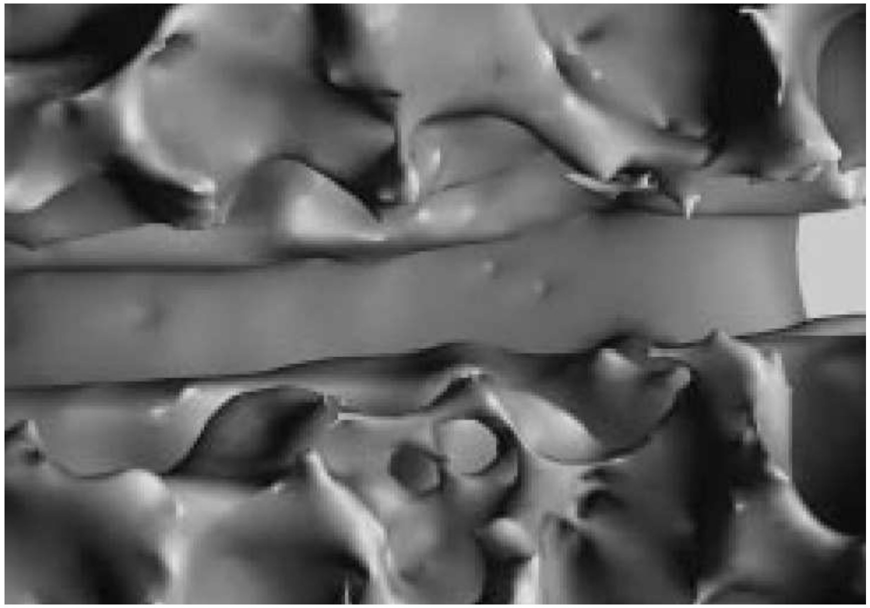


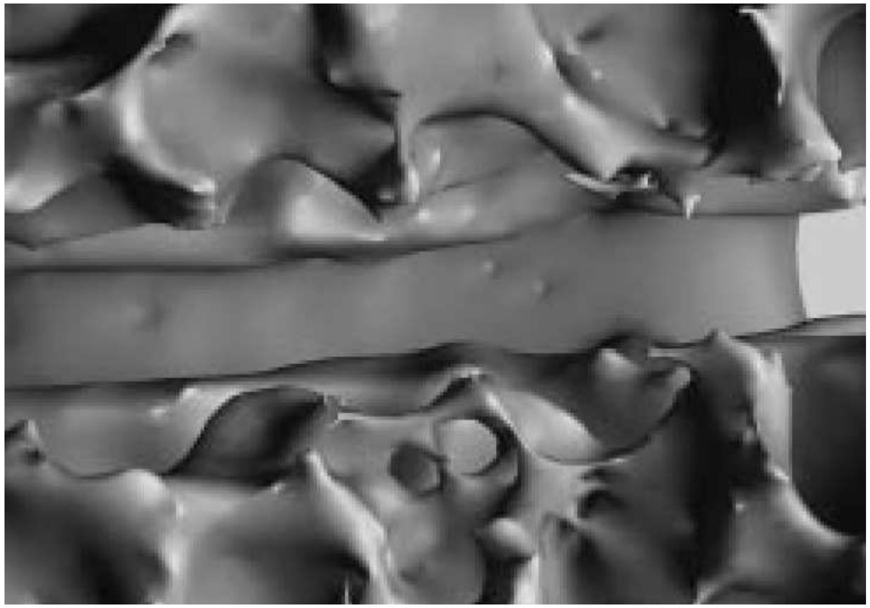
- How many components in $g^{-1}(-\infty, x]$?
- Count the components of $f^{-1}(-\infty, y]$ **induced** by those of $f^{-1}(-\infty, x]$.

Three-dimensional example



- What is the “actual” number of loops in this surface?
- More generally, how can we estimate the k-th **Betti number** of a sub-level set if the function is noisy?

Persistent Betti numbers

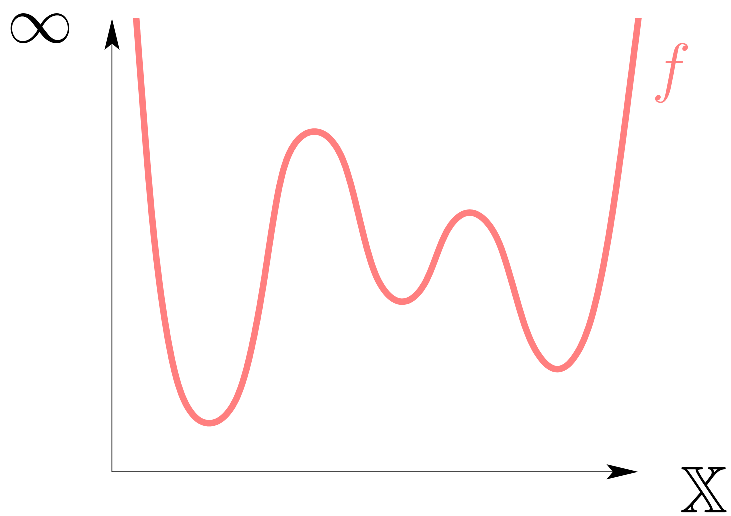


- **Persistent k-th Betti number** of $f : \mathbb{X} \to \mathbb{R}$:

$$\beta_k^{x,y}(f) = \mathrm{rk}(H_k(f^{-1}(-\infty, x]) \to H_k(f^{-1}(-\infty, y]))$$

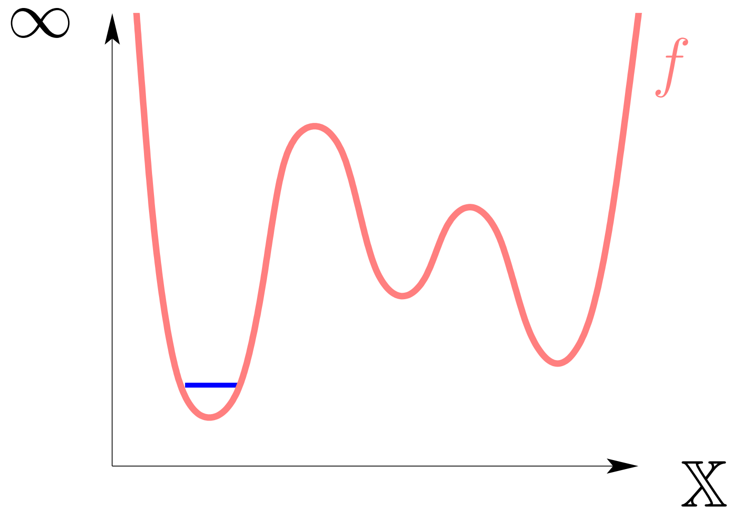
- Filter out **topological noise**.

Persistence intervals



- Track the evolution of the topology of sub-level sets as the threshold increases.
- Pair thresholds that create components with those that destroy them.

Persistence intervals



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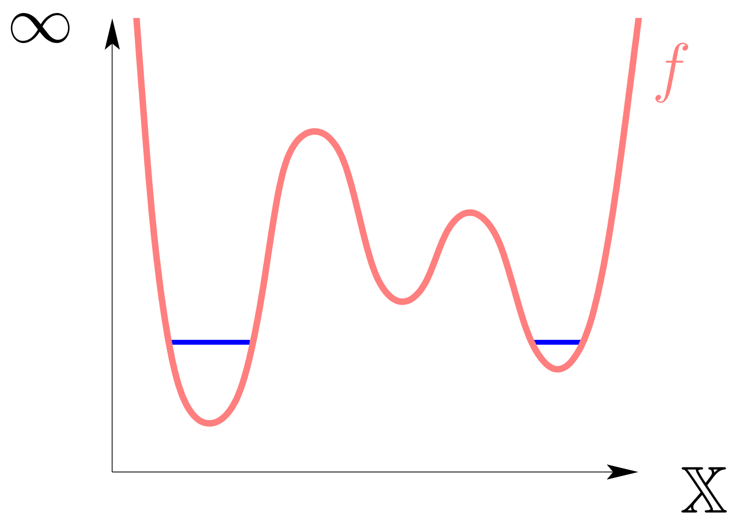

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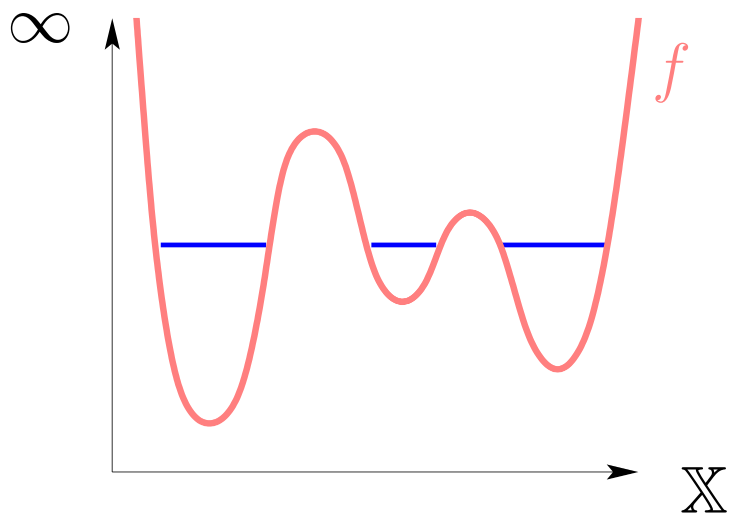

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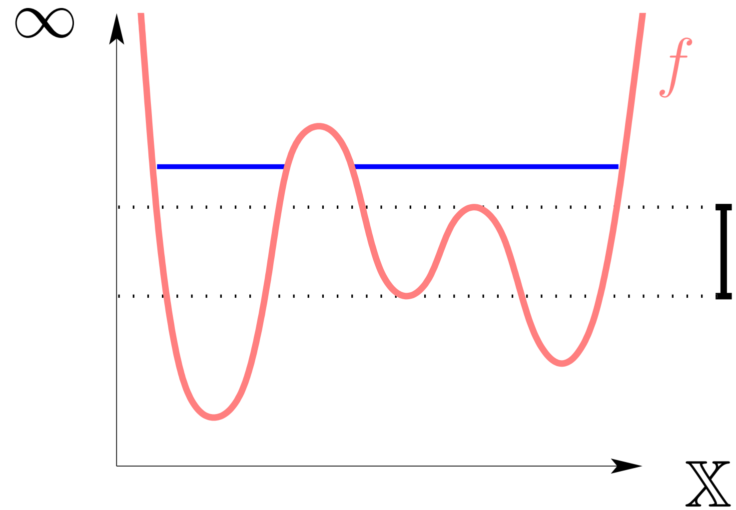

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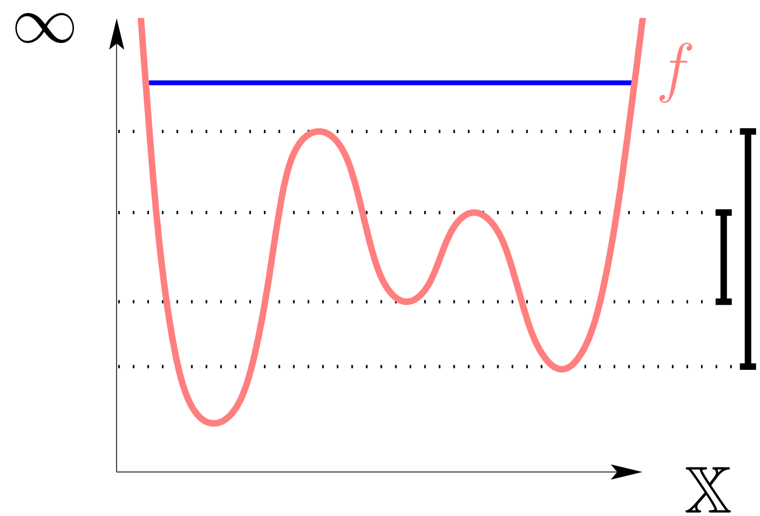

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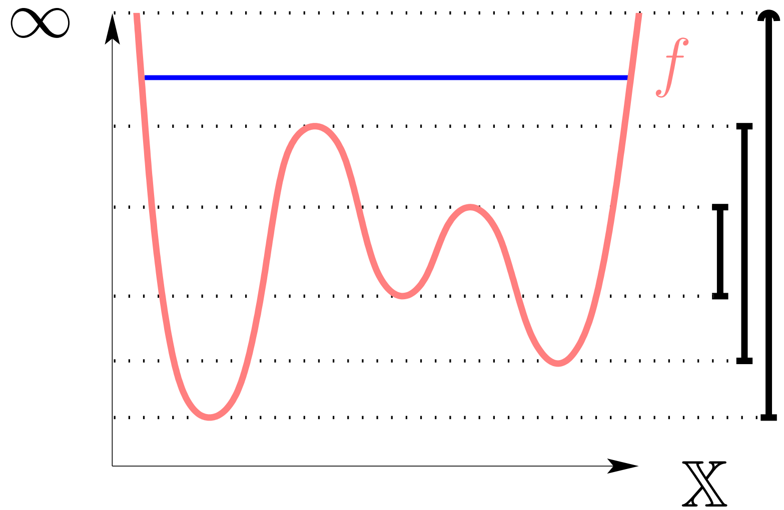

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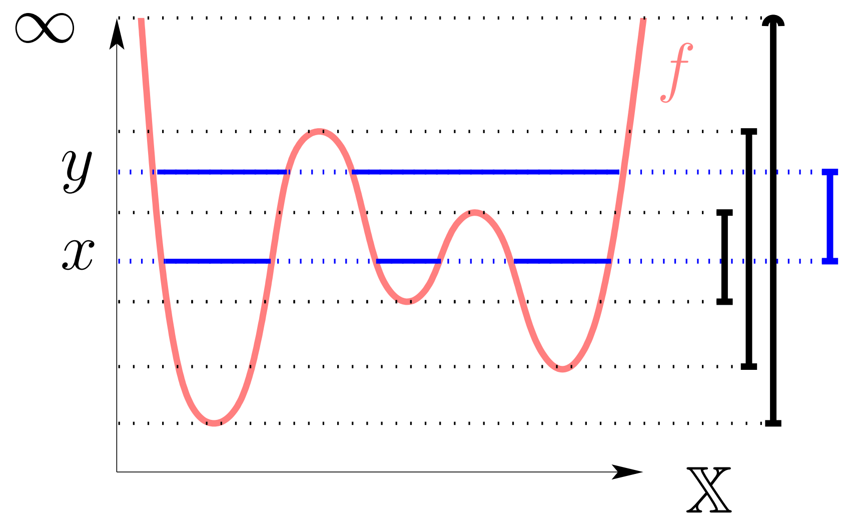
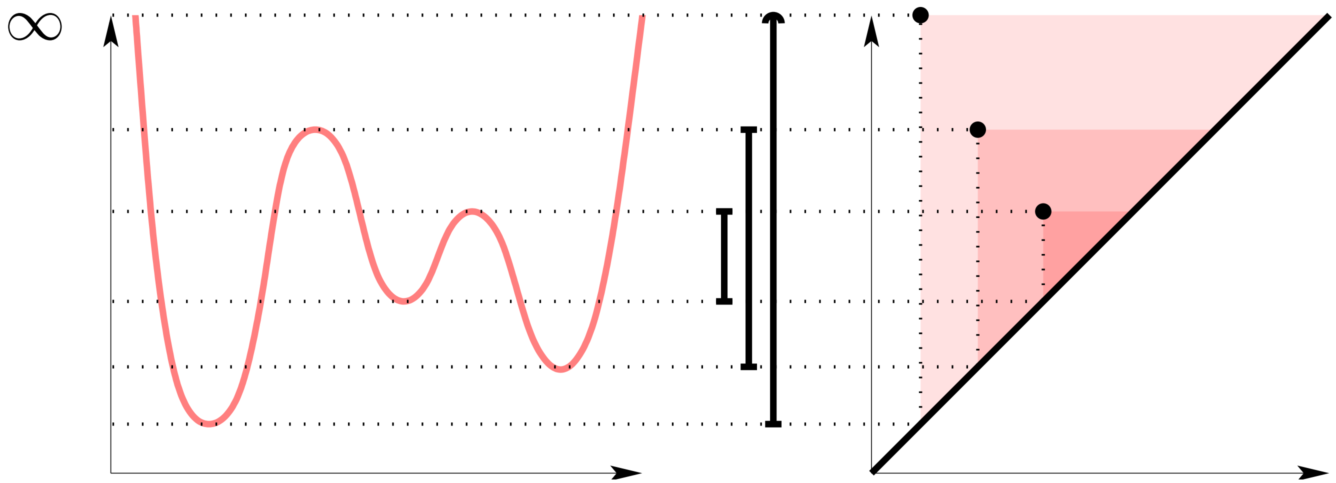


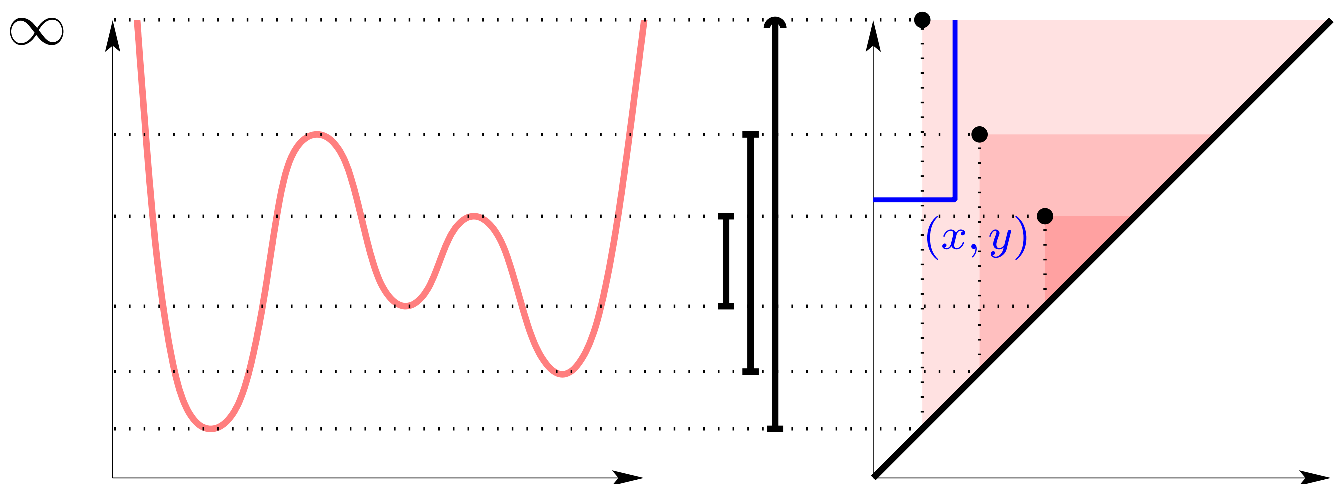
- The number of intervals containing $[x, y]$ is $\beta_k^{x,y}(f)$ ("k*-triangle lemma*" [ELZ02][CaZo04])
- In the case of components, persistence is equivalent to *Size theory*, developped by [Frosini, Landi].

Persistence diagrams



- Persistence intervals become points in the plane.
- The diagonal is included.

Persistence diagrams



- $\beta_k^{x,y}(f)$ is the number of points of $\mathrm{D}_k(f)$ within the upper left quadrant with corner (x, y).
- Persistence diagrams encode the topology of all sub-level sets at all *scales*.

Algorithm for PL functions

Persistence algorithm

Sort the simplices by increasing function values.

Build the mod 2 incidence matrix: $A_{ij} = 1$ iff $s_j \subsetneq s_i$.

while two columns have their last 1 on the same row

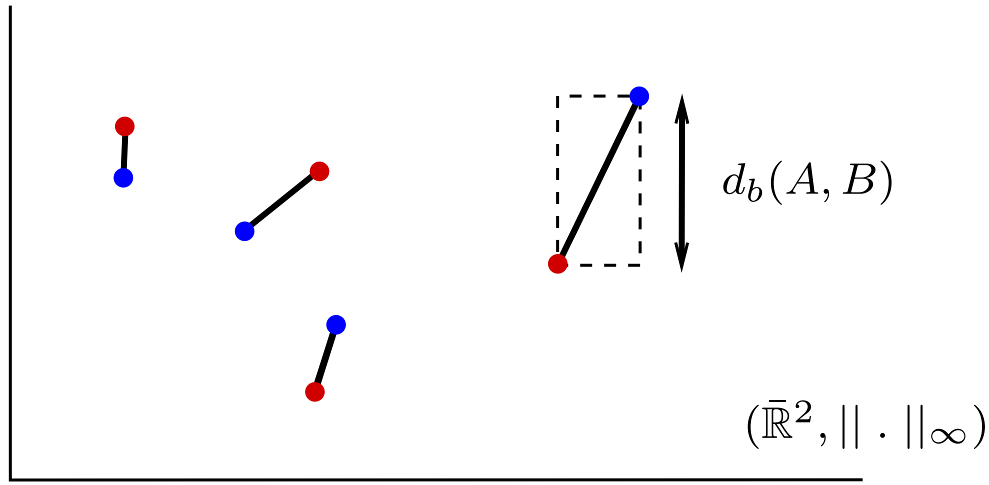
do

add the leftmost to the rightmost.

end while

return $\{(value(s_i), value(s_{last(i)})\}$

Metric on diagrams

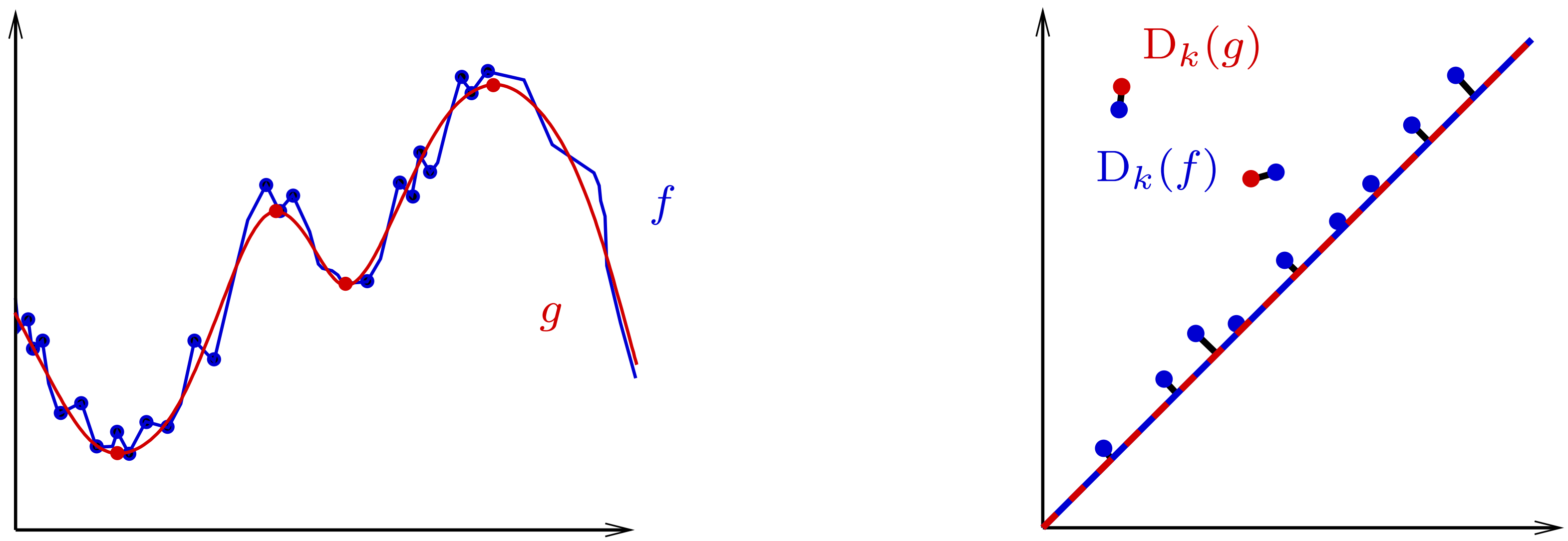


Definition. The *bottleneck distance* between sets A and B is:

$$d_b(A, B) = \inf_\gamma \sup_a \|a - \gamma(a)\|_\infty$$

over all $a \in A$ and all bijections $\gamma : A \to B$.

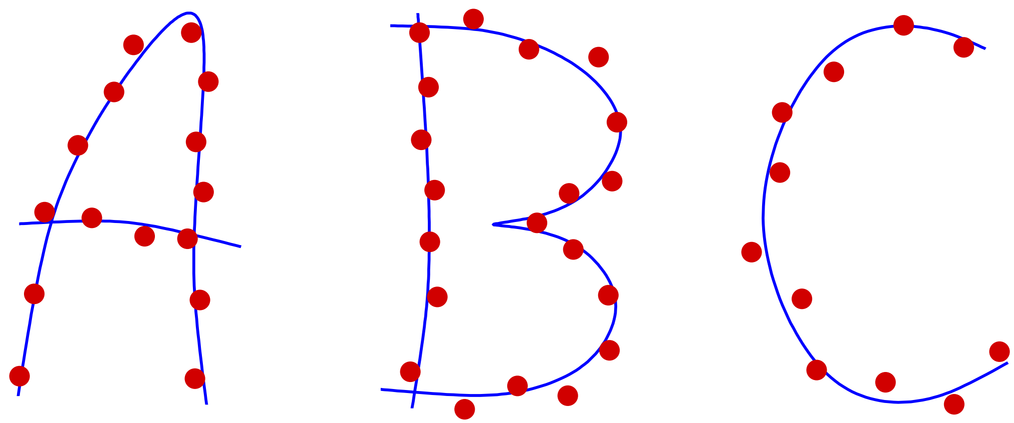
Stability theorem



Theorem. [CEH04].*For two continuous* tame *functions f and g on a finitely triangulable space:*

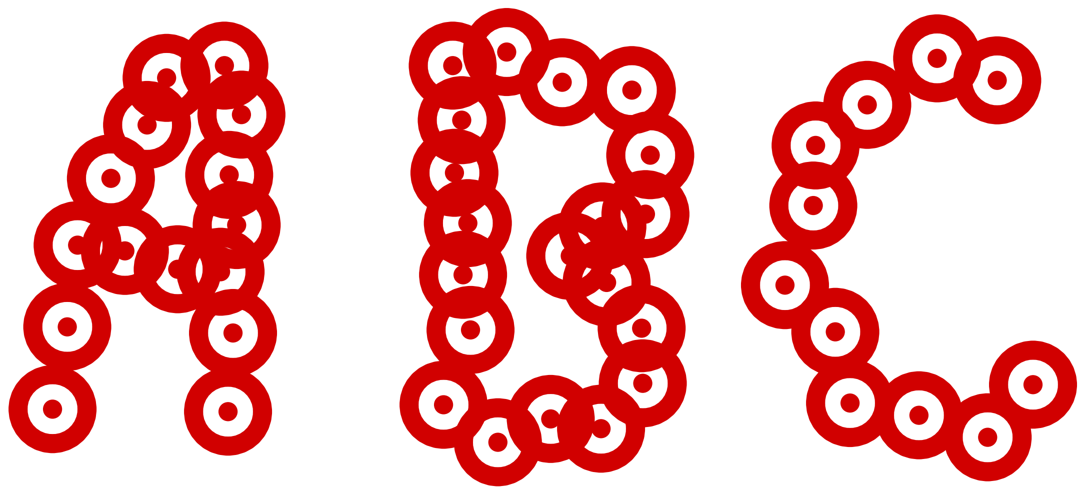
$$d_b(\mathrm{D}_k(f), \mathrm{D}_k(g)) \leq \|f - g\|_\infty$$

Betti numbers from samples



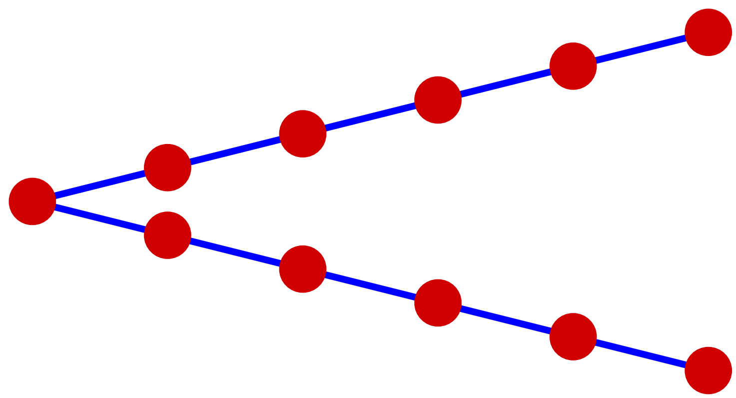
- Build a simplicial approximation of the unknown shape and compute its Betti numbers.
- Use offsets/alpha-shapes.

Reconstruction by offset



- Works for a large class of shapes in $\mathbb{R}^n$ [CCSL06].
- But might requires many data points.

“Sharp” angle problem



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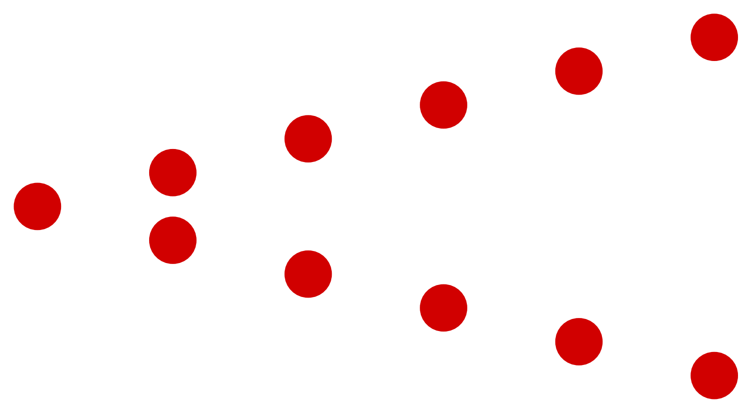

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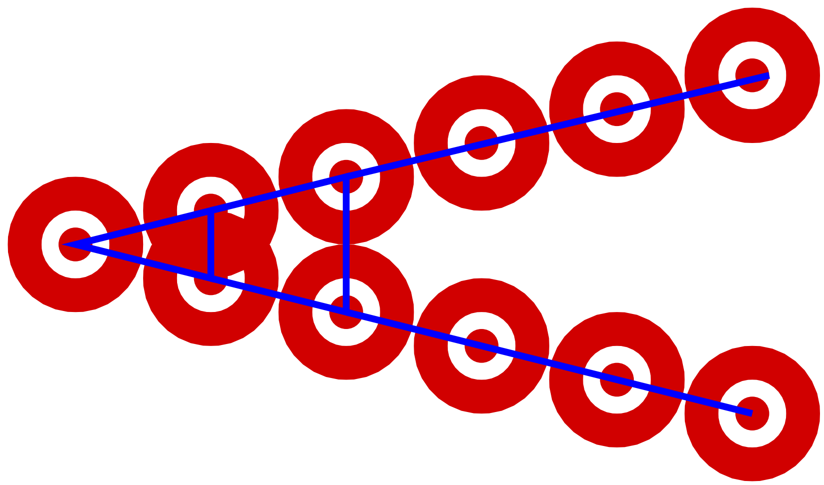

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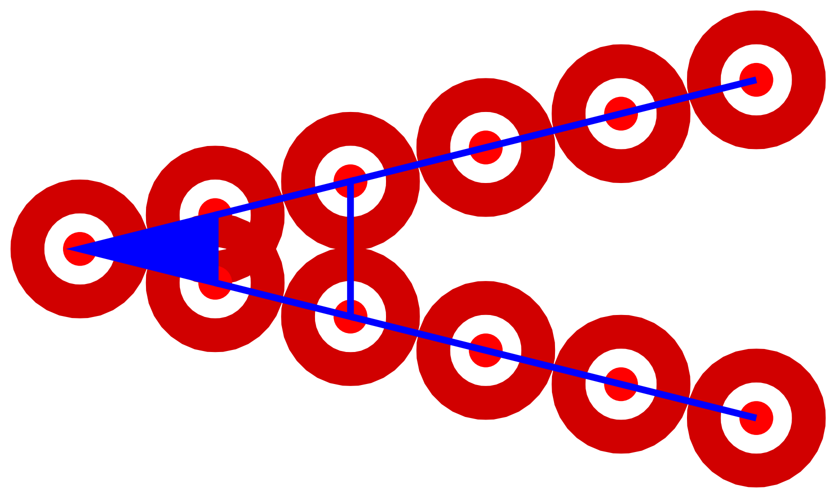

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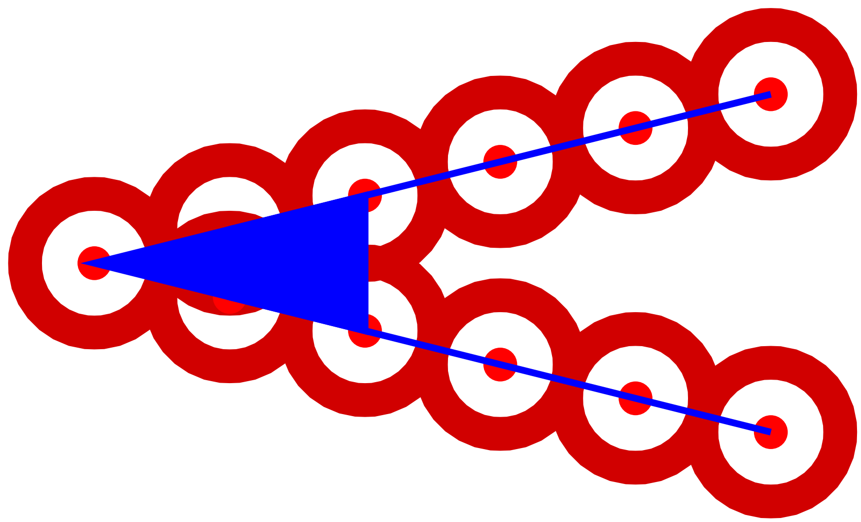

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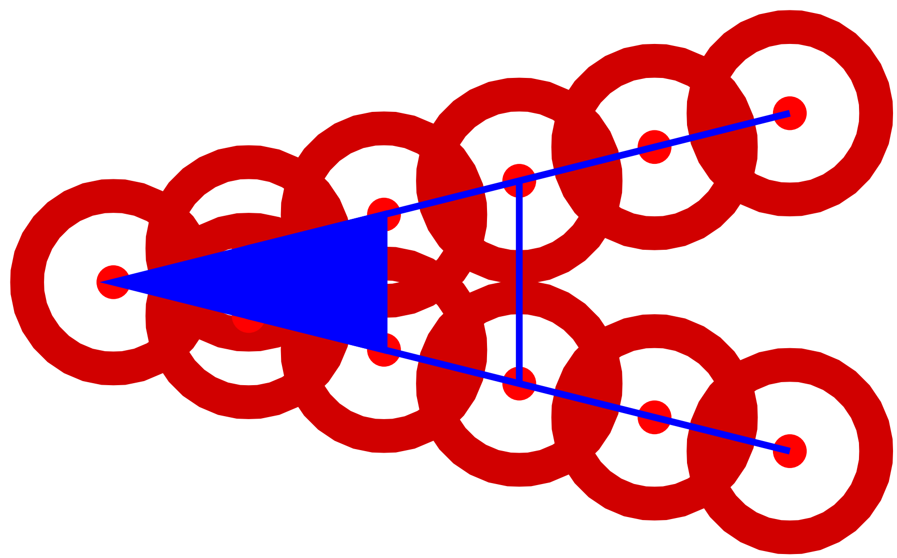

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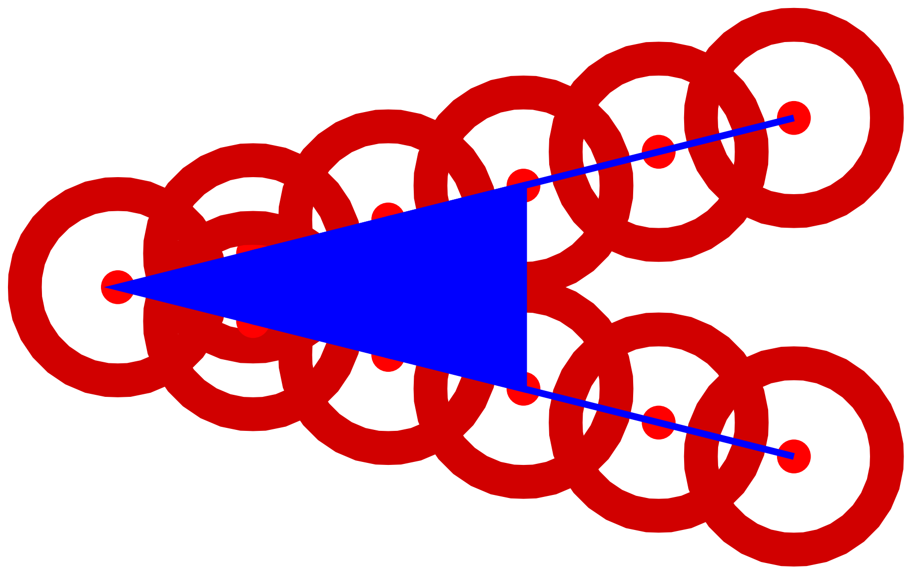

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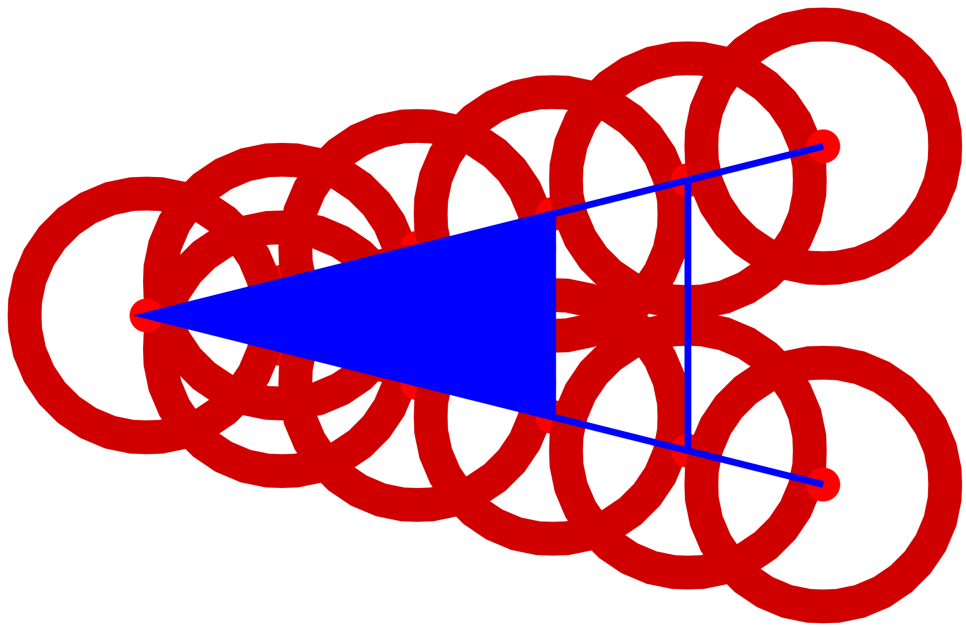

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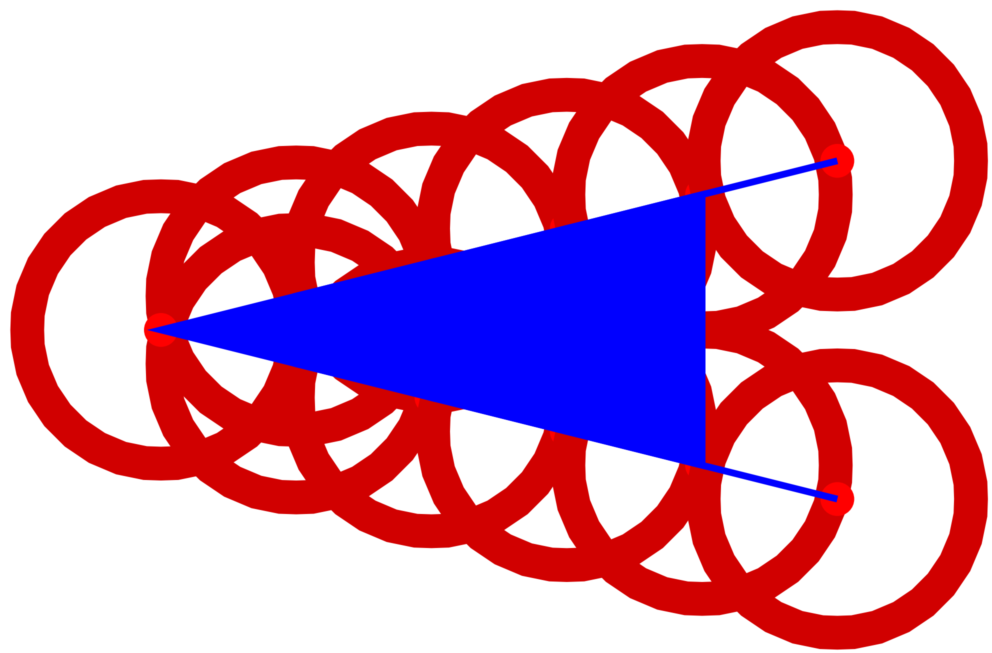
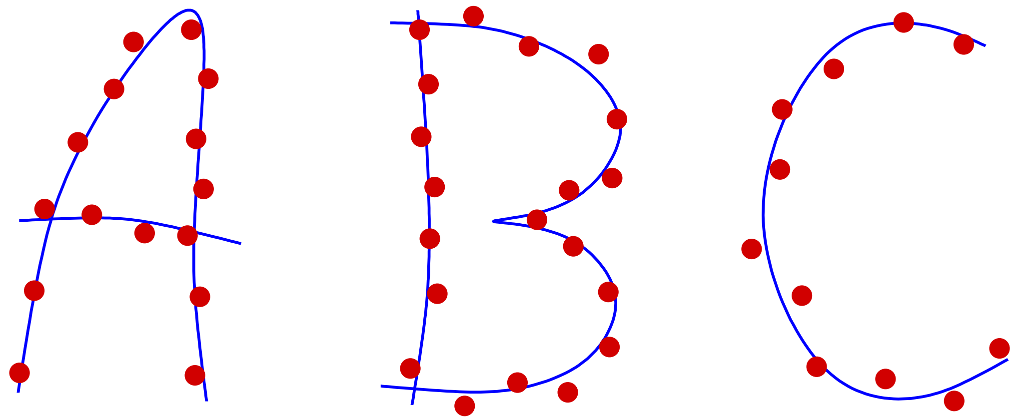


Persistence helps

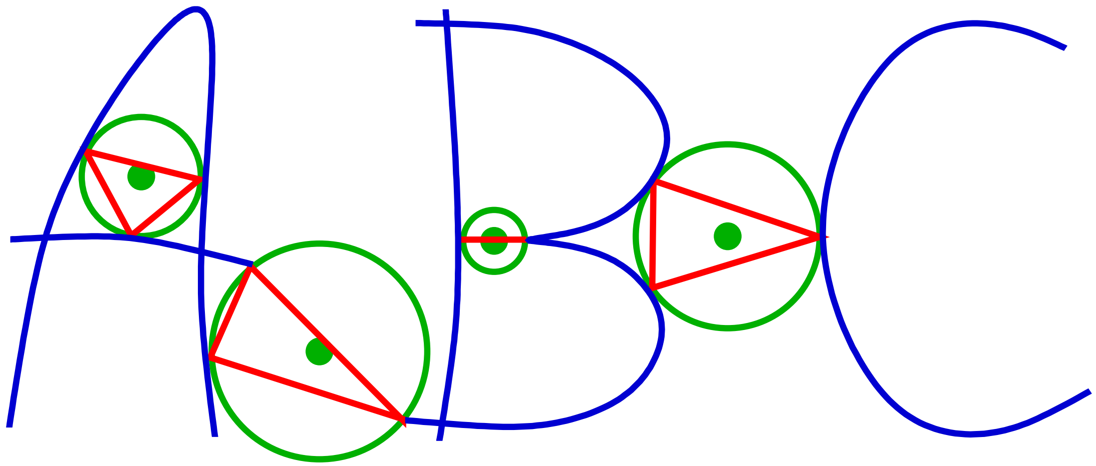
- *Distance function* to S:
  $\text{dist}_S(p) = \inf\{d(s,p) | s \in S\}$ for $p \in \mathbb{R}^n$.
- We were looking at the sublevel-sets of the distance function to the point cloud.
- Spurious loops are short-lived if the sampling is good enough.

Hausdorff distance



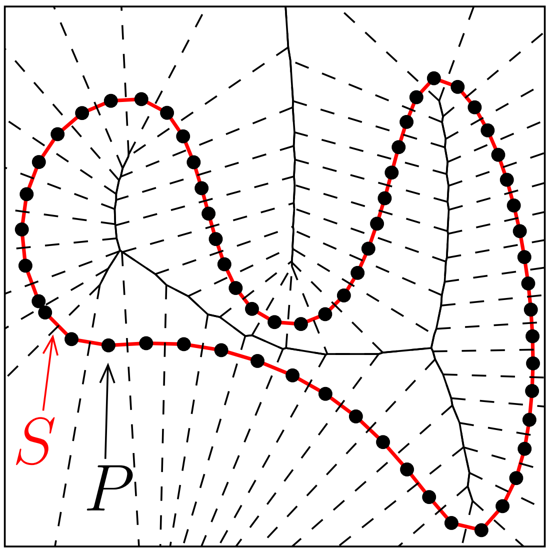
- The *Hausdorff distance* between two sets A and B is the minimum number r such that each point in A is at distance at most r from B and vice versa.
- $d_H(A, B) = ||\text{dist}_A - \text{dist}_B||_\infty$

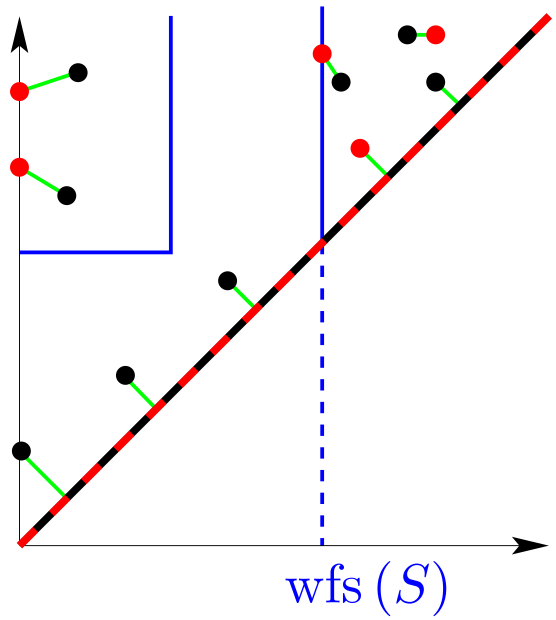
Weak feature size



- $\mathrm{wfs}\,(C) =$ inf $\{$ positive critical value of dist$_C\}$
- $\mathrm{wfs}\,(C) > 0$ if $C \subset \mathbb{R}^n$ is semi-algebraic [Fu95].

Betti numbers from samples





Theorem. [CEH/CL04]. Let S and P be closed subsets of $\mathbb{R}^n$.
If l is such that $d_H(S,P) < l < \mathrm{wfs}\,(S)/4$:

$$\beta_k(S) = \beta_k^{l,3l}(\text{dist}_P)$$

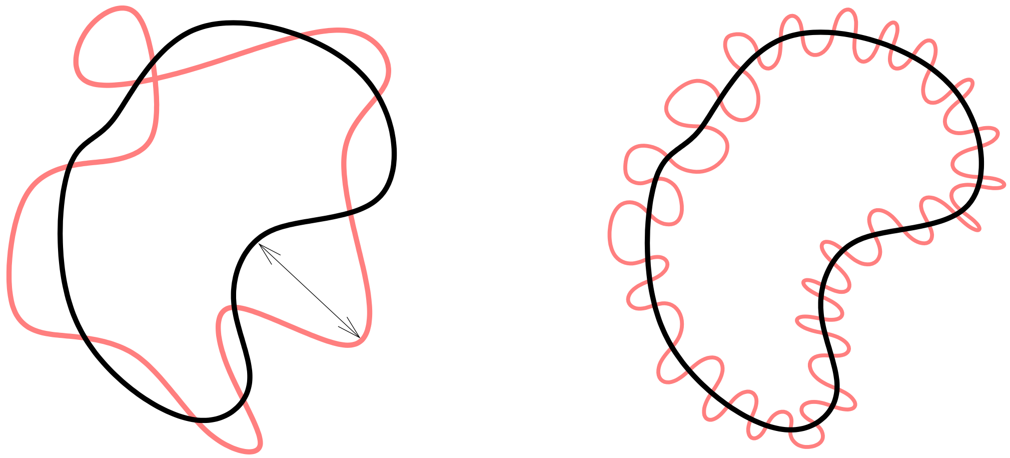
Comments

- Persistent Betti numbers/diagrams of distance function easily computable from the Delaunay triangulation of the sample points.
- You do not get any simplicial complex with the correct Betti numbers.
- Case of high dimensional ambiant space: witness complexes [CdS03]

Robust signatures of shapes

- Given two shapes, are they **approximately congruent**?
- Pick some rotation invariant function defined on shapes, *e.g.* distance function, curvature.
- Compare the persistence diagrams for the two shapes.

Problem for curves

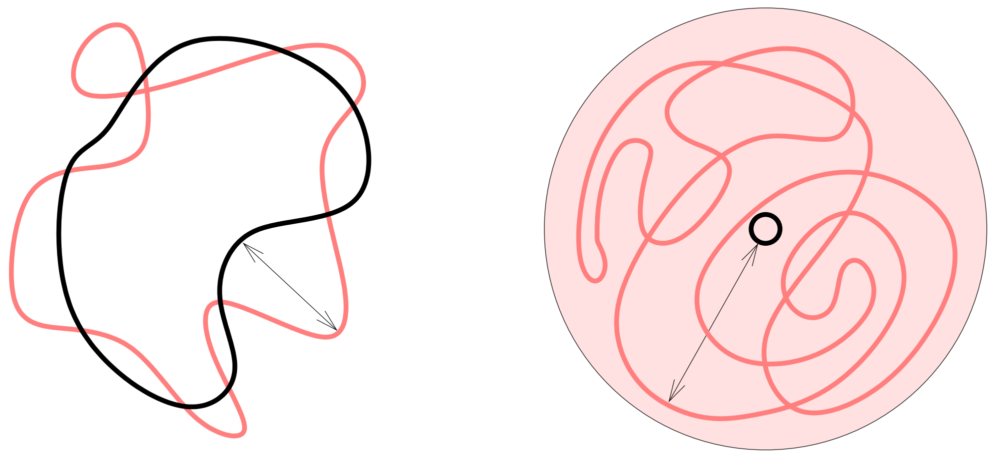


- If two curves are close, does it imply that their lengths are close?
- Fréchet distance between C_1 and C_2:

$$d_b(C_1, C_2) = \inf_{\phi_1, \phi_2} \sup_s d(\phi_1(s), \phi_2(s))$$

where ϕ_i ranges over all parameterizations of C_i.

Result



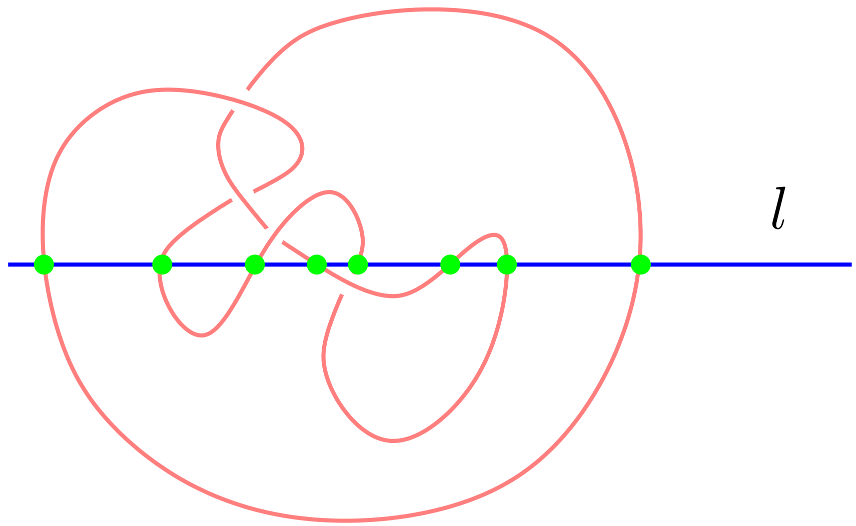
Theorem. *Let C_1 and C_2 be two closed curves in $\mathbb{R}^n$.*

Let L_i be the length of C_i, and K_i be the integral of its curvature.

One has:

$$|L_1 - L_2| \le \frac{2\mathrm{vol}(\mathbb{S}^{n-1})}{\mathrm{vol}(\mathbb{S}^n)}[K_1 + K_2 - 2\pi]\ d_b(C_1, C_2)$$

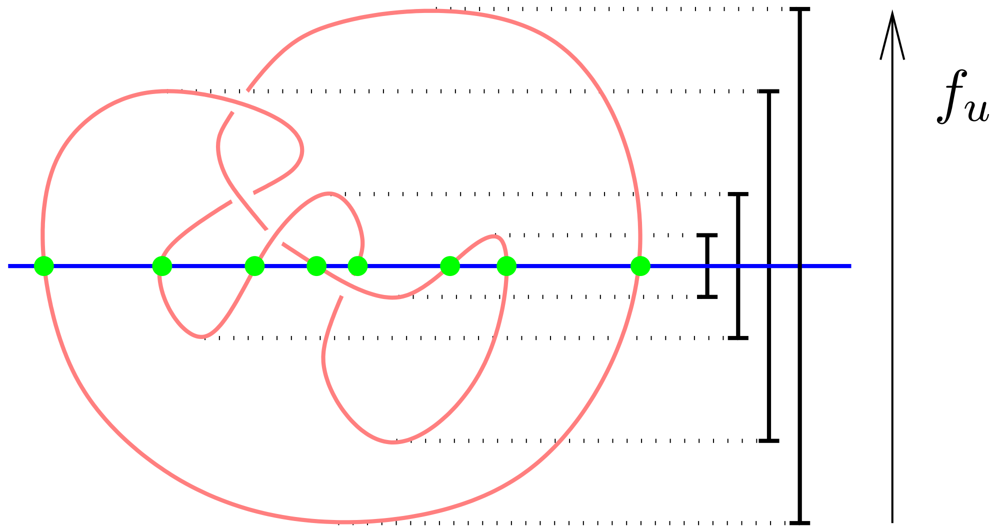
Proof



- Crofton formula:

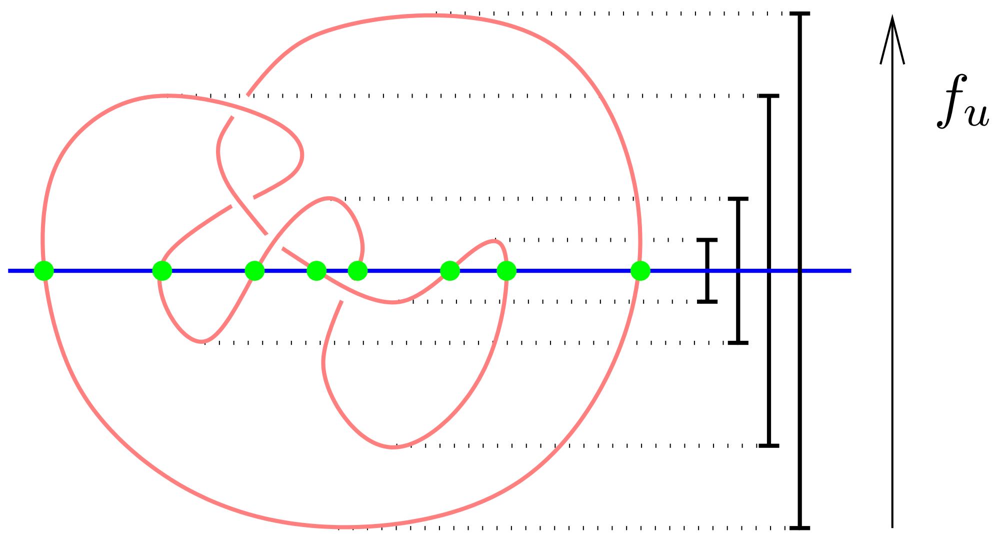
$$L(C) = \frac{\pi}{\mathrm{vol}(\mathbb{S}^n)} \int_{\text{hyperplane } l \subset \mathbb{R}^n} \sharp(l \cap C)$$

Proof



- Let $f^u : C \to \mathbb{R}$ be the height function in the direction u.
- If l has normal vector u, then $\sharp(l \cap C)$ is twice the number of "persistence intervals" of f_u stabbed by l.

Proof

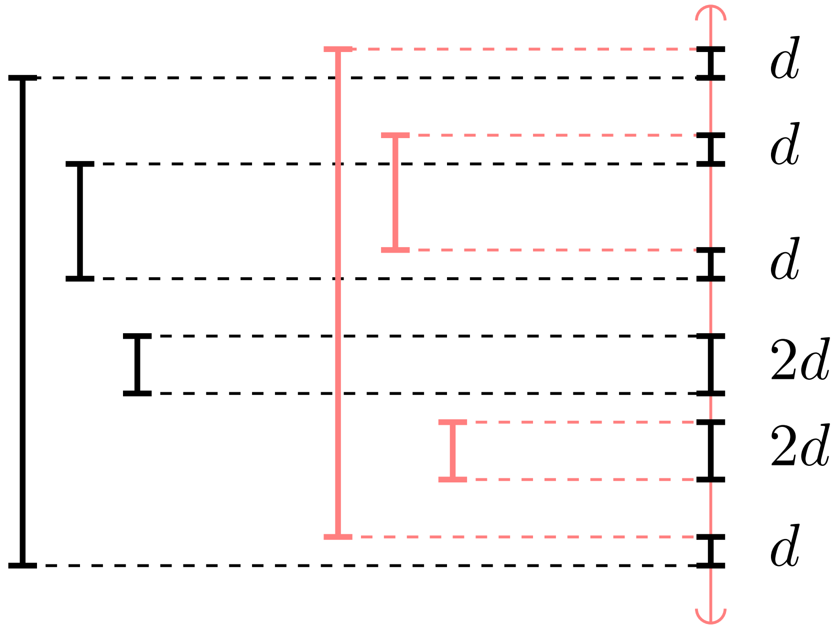


- Hence :

$$\int_{l \text{ hyperplane with normal } u} \sharp(l \cap C)$$

is twice the total length of the persistence intervals of f^u.

Proof



- Stability theorem : the bounds of the persistence intervals of f^u move by at most $d_b(C_1, C_2) = d$.
- Hence the total length of these intervals changes by at most $d(n_1^u + n_2^u - 2)$, where n^u is the number of critical points of f^u.

Proof

- By integrating over all directions :

$$|L_1 - L_2| \leq 2d \, \frac{\pi}{\mathrm{vol}(\mathbb{S}^n)} \int_{u \in \mathbb{S}^n} n_1^u + n_2^u - 2 \, du$$

- *Exchange theorem*:
  The integral of the number of critical points n_i^u over $u \in \mathbb{S}^n$
  is the integral of the curvature of C_i divided par $\pi/\mathrm{vol}(\mathbb{S}^{n-1})$
  $\rightarrow$ qed.

Result for surfaces

Theorem. *Let $S_1 = \partial V_1$ and $S_2 = \partial V_2$ be two closed surfaces in $\mathbb{R}^3$ with the same genus g. Let H_i be the integral of the* ***mean curvature*** *of S_i, and K_i be the integral of its* ***absolute Gauss curvature****. One has:*

$$|H_1 - H_2| \leq [K_1 + K_2 - 4\pi(1+g)]\, d_b(V_1, V_2)$$

- Holds for piecewise-linear surfaces, for which simple formula exist: accurate total mean curvature estimation from a mesh.
- Closeness between normals to the surfaces is not explicitly required, unlike in [CSM03].

Conclusion

- Thank you!