Chapter 2:
The General Theory of Relativity

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1. Newtonian versus Relativistic Gravity

The evolution of the Universe is ruled, dominated, by the force of gravity. Of the four known fundamental forces of nature, gravity is the only one whose effects are propagated over cosmological distances. As long as the force of gravity was described by Newton's second law,

$$\mathbf{g} = -\frac{GM}{r^2}\,\hat{e}_r \tag{1}$$

it was not possible to develop a proper cosmology. First, at hindsight of course, it remained a puzzling observation that gravity operated as an "interaction-at-a-distance", hardly a convincing concept of reality. In addition, in Newton's theory of gravity the geometry of spacetime is simple, a *"God-given", static and simple Cartesian geometry: a flat Universe, here, everywhere, and always*, with a simple spatial metric $g_{kl} = (+1, +1, +1)$. In other words, Newtonian space-time is a rigid medium against whose backdrop all physical processes take place.

It was with the publication in 1915 of his **Theory of General Relativity** that Albert Einstein revolutionized our ideas of space-time and of the force of gravity. He showed that spacetime is a dynamic medium whose geometry changes as consequence of the matter distribution in the Universe, and which in turn influences the matter content (or, rather, the energy-momentum content) of the Universe. In this view the force of gravity is nothing else but the manifestation of the local curvature of spacetime. For the description of the *dynamics* of the Universe the *energy-momentum content* is a *central and crucial factor* of which we need to have precise knowledge.

2. The Principle of Relativity

Albert Einstein (1905)

> The same laws of electrodynamics and optics are valid for all
> frames of reference for which the equations of mechanics hold good.
> We will raise the conjecture (the purport of which will hereafter
> be called the 'Principle of Relativity') to the status of a postulate
> and also introduce another postulate which is only apparently irreconcilable
> with the former, namely that light is always propagated in empty space
> with a definite velocity c which is independent of
> the state of motion of the emitting body."

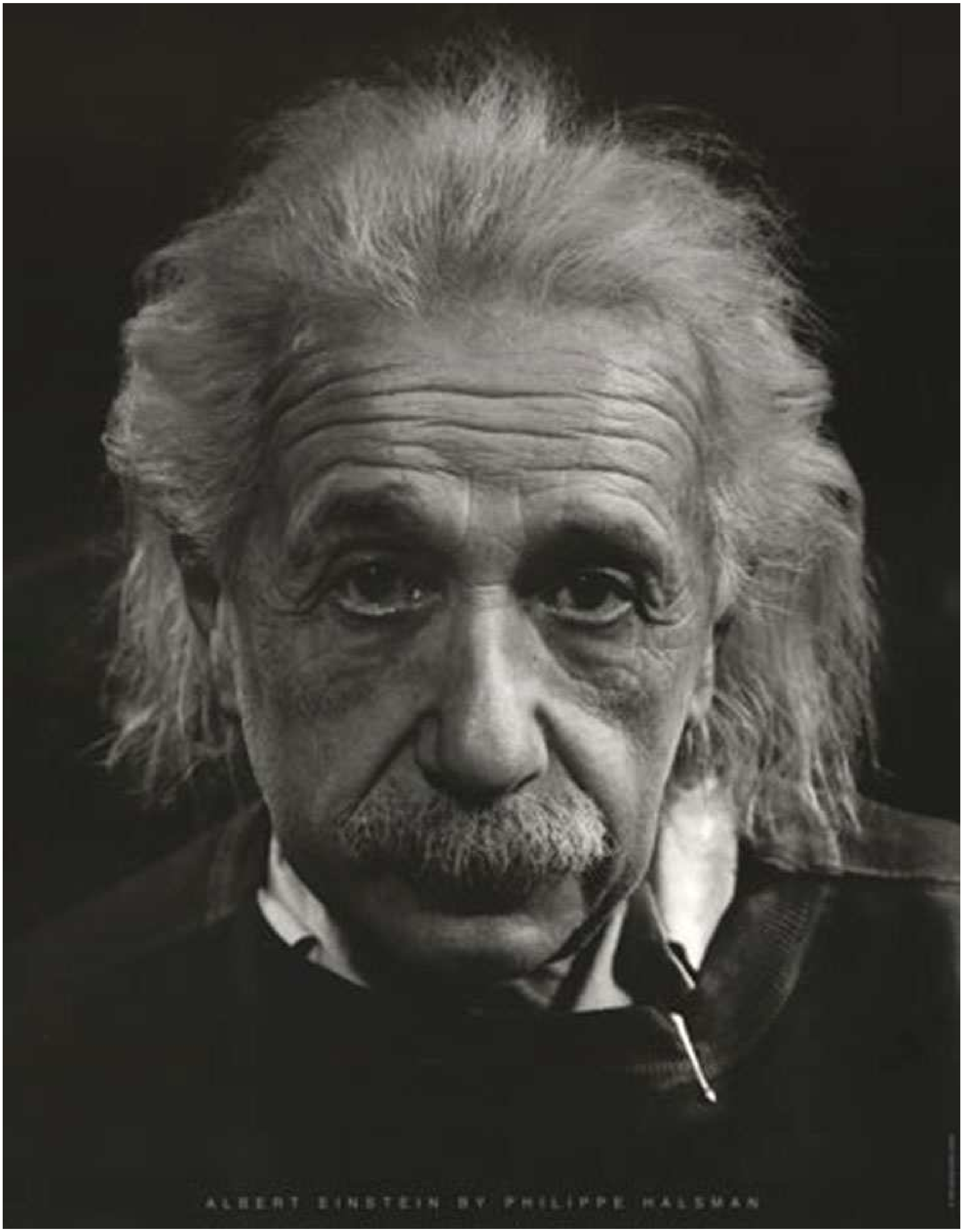

Figure 1. Albert Einstein (1879-1955)

3. Literature

Some recommendable books for further study of Special and General Relativity are contained in the following listing:

- **Relativity: The Special and the General Theory**

  A. Einstein; 1920, Henry Bolt, New York

  What better than an explanation by the master himself ! Highly accessible and nicely written popular account of his masterworks. Quote: "Who would imagine that this simple law [the constancy of the velocity of light] has plunged the conscientiously thoughtful physicist into the greatest intellectual difficulties"

  (*available in numerous bargain paperback editions*)

- **Theory of Relativity**

  W. Pauli; 1920 (available as Dover paperback)

  brilliant introduction on relativity, written when Pauli was 20 years old ! While slightly out of date, still a great historic document

- **General Relativity from A to B**

  R. Geroch; 1981, Univ. Chicago Press

  Excellent qualitative introduction. Beautifully illustrated and insightful description of Minkowski spacetime, followed by general relativistic curved spacetimes.

- **Introduction to Special Relativity**

  W. Rindler; 1991, Oxford Univ. Press

  A classic introductory text on special relativity

- **Gravitation**

  C.W. Misner, J.A. Wheeler, K.S. Thorne; 1973, Freeman

  Biblical, in content as well as in proportion. One of the great scientific books of all time. However, not suited for your introduction into General Relativity

- **General Relativity**

  R.M. Wald; 1984, Univ. Chicago Press

  Very good textbook for graduate students. While not always easy, it is well written and has a good selection of topics.

- **Problem Book in Relativity and Gravitation**

  A. Lightman, R. Price; 1975, Princeton Univ. Press

  great book for obtaining "hands-on" insight

- **Gravitation and Cosmology**

  S. Weinberg; 1972, Wiley

  A Classic !!! Cosmology with heavy focus on general relativistic background

- **General Relativity and Cosmology**

  Proc. International School of Physics "Enric Fermi", Course XLVII, Varenna, 1969

  ed. B.K. Sachs; 1971, Academic Press

  Wonderful summerschool proceedings. Fantastic lectures, particularly noteworthy lectures are the ones by J. Ehlers and G.F.R. Ellis. Interesting group photo !!!

- **Subtle is the Lord: The Science and Life of Albert Einstein**

  A. Pais; 1983, Oxford Univ. Press

  Thé biography on Albert Einstein, vividly narrating his journey of discovery, putting particular emphasis on the science involved. Fantastic description of how great discoveries and science are achieved.

4. Special Relativity and the Lorentz Transformation

Special Relativity is based upon two fundamental tenets,

- All **Laws of Nature** are equivalent in **reference frames** in uniform relative motion.
- The **(Vacuum) speed of light** is **c** in all such frames.

These two principles have profound implications for the structure and geometry of spacetime. It is easy to appreciate that in conventional Euclidian space it would be impossible to hold up in particular the second principle, the fact that the speed of light is the same for all frames. The geometry of inertial systems turns out not that of Euclidian space but that of **four-dimensional Minkowski spacetime**. The metric $\eta^{\alpha\beta}$ of Minkowski space is

$$\eta^{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{2}$$

so that distances between a point $x^\mu = (ct, x, y, z)$ and the origin $(0,0,0,0)$ is simply given by

$$s^2 = \eta_{\mu\nu} x^\mu x^\nu = c^2 t^2 - x^2 - y^2 - z^2 \,. \tag{3}$$

Einstein proved that two inertial systems are related by means of the Lorentz transformation. The Minkowski distance s^2 is invariant under the Lorentz transformation. Not only does the Lorentz transformation imply s^2 to be invariant, the same holds true for all scalars, including the velocity of light c.

When two intertial systems x^μ and x'^ν are moving with respect to each other with a velocity v in the x direction, the Lorentz transformation between them is

$$\begin{aligned} x' &= \gamma_v \, (x - \frac{v}{c}\, ct) \\ y' &= y \\ z' &= z \\ ct' &= \gamma_v \, (ct - \frac{v}{c}\, x) \end{aligned} \tag{4}$$

in which γ_v is the well-known Lorentz factor

$$\gamma_v = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} . \tag{5}$$

In proper tensor language the Lorentz transformation is a rank-2 tensor $\Lambda^{\mu}{}_{\nu}$, so that the transformation from the inertial system x^{ν} to a system x'^{μ} is expressed as

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} \, x^{\nu} . \tag{6}$$

As has been noted above, the Lorentz transform keeps the interval

$$c^2 d\tau^2 = c^2 dt^2 - \left(dx^2 + dy^2 + dz^2\right) \tag{7}$$

invariant. The significance of τ can be appreciated from considering two events at the same location, i.e. $dx = dy = dz = 0$. Then it measures the time interval between two events. It is therefore called the **proper time** of a moving system.

Naturally, also for a moving source τ corresponds to the time as it proceeds in its reference frame. Its relation to the time of the observer can be easily obtained from the inverse of expression 5, $cd\tau = \gamma_v^{-1} \, cdt$. **Proper time** is therefore given by the expression

$$\boxed{d\tau = dt \left(1 - \frac{v^2}{c^2}\right)^{1/2}} \tag{8}$$

Not only do the spacetime coordinates transform by means of the Lorentz transform, also all properly defined four-vectors A_{μ} do,

$$A'^{\mu} = \Lambda^{\mu}{}_{\alpha} \, A^{\alpha} . \tag{9}$$

and in general all well-defined tensors of arbitrary rank. The transformation of a tensor of rank 2 is for example

$$B'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha} \, \Lambda^{\nu}{}_{\beta} \, A^{\alpha\beta} . \tag{10}$$

Within the tensor formulation, the above Lorentz transform (eq. 5) tensor $\Lambda^{\mu}{}_{\nu}$ is given by

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & \beta\gamma_v & 0 & 0 \\ -\beta\gamma_v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{11}$$

By definition $\beta \equiv (v/c)$. If we wish to consider the more generic case of an arbitrary directed velocity $\mathbf{v}$, one has to distinguish between the spatial direction $\mathbf{x}_{\parallel}$ parallel to the velocity $\mathbf{v}$ and the perpendicular direction $\mathbf{x}_{\perp}$,

$$\mathbf{x} = \mathbf{x}_{\perp} + \mathbf{x}_{\parallel} . \tag{12}$$

Only the parallel component is influenced by the Lorentz transform,

$$\begin{aligned} ct' &= \gamma_v \left(ct - \frac{v}{c} \, x_{\parallel}\right) \\ \mathbf{x}' &= \mathbf{x}_{\perp} + \gamma_v \left(\mathbf{x}_{\parallel} - \frac{\mathbf{v}}{c} \, ct\right) , \end{aligned} \tag{13}$$

from which we may infer the general expression for the Lorentz transformation $\lambda^{\mu}{}_{\nu}$ from a frame x^{μ} to a frame x'^{ν} moving with a velocity $\mathbf{v} = (v_x, v_y, v_z)$ in an arbitrary direction,

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\frac{v_x}{c}\gamma_v & -\frac{v_y}{c}\gamma_v & -\frac{v_z}{c}\gamma_v \\ -\frac{v_x}{c}\gamma_v & 1+\frac{v_x^2}{v^2}(\gamma_v-1) & \frac{v_x v_y}{v^2}(\gamma_v-1) & \frac{v_x v_z}{v^2}(\gamma_v-1) \\ -\frac{v_y}{c}\gamma_v & \frac{v_y v_x}{v^2}(\gamma_v-1) & 1+\frac{v_y^2}{v^2}(\gamma_v-1) & \frac{v_y v_z}{v^2}(\gamma_v-1) \\ -\frac{v_z}{c}\gamma_v & \frac{v_z v_x}{v^2}(\gamma_v-1) & \frac{v_z v_y}{v^2}(\gamma_v-1) & 1+\frac{v_z^2}{v^2}(\gamma_v-1) \end{pmatrix} \tag{14}$$

Note that in the above expression we have assumed that the origin of frame x^{μ} did coincide with that of frame x'^{ν}.

4.1. Relativistic Mechanics: four-velocity

The Newtonian equations of motion are not **Lorentz-invariant** (they are **Galilean invariant**). It is therefore impossible to write them in a covariant form: upon a Lorentz transform they would change. To reformulate dynamics of a system such that it concerns a Lorentz-invariant formulation we need to find properly defined *four-vectors*. To do this we seek to define quantities which keep close to their conventional Newtonian relatives, to assure that asymptotically they keep their Newtonian form, but which at the same time transform in a proper covariant form.
The first such quantity is the **four-velocity** U^{μ}. The conventional velocity

$$\vec{u} = \frac{d\vec{x}}{dt}, \tag{15}$$

cannot be a proper four-vector, both $\vec{x}$ as well as t get transformed. However, because x^{μ} is a proper four-vector, and therefore also the differential dx^{μ}, the four-velocity U^{μ} defined according to

$$U^{\mu} \equiv \frac{dx^{\mu}}{d\tau}, \tag{16}$$

is indeed a proper covariant form while retaining close to the original idea of velocity. In this definition the proper time τ (eqn 8), invariant under a Lorentz transformation, is used instead of the time t. From this definition of the four-velocity we can easily see that

$$\begin{aligned} U^0 &= \frac{dx^0}{d\tau} = \frac{c\,dt}{d\tau} = c\,\gamma_u \\ U^i &= \frac{dx^i}{d\tau} = \frac{dx^i}{dt}\frac{dt}{d\tau} = \gamma_u u^i \end{aligned} \tag{17}$$

so that the relativistic four-velocity U^{μ} is

$$\boxed{U^{\mu} = \gamma_u \begin{pmatrix} c \\ \vec{u} \end{pmatrix}} \tag{18}$$

The length of the four-velocity U^{μ} has a value equal to

$$U^{\mu}U_{\mu} = -(\gamma_u c)^2 + (\gamma_u \vec{u})^2 = -\gamma_u^2 c^2 \left(1 - \frac{u^2}{c^2}\right) = -c^2, \tag{19}$$

evidently a Lorentz-invariant quantity.

4.2. Relativistic Mechanics: four-momentum

Following up on the definition of the relativistic four-velocity, we define the **relativistic four-momentum** in an analogous four-vector form, retaining close to its asymptotic Newtonian form. For a particle with rest mass m_0 (a scalar, and therefore Lorentz-invariant), we have

$$\boxed{P^\mu \equiv m_0 \, U^\mu} \tag{20}$$

Note that in the non-relativistic limit P^μ behaves like

$$P^\mu = m_0 \, U^\mu = m_0 \, \gamma_u \begin{pmatrix} c \\ \vec{u} \end{pmatrix} \approx m_0 \begin{pmatrix} c \\ \vec{u} \end{pmatrix} . \tag{21}$$

In other words, the spatial components $\vec{P}$ in the nonrelativistic limit are precisely equal to the ordinary three-momentum of Newtonian mechanics, $\vec{P} = m_0 \, \vec{u}$. In the relativistic limit, the spatial components of the four-momentum are equal to $\vec{P} = m_0 \vec{U}$. This brings us to the definition for the **relativistic spatial momentum** $\vec{p}$,

$$\vec{p} \equiv m_0 \gamma_u \, \vec{u} \, . \tag{22}$$

To settle the identity of the (0) component of the four-momentum we expand the expression for P^0,

$$P_0 c = m_0 c \, U^0 = m_0 c^2 \left(1 - \frac{v^2}{c^2}\right)^{-1/2} = m_0 c^2 + \frac{1}{2} m_0 v^2 + \ldots . \tag{23}$$

The second term is the nonrelativistic expression for the kinetic energy of the particle. The implication is that $P_0 c$ should be interpreted as the total energy E of the object. In summary, the four-momentum's identity is

$$P^\mu = \begin{pmatrix} E/c \\ \vec{p} \end{pmatrix} . \tag{24}$$

From the consideration of the Lorentz-invariant length of the four-momentum,

$$P^\mu P_\mu = m_0^2 U^\mu U_\mu = -m_0^2 c^2 = -\frac{E^2}{c^2} + |\vec{p}|^2 \tag{25}$$

one is therefore lead to the expression for the **relativistic energy**,

$$\boxed{E^2 = m_0^2 c^4 + c^2 \, |\vec{p}|^2} \tag{26}$$

Undoubtedly, this is one of the most important relations obtained by the theory of relativity. Its implications far surpassing anything Einstein could have foreseen in 1905. In particular the energy a particle has when it is at rest has caught the imagination of a large fraction of the world,

$$\boxed{E = m_0 c^2} \tag{27}$$

It is no exaggeration to say that this has become THE equation of the 20^{th} century, a symbol of human progress. But also a symbol of human failure and agression ...

5. Principle of Equivalence

The fundamental principle, or observation, underlying Einstein's theory of General Theory of Relativity is the **Equivalence Principle**. After some pondering, in 1907 he concluded that *free-falling frames* were entirely equivalent to *inertial frames* in terms of applicability of all physical laws. According to his own statement, "all local, freely falling, non-rotating laboratories are fully equivalent for the performance of all physical experiments". In general, we may express the (strong) equivalence principle as follows:

> The physics in the frame of a freely falling body
> is equivalent to that of
> an inertial frame in Special Relativity.

In other words, at any point *in a gravitational field*, in a frame of reference moving with the *free-fall acceleration* at that point, all **laws of physics** are entirely equivalent and have their usual **Special Relativistic**. This except the force of gravity, which disappears locally. For every conceivable physical experiment, special relativity will therefore apply to both inertial and free-falling frames. The consequences of this observation are momentous.

Recall that in the theory of Special Relativity the special relativistic descriptions of physical phenomena apply exclusively to **inertial frames**. These are the frames corresponding to the shortest distances between two events $\xi^\mu \equiv (ct, x, y, z)$ in Minkowski spacetime, i.e. frames corresponding to *"straight"* worldlines in Minkowski spacetime. Evidently, accelerated frames do *not* have straight worldlines in Minkowski space, they are *curved* !

By stating that Special Relativity applies also in free-falling frames, the *principle of equivalence* should be understood in a similar way as the special relativistic *principle of relativity.* While the worldlines of accelerated frames are *curved* in Minkowski spacetime, they are **"straight"** in **curved spacetime**. The consequence of the equivalence principle is that in a **curved spacetime** the paths of a free-falling body is **"straight"**. In other words, the paths of free-falling bodies are straight lines, **geodesics**, in curved spacetime. This assures that **locally**, at any point along their path, the geometry remains equivalent to Minkowski spacetime. Thus, on the basis of the principle of equivalence we find that

> Free-falling bodies follow straight worldlines
> – **geodesics** –
> in curved spacetime

To understand and specify this we need to resort to concepts from differential geometry. Indeed, it was not before Einstein's friend Marcel Grossman had taught Einstein about differential geometry that he was able to mould his physical insights into the proper theoretical formulation of his Theory of General Relativity.

5.1. Time Dilation and Gravitational Redshift

An immediate consequence of the *principle of equivalence* is the effect of **Gravitational Time Dilation**. Imagine an object that fell into a gravitational potential field ϕ from infinity. Starting from a position of rest it thus acquired a velocity v given by

$$\frac{1}{2}v^2 = -\phi(\mathbf{x})\,. \tag{28}$$

Because of the *equivalence principle* the free-falling object should be treated as if it would concern an inertial system. We can therefore also use the relation for the special relativistic time dilation for determining the resulting *gravitational time dilation.* The time interval Δt_0 measured by the observer at infinity is then simply related to the time interval Δt of the object in the gravitational field,

$$\Delta t_0 = \frac{\Delta t}{\sqrt{1+\frac{2\phi}{c^2}}}\,, \tag{29}$$

in which we replaced v^2 simply by -2ϕ, according to eqn. 28. Expression 29 is the time dilation which we observe of an object embedded in an gravitational field. If it would clock a time interval $\Delta t'$ it would occur to take a longer time Δt_0 to the outside observer ($\phi < 0$).

Closely related to the *gravitational time dilation* is the resulting **gravitational redshift** of radiation that was emitted from a gravitational field. When the source in the field ϕ emits radiation with a period Δt it has a frequency $\nu = 1/\Delta t$, while the observed frequency is equal to $\nu_0 = 1/\Delta t_0$. From eqn. 29 we may simply infer that this implies that

$$\boxed{\nu_0 = \nu\sqrt{1 + \frac{2\phi}{c^2}}}$$

which for a weak gravitational field, $\phi/c^2 \ll 1$, becomes

$$\nu_0 \approx \nu\left(1+\frac{\phi}{c^2}\right)\,. \tag{30}$$

The above relations specify how the frequency of electromagnetic radiation depends on the gravitational potential in which they propagate. Also, we see that if two points are a potential difference $\Delta\phi$ apart their corresponding relative gravitational redshift is equal to:

$$\Delta\nu \approx \frac{\Delta\phi}{c^2}\,, \tag{31}$$

so that the corresponding **gravitational redshift** z_{grav} ($= 1 - \nu_1/\nu_2$) amounts to

$$z_{grav} = \frac{\Delta\phi}{c^2}\,. \tag{32}$$

In most generic circumstances the gravitational redshift of light is a tiny effect. Within the earth's gravitational field the effect is only of the order $z_g \sim 2.5 \times 10^{-5}$.

Nonetheless, the effect is of utmost importance for our ability to study the structure of the Universe at the epoch of recombination: the temperature fluctuations in the microwave background are often a result of the effect of the gravitational potential perturbations at the surface of last scattering.

6. Metric

The most fundamental concept for the geometry, i.e. curvature, of any generally curved space is the **metric** of a reference frame x^μ. Locally, the curved geometry is specified by the **metric tensor** $g_{\alpha\beta}$. It specifies the *distance between points* in any *general curved space* for the reference frame x^α,

$$ds^2 = g_{\alpha\beta}\,dx^\alpha\,dx^\beta\,. \tag{33}$$

In the special case of Minkowski spacetime, for an inertial frame $\xi^\mu = (ct, x, y, z)$, the metric tensor is usually denoted by $\eta_{\alpha\beta}$. It is given by

$$\eta_{\alpha\beta} = (1, -1, -1, -1) . \tag{34}$$

On the basis of the *principle of equivalence* one may infer, rather straightforwardly, that free falling bodies follow straight worldlines. First one may derive that the motion of such a body is described by the **general relativistic equation of motion**,

$$\frac{d^2 x^\beta}{d\tau^2} + \Gamma^\beta_{\lambda\nu} \frac{dx^\lambda}{d\tau} \frac{dx^\nu}{d\tau} = 0 . \tag{35}$$

In this equation the quantity τ is the **proper time** in general relativistic spacetime,

$$d\tau \equiv dt \sqrt{1 + \frac{2\phi}{c^2}} , \tag{36}$$

measuring the time interval between two events at the same spatial location (within the frame). The crucial difference with the familiar equation of motion for an *inertial frame*,

$$\frac{d^2 x^\beta}{d\tau^2} = 0 . \tag{37}$$

is the righthand term. This involves the **Christoffel symbol** $\Gamma^\alpha_{\beta\gamma}$, (also called the *affine connection*) is given by,

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\nu} \left\{ \frac{\partial g_{\gamma\nu}}{\partial x^\beta} + \frac{\partial g_{\beta\nu}}{\partial x^\gamma} - \frac{\partial g_{\gamma\beta}}{\partial x^\nu} \right\} , \tag{38}$$

specifying the effect of the spatial variation of the geometry of space, i.e. its curvature.

7. Geodesic Equation

The intrinsic beauty of the theory of relativity may be appreciated from the fact that the above **General Relativistic Equation of Motion** is exactly equal to the geometric concept of the **Geodesic Equation**,

GENERAL RELATIVISTIC EQUATION OF MOTION

=

GEODESIC EQUATION

The geodesic equation specifies the **shortest path** a body can take in a general **curved space** (i.e. in 4-dimensional space-time).

$$\boxed{\frac{d^2 x^\beta}{d\tau^2} + \Gamma^\beta_{\lambda\nu} \frac{dx^\lambda}{d\tau} \frac{dx^\nu}{d\tau} = 0}$$

We have therefore arrived at the fundamental observation that there is a crucial connection between the dynamics in the theory of General Relativity and the geometry of curved spacetimes: the movement of free-falling bodies is equivalent to following the shortest path on a surface in curved spacetime.

Notice that the effect of the curvature of space is comes in via the derivative of the metric $g_{\alpha\beta}$. It is therefore easy to see that for the Minkowski metric, constant throughout space, we recover the familiar equation of motion for an inertial frame (eqn. 37).

7.1. Equation of Motion: Weak Field Limit

To appreciate the physical significance of the metric $g^{\mu\nu}$, it is highly instructive to study the relativistic equation of motion in the socalled **weak field limit**.

In this asymptotic limit we assume that the gravitational field ϕ is weak and stationary. The latter means that the potential and the metric do not change very much in time. Moreover, we take the situation in which the body is moving slowly. Explicitly, these conditions are:

- weak field: $\frac{\Delta\phi}{c^2} \ll 1$
- stationary: $\frac{\partial g_{\mu\nu}}{\partial x^0} = 0$
- slowly moving: $\frac{\dot{x}^k}{c} \ll 1$

For the spatial k component of the geodesic equation (eq. 35) we have,

$$\frac{d^2x^k}{d\tau^2} + \Gamma^k{}_{00} \frac{dx^0}{d\tau}\frac{dx^0}{d\tau} \approx 0 \,. \tag{39}$$

where we have used the fact that $\dot{x}^k/c \ll 1$ in order to ignore the factors including $\dot{x}^k/c$. The assumption of *stationarity* allows us to evaluate $\Gamma^k{}_{00}$:

$$\begin{aligned} \Gamma^k{}_{00} &= \frac{1}{2} g^{k\nu} \left\{ \frac{\partial g_{0\nu}}{\partial x^0} + \frac{\partial g_{0\nu}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^\nu} \right\} \\ &\approx -\frac{1}{2} g^{k\nu} \frac{\partial g_{00}}{\partial x^\nu} \approx -\frac{1}{2} g^{kk} \frac{\partial g_{00}}{\partial x^k} \approx \frac{1}{2}\frac{\partial g_{00}}{\partial x^k} \end{aligned} \tag{40}$$

where we have used $g^{kk} \approx \eta^{kk} = -1$ and $g^{kj} \approx 0$ $(k \neq j)$. The resulting relation is

$$\frac{d^2x^k}{d\tau^2} = -\frac{1}{2}\left(\frac{dx^0}{d\tau}\right)^2 \frac{\partial g_{00}}{\partial x^k} \,, \tag{41}$$

leaving us with the equation of motion for the weak field limit,

$$\boxed{\ddot{x}^k \approx -\frac{1}{2} c^2 \vec{\nabla} g_{00}} \tag{42}$$

The weak field limit of course corresponds to the familiar Newtonian limit. In this asymptotic limit thethe equation of motion we obtained above should therefore be identified with the regular Newtonian equation of motion in a gravitational field,

$$\ddot{x}^k = -\vec{\nabla}\phi \,. \tag{43}$$

The final conclusion is that on the basis of the weak field limit of the geodesic equation we have found a physical interpretation for the (00)-component of the metric tensor $g^{\mu\nu}$. It is nothing less than the gravitational potential ϕ,

$$\boxed{g_{00} = \frac{2\phi}{c^2}} \tag{44}$$

The identification of g_{00} will prove to be a very useful relation for further analyses. Far more important though is that it provides us with a direct link between the gravitational field and the geometry of space. Exactly that what General Relativity is all about !

8. Covariant Derivatives

Before we can make any further progress, we need to pay some more attention to a fundamental operation in the description of nature and the scripting of its governing physical laws. In nearly all physical theory we are comparing physical quantities of a system at different locations and different times. This is subsequently drafted in terms of *derivatives*, involving the comparison in infinitesimal intervals. One would think the issue a rather straightforward one. However, in the General Theory of Relativity the issue of derivatives is not an entirely trivial one and we need to devote more specific attention to the difference between *regular derivatives* and *covariant derivatives*.

In relativity we employ a shortcut in the notation for a regular derivative of a quantity F with respect to a coordinate x^μ,

$$F_{,\nu} \equiv \frac{\partial F}{\partial x^\mu} \,. \tag{45}$$

Within a rigid Newtonian spacetime or in "flat" special relativistic Minkowski space this is a perfectly well-defined operation. Derivative quantities like

$$B^\mu \equiv \frac{\partial A^\mu}{\partial x^\nu} \tag{46}$$

are themselves well-defined **tensor** quantities which transform nicely from one inertial reference system into another:

$$\tilde{A}^\mu{}_{,\nu} = \frac{\partial \tilde{x}^\mu}{\partial x^\alpha} A^\alpha{}_{,\nu} \tag{47}$$

In other words, expressions for a physical law involving derivatives will transform without a problem into the same tensor expression in another inertial reference system.

However, in a generally curved space this is no longer true. The concept of derivative needs further attention. We need to define more careful what we mean with *comparison* of physical quantities at different locations. When moving a quantity over a curved space something more changes then just the "intrinsic" change $F_{,\nu}$: we need to specify what the influence is of the path along curved space for the change of the quantity. This brings us to the concept of **parallel displacement**: the quantity A^α at one location x_1^μ is moved to the other location x_2^μ before one can make the comparison, i.e. the proper derivative. In a flat space, parallel displacement would not induce any change. In a curved space it does, and the effect enters, perhaps not unexpected, via the *Christoffel symbol.* A vector A^α moved parallel along a path λ, in a general reference system x^μ, transforms into

$$A^\alpha(\lambda + d\lambda, \lambda) = A^\lambda - \Gamma^\alpha_{\beta\gamma} A^\beta \, dx^\gamma \,. \tag{48}$$

The effective derivative is that corresponding to the differential between the transported vector $A^\alpha(\lambda + d\lambda, \lambda)$ and the local vector $A^\alpha(\lambda + d\lambda)$. This leads to the definition for the **covariant derivative**,

$$\frac{DA^\alpha}{d\lambda} = \frac{dA^\alpha}{d\lambda} + \Gamma^\alpha_{\beta\gamma} \, A^\beta \, \frac{dx^\gamma}{d\lambda} \,. \tag{49}$$

From this we can easily infer the relation between the **covariant derivative** $A^\alpha{}_{;\nu}$ and the **regular derivative** $A^\alpha{}_{,\nu}$,

$$A^\alpha{}_{;\nu} \equiv A^\alpha{}_{,\nu} + \Gamma^\alpha_{\beta\gamma} \, A^\beta \,. \tag{50}$$

Note that we have introduced the index notation $(;\nu)$ as a shorthand for a covariant derivative. For the covariant derivative of the **covariant component** A_μ of the vector A we have

$$A^{\mu\,;\nu} \equiv A_{\mu\,,\nu} - \Gamma^\alpha_{\mu\gamma} \, A_\alpha \,. \tag{51}$$

It is important to realize that the covariant derivative $A_{\mu\,;\nu}$ is a genuine tensor, unlike the regular derivative $A_{\mu\,,\nu}$, so that any expression involving this quantity will also hold in any reference frame.

As an interesting illustration of the significance of the concept of covariant derivatives, one can turn to the geodesic equation, the equation of motion for a free-falling body. Using the expression for the four-velocity U^μ of a body (eqn. 16) in combination with the geodesic equation Eqn 35 leads to the observation that

$$\frac{d}{d\tau}U^\mu + \Gamma^\mu_{\alpha\beta}\,U^\alpha\,\frac{dx^\beta}{d\tau} = 0\,, \tag{52}$$

which is nothing more than the statement that the covariant derivative of the four-velocity U^μ is equal to zero (see eqn. 49),

$$\frac{DU^\mu}{d\tau} = 0\,. \tag{53}$$

In other words, the equation of motion under the force of gravity is simply that the *covariant derivative of the four-velocity* disappears. And as

As an illustration of the use of fact that covariant derivatives are tensors consider the covariant derivative of the metric tensor $g^{\mu\nu}$. It is easy to see that in a *local inertial frame* $g^{\mu\nu}{}_{;\nu} = \eta^{\mu\nu}{}_{;\nu} = 0$. Because it involves a tensor relation, we find that in any arbitrary reference always

$$g^{\alpha\beta}{}_{;\beta} = 0\,. \tag{54}$$

This relation is a very useful one, returning in variety of contexts of General Relativity.

9. Energy-Momentum Tensor

We have seen that in General Relativity we have the *geodesic equation* (eqn 35) as equation of motion. The significance was that bodies move along the shortest paths in curved space, the geodesics. Earlier we noticed that Einstein had remarked that the curvature of spacetime is set by its material content. We therefore need to assess the proper General Relativistic description of the matter (i.e. energy) content of spacetime. In this section we will see that the 2-rank *energy-momentum tensor* is the key concept.

In relativity **energy** and **momentum** are intimately linked physical quantities, as both are components of the **energy-momentum** four-vector P^μ,

$$P^\mu = m_0\,U^\mu = m_0\,\gamma_u \begin{pmatrix} c \\ \vec{u} \end{pmatrix} \approx m_0 \begin{pmatrix} c \\ \vec{u} \end{pmatrix} \tag{55}$$

in which U^μ is the *four-velocity* of an object with ordinary velocity $\vec{u} = d\mathbf{x}/dt$ (see eqn. 16).
Relativity is concerned with formulating physical laws and relations in a coordinate-free form, ie. in a **covariant** form. We therefore look like formulations in terms of tensor equations that are valid in any reference frame. For the energy-momentum four-vector we therefore look for the equivalent of the Newtonian fluid equations, expressing the conservation of mass, energy and momentum. If we work out the relativistic equivalents of the Newtonian fluid equations, we end up with the tensor equations,

$$T^{\mu\nu}{}_{;\nu} = 0 \tag{56}$$

in which the tensor $T_{\mu\nu}$ is the **energy-momentum tensor**,

$$\boxed{T_{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) U^\mu U^\nu - p g^{\mu\nu}} \tag{57}$$

In the equation above the term ρ concerns the density of matter (and energy) and p the pressure of the medium. The pressure term enters this expression via the equation for transport of momentum. Note that in the *restframe* the energy-momentum tensor works out as:

$$T^{\mu\nu} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \tag{58}$$

which we will notice later is a particularly insightful expression within the context cosmology (matter and energy following the Hubble expansion are in rest with respect to the expanding spacetime of the Universe).

In a cosmological context we also often consider a pressureless medium, denoted by the term of "dust". Take care not to confuse this with dust in another astronomical context, that of the dust in the interstellar medium. For an appreciation of the concept of the energy-momentum tensor it is quite insightful to consider the expressions for the components of the *energy-momentum* tensor for such a "dust" medium with a velocity $\vec{v}$,

$$\begin{aligned} T^{00} &= \rho c^2 \gamma_v^2 \\ T^{0i} &= \rho c \gamma_v^2 v_i \\ T^{i0} &= \rho c \gamma_v^2 v_i \\ T^{ij} &= \rho \gamma_v^2 v_i v_j , \end{aligned} \tag{59}$$

in which γ_v is the well-known Lorentz factor (see eq. 5). Having these explicit expressions for the "dust" energy-momentum tensor, we can observe that for an inertial frame with $v \ll c$ we should recover the Newtonian fluid equations (i.e. the *continuity equation* and the *Euler equation*),

$$\begin{aligned} \partial_0(\rho c) + \partial_i(\rho v_i) &= 0 \\ \partial_0(\rho c v_i) + \partial_j(\rho v_i v_j) &= 0 \end{aligned} \tag{60}$$

In this inertial frame we therefore find that these equations are equivalent to the tensor equality, $T^{\mu\nu}{}_{,\nu} = 0$. Because tensor equalities are equally valid in inertial as in free-falling frames we arrive at the observation that for a "dust" medium the energy-momentum tensor will always obey the relation

$$T^{\mu\nu}{}_{;\nu} = 0 . \tag{61}$$

This finding is of fundamental importance. It basically contains *all you always wanted to know about "dust physics"*. More significantly, it is the one relation that allows us to establish the link between the energy-momentum content of space and curvature. Not until after making a final remark that Eqn. 61 is not only true for a "dust" medium but in general for any medium with energy-momentum tensor $T^{\mu\nu}$ specified by Eqn. 57 we therefore return to the description of curvature.

10. Curvature Tensors

Have managed to specify the contents of space by means of the energy-momentum tensor, we need to return to the question of the proper encryption of the curvature of space. Once this has been clarified we are set for establishing the profound link between the content of spacetime and its curvature.

Curvature of a space might be observed in a variety of ways. It would be rather trivial if we were able to move out of the space in which we live and look from the outside. A telling example of such an approach is measuring the surface of the earth when looking down from a space ship, or making a map from a mountain range on the basis of airplane photography. However, it is an entirely different exercise when one has to stay on the Earth's surface, or are constrained to merely walking around the mountains. One may understand the complexity of the problem by realizing that it took humanity centuries, or even millennia, before it was accepted that the Earth was not flat but a sphere. Apparently, some effort was needed to prove this. The question therefore arises how one may infer and characterize the curvature of a space when one is constrained to living within this space. A good starting point is to look for local evidence. This is indeed what Johann Gauss set out to do.

The solution suggested by Gauss was an ingeneous one, at the same time a rather simple one (i.e. at hindsight). He proved that one can describe curvature by following a vector along a path in a curved space. By *parallel displacement* of the vector A^α along the path, one would find upon return at the initial position a total *parallel displacement* ΔA^γ. The displacement of the vector A^γ along a closed path corresponding to small coordinate intervals Δx^γ is

$$\Delta A^\gamma \,=\, R^\gamma_{\ \alpha\beta\delta}\, A^\alpha\, \Delta x^\beta\, \Delta x^\delta\,, \tag{62}$$

in which the four-rank tensor $\mathbf{R}^\mu_{\ \alpha\beta\gamma}$ is the **Riemann tensor**. For a closed macroscopic loop the contributions by small loops filling up the large loop have to be added, yielding a total *parallel displacement*

$$\Delta A^\gamma \,=\, \frac{1}{2}\, R^\mu_{\alpha\lambda\beta}\, A^\lambda\, \oint \left(\Delta x^\beta dx^\alpha \,-\, \Delta x^\alpha dx^\beta\right) \tag{63}$$

The *Riemann tensor* is one of the most fundamental concepts in *differential geometry*. It is defined by

$$R^\mu_{\ \alpha\beta\gamma} \,=\, \Gamma^\mu_{\ \alpha\gamma,\beta} - \Gamma^\mu_{\ \alpha\beta,\gamma} + \Gamma^\mu_{\ \sigma\beta}\Gamma^\sigma_{\ \gamma\alpha} - \Gamma^\mu_{\ \sigma\gamma}\Gamma^\sigma_{\ \beta\alpha}\,, \tag{64}$$

and is the one fundamental way of directly and locally measuring curvature. It is rather straightforward to see that $R^\mu_{\ \alpha\beta\gamma} = 0$ for a flat space.

From the *Riemann tensor* one may derive a few related quantities, tensors of lower rank, by contraction over indices of the Riemann tensor. By contracting on the 1^{st} and 3^{rd} index, we obtain the **Ricci tensor**,

$$R_{\alpha\beta} \,\equiv\, R^\mu_{\ \alpha\mu\beta} \,=\, R_{\beta\alpha}\,. \tag{65}$$

While other contractions of the Ricci tensor are possible, on the basis of (anti)symmetries one may probe that they either vanish or may be reduced to the Ricci scalar. On the basis of the *Ricci tensor* we may then infer the **curvature scalar**, also known as **Ricci scalar**,

$$R \,\equiv\, g^{\mu\nu}\, R_{\mu\nu} \,=\, g^{\mu\nu}\, g^{\alpha\beta}\, R_{\alpha\mu\beta\nu}\,, \tag{66}$$

which appeals closely to our imagination of curved surfaces. We would recognize it as the radius of curvature of a spherical surface.

By using intrinsic symmetries of the *Riemann tensor*, the socalled **Bianchi Identities**,

$$R_{\alpha\beta\mu\nu;\lambda} \,+\, R_{\alpha\beta\lambda\nu;\nu} \,+\, R_{\alpha\beta\nu\lambda;\mu} \,=\, 0 \tag{67}$$

one may prove, as Einstein did, that there is one, and only one, curvature related tensor of rank 2 $G^{\alpha\beta}$ for which

$$G^{\alpha\beta}_{\ \ ;\beta} \,=\, 0\,. \tag{68}$$

Tellingly, this tensor is called the **EinsteinTensor**,

$$\boxed{G^{\alpha\beta} \,=\, R^{\alpha\beta} \,-\, \frac{1}{2}\, g^{\alpha\beta}\, R} \tag{69}$$

Having observed this unique property, we are nearing awfully close to the ultimate synthesis made by Albert Einstein.

11. the Einstein Field Equations

How did Einstein arrived at his final synthesis ? He noted that he had identified two rank-2 tensors for which their contracted covariant derivative is equal to zero,

$$\begin{aligned} G^{\alpha\beta}{}_{;\beta} &= 0 \\ T^{\alpha\beta}{}_{;\beta} &= 0 \end{aligned} \tag{70}$$

Subsequently, he made the stout jump, a stroke of genius, to realize that this meant nothing else but that the Einstein tensor $G^{\alpha\beta}$ should be equal to the energy-momentum tensor $T^{\alpha\beta}$, i.e.

$$\boxed{G^{\alpha\beta} \propto T^{\alpha\beta}} \tag{71}$$

One can do nothing else but marvel at what had happened here. Indeed, as you may read from the highly recommendable book by Abraham Pais, “Subtle is the Lord”, the scientific biography of Albert Einstein, upon realizing this breakthrough in his mind for three days Einstein was in the highest state of excitement !!! One may only dare to imagine how one would feel by going back in one’s memory to those rare moments one suddenly realizes the solution to a nagging science or maths problem. I guess anyone of us has an experience of such a “minor” breakthrough ... Undoubtedly, it concerned one of the greatest achievements in humanity’s intellectual and scientific history.

What he had achieved is to find the connection between the curvature of space and the “content” of space, and by one stroke of genius he had transformed spacetime into a dynamic medium:

SPACETIME REACTS TO CONTENT OF THE UNIVERSE
$\Longleftrightarrow$
CONTENT OF UNIVERSE REACTS TO CURVATURE

Having made the bold step describe above, and having established that $R^{\alpha\beta} \propto T^{\alpha\beta}$, the constant of proportionality between curvature and energy-momentum tensor should be computed.

The proportionality constant may be inferred in a similar way as followed in establishing the link between gravitational potential ϕ and metric tensor $g^{\alpha\beta}$ (see eq. 44). By taking the limit of the non-relativistic **Newtonian** regime, Einstein firmly linked his General Theory of Relativity to Newtonian physics. In other words, he assumed Newton’s theory was in fact an asymptotic limit of the more general theory of gravity he was formulating. We will follow his reasoning partially, and restrict ourselves to a pure *dust* medium. Formulated explicitly the assumptions are

- weak field: $\dfrac{\phi}{c^2} \ll 1$
- stationary: $\dfrac{\partial g_{\mu\nu}}{\partial x^0} = 0$

- dust: $p \ll \rho c^2$

The only nonzero component of the energy-momentum tensor in this regime is T^{00},

$$T^{00} = \rho c^2 \,, \tag{72}$$

while for the pure spatial part $T^{ij} = 0$. By implication $G^{ij} = 0$, so that $R^{ij} = \frac{1}{2} g^{ij} R$. This allows us to determine R: $R = R^0{}_0 + R^i{}_i$. Because $R^i{}_j = \frac{1}{2} g^i{}_j R$, and in this regime $g^i{}_j \approx \delta^i{}_j$, $R^i{}_i = \frac{3}{2}$R. In addition, $R^0{}_0 = R_{00}$, so that $R = -2R_{00}$, yielding

$$G^{00} = 2R^{00} \,. \tag{73}$$

Somewhat more challenging is the determination of R^{00}. Because in the Newtonian limit the Christoffel symbols are all small, $\Gamma \ll 1$, the higher order terms $\Gamma\Gamma$ in the Riemann tensor may be neglected (see eq. 64, leaving the Ricci tensor

$$R_{\alpha\beta} = \frac{\partial \Gamma^\mu{}_{\alpha\mu}}{\partial x^\beta} - \frac{\partial \Gamma^\mu{}_{\alpha\beta}}{\partial x^\mu} \tag{74}$$

Because of the approximation of stationarity, we then find that $R_{00} = -\Gamma^i{}_{oo,i}$. Earlier, in equation 41, we had found that in the limit of a weak field and stationary conditions that

$$\Gamma^k{}_{00} \approx \frac{1}{2} \frac{\partial g_{00}}{\partial x^k} = \frac{1}{c^2} \frac{\partial \phi}{\partial x^k} \tag{75}$$

where we have used the relation between metric and potential, $g_{00} = 2\phi/c^2$ (eqn. 44). This implies that

$$R_{00} = -\Gamma^k{}_{00,k} = -\frac{1}{c^2} \nabla^2 \phi \,. \tag{76}$$

The above equation should call up some vague memories of Newton days: the **Poisson Equation** specifies the relation between the density ρ of a medium and the induced *gravitational potential* ϕ,

$$\nabla^2 \phi = 4\pi G \, \rho \,. \tag{77}$$

Because $T^{00} = \rho c^2$ (see eqn. 72) and $G^{00} = 2R^{00}$, we find that

$$G^{00} = -\frac{8\pi G}{c^4} T^{00} \,, \tag{78}$$

on the basis of which we have identified the constant of proportionality between the Einstein tensor and the energy-momentum tensor. And therefore, esteemed students of the cosmos, we have arrived at what should be rightfully considered a work of art, the Einstein Field Equations. The equations of nature ruling the structure, evolution and fate of our own world, the Universe.

$$\boxed{R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R = -\frac{8\pi G}{c^4} T^{\alpha\beta}} \tag{79}$$

In my view it is no exaggeration to state that this equation embodies amongst the highest and most supreme of intellectual and artistic (!) achievements of human civilization, a triumph of human genius. And by many measures perhaps also the most beautiful and impressive equation of science and nature known to humankind. And, in my personal view, of a beauty comparable to only the very most precious works of art ever. Rightfully in line with the cave paintings of Lascaux, Phidias' Parthenon, Michelangelo's Piéta, his David and the "Creation" on the Sistine Chapel ceiling, as well as Raffaello's School of Athens.

12. Relativistic Cosmology: Pressure

In the above we have come to appreciate the fact that the curvature of spacetime is an essential component of the theory of gravity described by the General Theory of Relativity. In addition, there are a few other significant contributions to relativistic gravity which we do not encounter in a Newtonian context. Implicitly, they are of course already included in the Einstein field equations. It is nonetheless instructive to explore explicitly the influence of both **pressure** and a socalled **cosmological constant** on the dynamics of spacetime. As both issues are of utmost importance within a cosmological context, we devote special attention to them in this and the following section.

First, we investigate the issue of pressure. In the definition of the *energy-momentum* tensor, eqn. 57, we have already seen that it contains a contribution by the *pressure* of the medium. The Einstein field equations (eqn. 79) therefore tell us that pressure will influence the geometry of spacetime and therefore plays a role in the resulting force of gravity.

To elucidate this role, we will first derive an alternative formulation of the Einstein field equation. First some algebraic manipulation: multiplying the Einstein tensor $G^{\mu\nu}$ by the metric tensor $g_{\alpha\beta}$ and subsequently contracting over both the first and second index yields the relation

$$g_{\mu\nu} R^{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\mu\nu} R = -\frac{8\pi G}{c^4} g_{\mu\nu} T^{\mu\nu} . \tag{80}$$

The result is a relation between the Ricci scalar R and T,

$$R = \frac{8\pi G}{c^4} T , \tag{81}$$

where we have defined the energy-momentum scalar $T \equiv g_{\mu\nu} T^{\mu\nu}$ and used the fact that $g^{\mu}_{\mu} = 4$ (note that this is a scalar and that in the restframe and $g^{\mu}_{\mu} = \eta^{\mu}_{\mu} = 4$). This leads to the alternative formulation of the Einstein field equations,

$$\boxed{R^{\alpha\beta} = -\frac{8\pi G}{c^4} \left(T^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} T \right)} \tag{82}$$

The curvature term now only consists of the Ricci tensor $R^{\alpha\beta}$ itself. The energy-momentum term has been expanded and includes the extra factor T. Apparently the Einstein field equations do allow some measure of flexibility in specifying the contributions by the curvature and energy-momentum content.

To pursue the argument, we seek to obtain an expression for T. To this end, we should realize it is a scalar and therefore independent of the reference frame in which it is measured. We may therefore evaluate T in the restframe. For the restframe the explicit expression for the energy-momentum tensor $T^{\mu\nu}$ has been specified in eqn. 58, from which we can immediately infer that

$$T = \rho c^2 - 3p . \tag{83}$$

If we concentrate on the "(00)" component of the Einstein field equation, eqn. 82, and use the fact that $\nabla^2 \phi = -c^2 R_{00}$ (eqn. 76), we find that

$$\begin{aligned} R_{00} &= -\frac{8\pi G}{c^4} \left(T_{00} - \frac{1}{2} g_{00} T \right) \\ &= -\frac{8\pi G}{c^4} \left(T_{00} - \frac{1}{2} \left\{ \rho c^2 - 3p \right\} \right) \\ -\frac{1}{c^2} \nabla^2 \phi &= -\frac{8\pi G}{c^4} \left(\rho + \frac{3p}{c^2} \right) , \end{aligned} \tag{84}$$

where we have also used the fact that in the restframe $T_{00} = \rho c^2$ (see eqn. 58) and $g_{00} \approx 1 + \mathcal{O}(\phi/c^2)$. In summary, we find the following relation between gravitational potential ϕ and the content of the Universe,

$$\boxed{\nabla^2 \phi \,=\, 4\pi G \left(\rho \,+\, \frac{3p}{c^2}\right)} \tag{85}$$

In other words, in addition to the regular density term ρ known from Newtonian gravity the Poisson equation for the gravitational potential ϕ contains an extra pressure term. This is a fundamental difference between Newtonian gravity and Relativistic gravity, with tremendous repercussions for cosmological evolution. We will soon appreciate it is precisely the factor $(\rho + 3p/c^2)$ which will enter the Friedmann equations for cosmological evolution.

13. Relativistic Cosmology: Cosmological Constant

A third major difference between Newtonian and Relativistic gravity, in addition to the role of *spacetime curvature* and *pressure*, is the fact that the equations of gravity allow for the presence of an elusive constant which has become known as the **Cosmological Constant**.

To appreciate its providence we should have a closer look at the Einstein field equations,

$$G^{\mu\nu} \,=\, -\frac{8\pi G}{c^4}\, T^{\mu\nu}\,. \tag{86}$$

Einstein established this equality between $G^{\mu\nu}$ and $T^{\mu\nu}$ on the fact that for both tensors

$$G^{\mu\nu}{}_{;\nu} \,=\, T^{\mu\nu}{}_{;\nu} \,=\, 0\,. \tag{87}$$

In fact, the Einstein tensor $G^{\mu\nu}$ is the only rank 2 curvature related tensor for which this relation holds. However, we have also noted (eqn. 54) that also the metric tensor $g^{\mu\nu}$ obeys the same relation, i.e. $g^{\mu\nu}{}_{;\nu} = 0$. In other words, we have the freedom to add a term equal to a multiple of the metric tensor to the Einstein tensor. The multiple of the metric tensor $g^{\mu\nu}$ is called the **cosmological constant** Λ, and the corresponding change of the Einstein field equations is,

$$\boxed{G^{\alpha\beta} \,+\, \Lambda\, g^{\alpha\beta} \,=\, -\frac{8\pi G}{c^4}\, T^{\alpha\beta}} \tag{88}$$

At first this extra term of the cosmological constant was extremely welcome to Einstein. It allowed him to construct a static Universe. Nonetheless, the factor was entirely ad-hoc, and its (physical) nature hardly understood. While he discarded his inclusion of Λ as his "biggest blunder" upon the discovery of the cosmic expansion by Hubble in 1929, in 1998 we have found to our great astonishment that Λ is indeed a physical reality which appears to dominate our Universe.
As written in the above equation the Λ term corresponds to an extra curvature term. However, as may be judged from eqn. 87, one has equal freedom to reorden the field equation and transfer the Λ term to the righthand side. In other words, the Λ term is included in the energy-momentum tensor $T^{\mu\nu}$ and assumes the status of an extra energy term,

$$\boxed{\begin{aligned} G^{\mu\nu} &= -\frac{8\pi G}{c^4}\left(T^{\mu\nu} + T^{\mu\nu}_{vac}\right) \\ T^{\mu\nu}_{vac} &\equiv \frac{\Lambda c^4}{8\pi G}\, g^{\mu\nu} \end{aligned}} \tag{89}$$

This is the way in which most astrophysicists have adopted the cosmological constant. In this disguise it is usually denoted by the name of **Dark Energy** or, even more specific, as **Vacuum Energy**. As for the latter, note that this name already presumes knowledge of the nature of this *dark energy* in speculating about its background as energy of the vacuum. However, no theory of physics has even come close to a convincing argument for it to be due to vacuum fluctuations. In fact, "not close" here means off by a factor in the order of 10^{120} !!!

13.1. Equation of State

While unable to make a sensible guess at the identity of Λ's *dark energy*, we can infer one important aspect of its physical nature. This concerns its **equation of state**. To this end, turn to the restframe energy-momentum tensor (eqn. 58). In such a Minkowski frame, the dark energy-momentum tensor is equal to

$$T^{\mu\nu}_{vac} = \frac{\Lambda c^4}{8\pi G}\, \eta^{\mu\nu} \tag{90}$$

with $\eta^{00} = 1$ and $\eta^{ii} = -1$. From this we then find that:

$$\begin{aligned} T^{00}_{vac} &= \rho_{vac} c^2 \qquad \Rightarrow \qquad \rho_{vac} c^2 = \frac{\Lambda c^4}{8\pi G} \\ T^{ii}_{vac} &= p \qquad \Rightarrow \qquad p = -\frac{\Lambda c^4}{8\pi G} \end{aligned} \tag{91}$$

so that we end up with the astonishing finding that this elusive mysterious medium has a **negative pressure**:

$$\boxed{p_{vac} = -\rho_{vac} c^2} \tag{92}$$

Note that tacitly we have started to index the cosmological constant energy with the term *vacuum energy*. In this we adhere to a regularly employed description, but with the understanding that it by no means implies it has significance (see discussion above).

13.2. Dynamics

By combining the *relativistic Poisson equation* (eq. 85) and the *equation of state* (eq. 92) for the "vacuum energy" we can infer the dynamical repercussions for the presence of such a medium,

$$\rho_{vac} + \frac{3p_{vac}}{c^2} = -2\rho_{vac} < 0\,. \tag{93}$$

The implication from the relativistic expression for the Poisson equation (eq. 85) is that

$$\nabla^2\phi = -8\pi G \rho_{vac} < 0; \qquad \rho_{vac} \equiv \frac{\Lambda}{8\pi G}$$

Table 1. Relativistic and Newtonian Cosmology

Relativistic Cosmology	Newtonian Cosmology
10 field equations	1 field equation
10 potentials	1 potential
Non-linear equations	Linear equation
Intrinsically geometric	Absolute space and time
Can cope with ∞ space	Requires finite space
All energies gravitate	Gravitation mass-density only
Pressure gravitates	Gravitation mass-density only
Cosmological Constant feasible	Repulsive action gravity impossible
Hyperbolic propagation	Instantaneous propagations
Singularities spacetime	Singularities space
Horizons & Black Holes	No horizons/black holes
Gravitational Waves	No gravitational waves

and that therefore the gravitational effect of a **positive Λ** is that of a **repulsion** instead of an **attraction** !!!! The expansion of a Universe dominated by a cosmological constant would accelerate instead of the usual deceleration. Interestingly and strangely enough, this is exactly what our Universe appears to do ...

14. General Relativity versus Newtonian Gravity

While it is clear that with respect to the old Newtonian theory the General Theory of Relativity represents a tremendous advance in our understanding of the force of gravity, it is by no means a simple theory. In fact, for generic circumstances without specific symmetric configurations it is beyond hope to be able to find suitable solutions for the Einstein field equations.

The cosmologist is blessed that in a sense he/she is dealing with a highly symmetric system. On the basis of this, the Einstein field equations are restricted to a few (two) equations, the **Friedmann-Robertson-Walker-Lemaître equations**, which may indeed be succesfully analyzed and understood. For a lot of realistic circumstances this may even be done in a fully analytically fashion. Not only was understanding the Universe one of the most challenging questions forwarded to the new theory of relativity, it was and still is one of the most accessible systems.

To appreciate the complexities involved with general relativistic analyses, John Barrow (J. Barrow, 1986, in “The Early Universe”, summerschool Vancouver Island, ASI/NATO C219, eds. Unruh & Semenoff, Reidel) provided an insightful inventory of relevant issues and the way in which they were approached in general relativistic versus Newtonian approaches. I have copied this listing in table 13.2 and included a few more relevant items.