Tutorial I General Relativity

Exercise I: The Metric Tensor

To describe distances in a given space for a particular coordinate system, we need a distance recepy. The metric tensor is the translation for a coordinate system

$$ds^2 \equiv c^2 d\tau^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu \tag{1}$$

with

$$g_{\mu\nu} \equiv \frac{\partial \vec{r}}{\partial x^\mu} \cdot \frac{\partial \vec{r}}{\partial x^\nu} \tag{2}$$

and in which τ is the coordinate time, the proper time of the system,

$$ds^2 \equiv c^2 dt^2\,, \tag{3}$$

.

- For a 3-dimensional space $\vec{r} = (x, y, z)$ derive the elements of the metric tensor $g_{\mu\nu}$, and write in matrix form, for:
  - Euclidian coordinates (x,y,z)
  - cylindrical coordinates (ρ, ϕ, z)
  - spherical coordinates (r, θ, ϕ)

- In addition, give the covariant metric tensor $g^{\mu\nu}$ (the inverse of $g_{\mu\nu}$).
- What is the metric tensor for Minkowski space in coordinate system $x^\mu = (ct, x, y, z)$.

To describe the curvature of space we need to specify the spatial variation of the geometry of space. This brings us to a key quantity in differential geometry, the **Christoffel symbol** $\Gamma^\alpha{}_{\beta\gamma}$ (also called the *affine connection*),

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\nu} \left\{ \frac{\partial g_{\gamma\nu}}{\partial x^\beta} + \frac{\partial g_{\beta\nu}}{\partial x^\gamma} - \frac{\partial g_{\gamma\beta}}{\partial x^\nu} \right\}. \tag{4}$$

- Derive all Christoffel Symbol elements $\Gamma^\alpha{}_{\beta\gamma}$ for
  - Euclidian coordinates (x,y,z)
  - cylindrical coordinates (ρ, ϕ, z)
  - spherical coordinates (r, θ, ϕ)

- Subsequently derive the equation of motion of a freely moving particle with mass m,

$$\frac{d^2x^\beta}{dt^2} = 0$$

in each of these coordinate systems,

$$\frac{d^2x^\beta}{d\tau^2} + \Gamma^\beta{}_{\lambda\nu}\frac{dx^\lambda}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

Exercise II: Lorentz Transformations

The Lorentz transformation is usually defined for an object moving with a velocity v in the x-direction,

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu} \tag{5}$$

with

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\beta\gamma_v & 0 & 0 \\ -\beta\gamma_v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{6}$$

- Derive the "general" Lorentz matrix $\Lambda^{\mu}{}_{\nu}$ for a body moving with velocity $\vec{v} = (v_x, v_y, v_z)$.

In order to frame proper physical relations and laws we have to express them as tensor relations. Only then we are assured that they are valid in all inertial frames (and for GR: in all freely falling frames). We define the tensor four-velocity U^{μ} as

$$U^{\mu} \equiv \frac{dx^{\mu}}{d\tau} \tag{7}$$

- Express the four velocity in terms of the velocity $\vec{v}$ (and γ_v and c)
- Demonstrate that U^{μ} is a proper tensor, ie. that it transforms according to the Lorentz transform

$$U'^{\mu} = \Lambda^{\mu}{}_{\nu}\, U^{\nu} \tag{8}$$

- Argue why the conventional velocity vector $\vec{v}$ is not a proper tensor.

The **contravariant** four-vector A^{μ} has a **covariant** equivalent A_{ν},

$$A_{\nu} = \eta_{\nu\mu}\, A^{\mu} \tag{9}$$

in which $\eta_{\nu\mu}$ is the covariant Minkowski metric for an inertial system, the inverse of the contravariant one,

$$\eta_{\nu\alpha}\, \eta^{\alpha\mu} = \delta_{\nu}{}^{\mu} \tag{10}$$

Its Lorentz transformation between two inertial systems is specified by

$$A'_{\nu} = \tilde{\Lambda}_{\nu}{}^{\mu}\, A_{\mu} \tag{11}$$

where the covariant Lorentz transformation is given by

$$\tilde{\Lambda}_{m}u^{\ \nu} = \eta_{\mu\alpha}\, \Lambda^{\alpha}{}_{\beta}\, \eta_{\beta\nu} \tag{12}$$

- Demonstrate that the inproduct $A_{\nu}A^{\nu}$ is an invariant scalar.
- What is the invariant inproduct of the four velocity $U^{\nu}U_{\nu}$?

Exercise III: Metric and Potential

General Relativity states that the gravitational force is a manifestation of the curvature of space. In other words, the metric $g_{\mu\nu}$ should in some way contain information on the gravitational field.

Here we are going to see how in the asymptotic limit of a 1) stationary 2) weak gravitational field - a situation which pertains to nearly all astrophysical systems we know, with the exception of e.g. the vicinity of black holes and neutron stars - we may infer the relation between gravitational potential ϕ and metric.

In the case of a weak field the depth of the gravitational potential is small,

$$\frac{\Delta\phi}{c^2} \ll 1\,, \tag{13}$$

while the condition of stationary field implies the accompanying curvature of space to change only slowly in time,

$$\frac{\partial g_{\mu\nu}}{\partial x^0} = 0 \tag{14}$$

- On the basis of the conditions above, for a slow (non-relativistically) moving object

$$\frac{\dot{x}^k}{c} \ll 1 \tag{15}$$

  do derive the approximate relationship for $k = (1, 2, 3)$,

$$\Gamma^k{}_{00} \approx -\frac{1}{2}\frac{\partial g_{00}}{x_k}\,. \tag{16}$$

  In this you should take into account that $g^{kk} \approx \eta_{kk}$.

- On the basis of the relativistic equation of motion,

$$\frac{d^2x^\beta}{d\tau^2} + \Gamma^\beta{}_{\lambda\nu}\frac{dx^\lambda}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

  show that

$$\ddot{x}^k \approx -\frac{1}{2}c^2\,\vec{\nabla}g_{00} \tag{17}$$

  where we may use the fact that in a weak gravitational field the measured time t is nearly equal to the proper time τ,

$$d\tau = dt\left(1 + \frac{\Delta\phi}{c^2}\right) \tag{18}$$

- Equating the above result for a stationary, weak field to the Newtonian limit, derive the crucial relation

$$g_{00} = \frac{2\phi}{c^2} \tag{19}$$

Exercise IV: Gravitational Redshift

Einstein's Strong Equivalence Principle extends the fact that the laws of physics are the same in inertial reference frames in uniform relative motion to that of accelerating frames. It states that

> The physics in the frame of a freely falling body
> is equivalent to that of
> an inertial frame in Special Relativity.

In other words, in a frame of reference moving with the free-fall acceleration at that point, all laws of physics are entirely equivalent and have their usual Special Relativistic form. Evidently, this remains true for a feeely falling body in a gravitational field. The direct implication is that you cannot really make a distinction between whether you are a freely falling body in e.g. an accelerating elevator or whether you find yourself with this cabin in a gravitational field. We are going to explore the wide-reaching ramifications of this idea ...

Look at the drawing of the accelerating elevator. The elevator cabin has a height h. We have just seen that you will not be able to distinguish between the situation in which the cabin is stationary in a gravitational field (left) and the situation in which the cabin is accelerated with an upward acceleration $\vec{a} = \vec{g}$ (right). To assure the survival of the elevator we assume the acceleration to be rather modest.

- Consider a lamp, at $t = 0$ freely dispensed at the ceiling of the cabin, emitting radiation with a frequency ν. What is the Doppler shift of the radiation observed by a person situated at the floor of the elevator (hint: calculate the velocity of the lamp with respect to the floor !).

  Note: the Doppler shift due to the emitting object moving with a velocity v towards the observer is

$$\nu' = \gamma_v \, \nu \left(1 + \frac{v}{c}\right) \tag{20}$$

- Relate the acceleration $\vec{g}$ to the potential difference $\Delta\phi$ between the ceiling and floor of the cabin at $t = 0$, ie. the distance the floor of the elevator went between emission and observation ? Use this result to prove that gravitational frequency shift is given by

$$\nu' = \gamma_v \, \nu \left(1 - \frac{\Delta\phi}{c^2}\right) \tag{21}$$

- On the basis of the result above, derive the expression for gravitational redshift z_{grav}.

- With a friend you visit Toronto. You chicken out and stay on the ground while your friend takes the courage to climb the mighty CN tower to the top floor, at 446.5 meter above the ground (it is forbidden to go up the antenna, with its tip at 553.3 m). You have a laser lamp and send a signal up to the glass floor of the CN tower ... What is the gravitational redshift your friend will measure ? Also calculate the gravitation redshift measured by an alien who happens to pass by at some (large) distance and receives the signal ? (ie. what is the z_{grav} of the Earth's surface $R_{\oplus} = 6,372.797 km$, $M_{\oplus} = 5.9736 \times 10^{24} kg$.

- Similarly, calculate the gravitational redshift ...
  - for the Sun ($R_{\odot} = 6.955 \times 10^8$ m, $M_{\odot} = 1.9891 \times 10^{30}$ kg)
  - a white dwarf (with $M_{wd} \approx 1 M_{\odot}$, $R_{wd} \approx 0.0075 R_{\odot}$)
  - a neutron star (with $M_{ns} \approx 1.44 M_{\odot}$, $R_{ns} \approx 19$ km)
  - ... and .. a black hole

- Relating the frequency nu to the period time T of the waves, what gravitational time dilation T' do you infer for someone sitting at a gravitational potential well depth $\Delta\phi$?

- Embedded in the gravitational potential of a point mass M, at a distance r of the mass,

$$\phi(r) = -\frac{GM}{r} \tag{22}$$

and making the assumption that you are sufficiently far away that the gravitational field ϕ is weak enough,

$$\frac{\phi(r)}{c^2} \ll 1 \tag{23}$$

show that the gravitational time dilation at r is given by

$$dt'^2 = dt^2 \left\{1 - \frac{2GM}{rc^2}\right\} \tag{24}$$

Notice that this is exactly the expression given by the Schwarzschild metric:

$$c^2 d\tau^2 = (1 - \frac{2GM}{c^2 r})c^2 dt^2 - (1 - \frac{2GM}{c^2 r})^{-1} dr^2 - r^2 d\Omega^2 \tag{25}$$

- The radius

$$R_s = \frac{2GM}{c^2} \tag{26}$$

is the Schwarzschild radius. Calculate your own Schwarzschild radius, that of the Earth, of the Sun, of a white dwarf and a neutron star. Also in terms of the physical radius of the system.