Tutorial III
Cosmic Solutions & Observations

Exercise 1:

Expansions and Transitions of some specific FRW Universes

In the former task set we looked at solutions of *matter-dominated* Universes. Today, we are going to take a step further and evaluate the solutions for general Friedman-Robertson-Walker-Lemître Universes with a non-zero cosmological constant,

$$\begin{aligned} \ddot{a} &= -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) a + \frac{\Lambda}{3} a \\ \dot{a}^2 &= \frac{8\pi G}{3}\rho a^2 - \frac{kc^2}{R_0^2} + \frac{\Lambda}{3} a^2 \end{aligned} \tag{1}$$

First, we are going to investigate a set of simple solutions.

- First, derive the following expression for the evolving Hubble parameter $H(t)$,

$$H(t) = H_0 \sqrt{\Omega_{r,0} a^{-4} + \Omega_{m,0} a^{-3} + (1-\Omega_0) a^{-2} + \Omega_{\Lambda,0}} \tag{2}$$

  with

$$\Omega_0 \equiv \Omega_{r,0} + \Omega_{m,0} + \Omega_{\Lambda,0} \tag{3}$$

- Also derive the general expression for the acceleration/deceleration parameter q,

$$q \equiv -\frac{\ddot{a}\, a}{\dot{a}^2} \tag{4}$$

- Take a flat, matter-dominated Universe, and derive the solution $a(t)$ for the FRW equation. Flat, matter-dominated Universe means: $k = 0$, $\Omega_{r,0} = 0$, $\Lambda = 0$. Make a graph of $a(t)$ vs. t. Do you happen to know the name for this Universe ?

- What is the age of this Universe, in terms of H_0 ? Given the present-day accepted value of $H_0 = 71\,\mathrm{km\ s^{-1}\ Mpc^{-1}}$, what is the age of this Universe in years ?

- How does the Hubble parameter $H(t)$,

$$H(t) \equiv \frac{\dot{a}}{a} \tag{5}$$

  change in time for the flat, matter-dominated Universe. Both in terms of its expansion factor $a(t)$, as well as in terms of the time t. Also, what would be the value of q ?

- We now turn to a flat, radiation-dominated Universe: $k = 0$, $\Omega_{m,0} = 0$, $\Lambda = 0$. Derive the solution $a(t)$ for this Universe.

- Derive the evolution of the Hubble parameter $H(t)$ in a flat, radiation-dominated Universe, in terms of $a(t)$ and in terms of time t. Also, find the general expression for the age of the Universe in such a Universe.

- Next solution: a Lambda dominate Universe. Discard all other contributions, how does a Lambda dominated Universe expand. How does the Hubble parameter evolve in such a Universe ?

- Given a galaxy with measured redshift z, how long ago would it have emitted its radiation in a Λ-dominated Universe ?

- Finally, imagine an empty (!!!!!) negatively-curved Universe ($k = -1$). How does this Universe expand ? This is called free expansion. Make a graph of a(t) vs. t.

While an empty Universe in itself does not make much sense, it is a very asymptotic situation: any low-density open matter-dominated Universe will evolve into such a Universe. To see this, consider the following:

- Derive the explicit expression for $H^2(t)$ in a matter-dominated Universe, ie. for a Universe with $\Omega_{r,0} = 0; \Lambda = 0$, i.e. show that:

$$H^2(t) \;=\; H_0^2 \left\{ \Omega_0 a^{-3} \;+\; (1 - \Omega_0) a^{-2} \right\} \tag{6}$$

- Carefully inspect the equation above. Let's restrict ourselves to the low-density open situation (e.g. $\Omega_0 \approx 0.3$). What will happen when $a(t) \downarrow 0$, and what if $a(t) \rightarrow \infty$. Do you recognize the solutions for these asymptotic solutions ? In other words, how do you see the expansion history of a general matter-dominated Universe (hint: it transits from one expansion phase into another ...).

- As you see above, apparently, a low-density matter-dominated Universe undergoes a phase transition, ultimately becoming a freely expanding Universe. Derive an expression, in terms of Ω_0 for the expansion factor a and the related redshift z. If we take the present-day matter value of $\Omega_{m,0} \approx 0.27$, when would this transition have happened (ie. value of expansion factor $a(t)$ and redshift z).

- Another major transition epoch in the expansion history of the Universe is the radiation-matter epoch. At that point the Universe went from a radiation-dominated expansion to a matter-dominated expansion. Given the values for $\Omega_{m,0}$ and $\Omega_{r,0}$ (we will forget, for the moment, about the contribution of neutrino's),

$$\begin{aligned} \Omega_{m,0} &\approx 0.27 \\ \Omega_{r,0} &= 1.3 \times 10^{-4} \end{aligned} \tag{7}$$

derive the exact value for expansion factor a_{rm} and redshift at which this transition happened. In this, we assume you can take the epoch at which the energy density in radiation becomes equal to the matter density as a good representation of this epoch.

- Finally, there is the major (recent) transition epoch of the Universe going from a matter-dominated to a Lambda-dominated Universe. In this case take the expansion factor $a_{m\Lambda}$ at which the acceleration of the Universe is taken over by the cosmological constant. Derive the expression for $a_{m\Lambda}$. For our concordance Universe, $\Omega_{m0} = 0.27$, $\Omega_{\Lambda 0} = 0.73$, when did this transition occur. In terms of expansion factor, redshift ... and how many years ago ? (the latter simply assuming you may use the expansion law for a completely Lambda dominated Universe).

- Make a sketch of the expansion of our Universe, i.e. $a(t)$ vs. t, clearly marking each different expansion period and each important transition epoch.

Exercise 2:
Solutions to general FRW Universes

In general it is not possible to find analytical expressions for the expansion history $a(t)$ of the Universe. You will need to solve numerically, and the help of a computer is almost imperative. Let's first have a look at how to find a solution. Turn to the equation for the Hubble parameter $H(t)$,

$$H(t) \;=\; H_0 \sqrt{\Omega_{r,0} a^{-4} \,+\, \Omega_{m,0} a^{-3} \,+\, (1-\Omega_0) a^{-2} \,+\, \Omega_{\Lambda,0}} \tag{8}$$

- Derive the expression for the time t for this generic situation,

$$H_0\,(t - t_{min}) \;=\; \int_{a_{min}}^{a} \frac{dx}{\sqrt{\Omega_{r,0} x^{-2} \,+\, \Omega_{m,0} x^{-1} \,+\, (1-\Omega_0) \,+\, \Omega_{\Lambda,0} x^2}} \tag{9}$$

  with $a_{min} = 0$ for nearly all solutions, ie. the "normal" ones with a Big Bang, and some finite value in the case of the exceptional cases of a *Big Bounce* universe.

We are going to discard the contribution by radiation: we are going to study the behaviour of generic Universes filled with matter and a cosmological constant, and not generically flat. To find a solution for a given Universe with matter density Ω_{m0}, cosmological constant contribution $\Omega_{\Lambda,0}$ and curvature dictated by $\Omega_0 = \Omega_{m0} + \Omega_{\Lambda,0}$, you have to numerically integrate the above integral for a range of values $a = [0, 1]$ (or further, e.g. $a = [0, 10]$). You then obtain a long list of numbers $(a_j, H_0 t_j)$ (j=1,N). Subsequently, invert this relation to $(H_0 t_j, a_j)$ and make a plot of $a(t)$ vs. t.

Make sure that always your solutions have today's cosmic parameters, ie. today $a = 1$ and $H = H_0$, in the plots of $a(t)$ vs. t, take today as the "origin", ie. if you plot two different models on top of each other, they should intersect at $t = t_0$ (note $t = 0$ is a different time ago for different universes).

- Solve numerically the above equation for a range of Universes, and make a figure of expansion factor $a(t)$ vs. time $H_0 t$ (notice that time is plotted in terms of its dimensionless value $H_0 t$) and as well a figure of age of the Universe vs. redshift z. Do this for the following configurations:

- **Flat matter-dominated Universe**:

  $\Omega_{m,0} = 1$, $\Omega_{\Lambda,0} = 0$.

  Compare this to the theoretically derived $a(t)$ for an Einstein-de Sitter Universe.

- **Flat Lambda dominated Universe**:

  $\Omega_{m,0} = 0$, $\Omega_{\Lambda,0} = 1$.

  Compare this to the theoretically derived $a(t)$ for a Lambda-dominated Universe.

- **Generic flat matter+Lambda Universes**

  $\Omega_0 = \Omega_{m,0} + \Omega_{\Lambda,0} = 1$:

  $\Omega_{m,0} + \Omega_{\Lambda,0} = 1$:
  $(\Omega_{m,0}, \Omega_{\Lambda,0}) = (0.1, 0.9)$
  $(\Omega_{m,0}, \Omega_{\Lambda,0}) = (0.27, 0.73)$
  $(\Omega_{m,0}, \Omega_{\Lambda,0}) = (0.5, 0.5)$
  $(\Omega_{m,0}, \Omega_{\Lambda,0}) = (0.75, 0.25)$
  $(\Omega_{m,0}, \Omega_{\Lambda,0}) = (0.9, 0.1)$

  Compare these to the theoretically derived $a(t)$ for flat matter+Lambda Universes:

$$H_0\, t \;=\; \frac{2}{3\sqrt{1-\Omega_{m,0}}}\;\ln\{\left(\frac{a}{a_{m\Lambda}}\right)^{3/2} \;+\; \sqrt{1+\left(\frac{a}{a_{m\Lambda}}\right)^{3}}\} \tag{10}$$

  with

$$a_{m\Lambda} \;\equiv\; \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda 0}}\right)^{1/3} \;=\; \left(\frac{\Omega_{m,0}}{1-\Omega_{m,0}}\right)^{1/3} \tag{11}$$

- **Generic non-flat Universe**:

  $(\Omega_{m,0}, \Omega_{\Lambda,0}) = (1.0, 0.3)$
  $(\Omega_{m,0}, \Omega_{\Lambda,0}) = (1.0, 0.5)$
  $(\Omega_{m,0}, \Omega_{\Lambda,0}) = (1.0, 1.0)$
  $(\Omega_{m,0}, \Omega_{\Lambda,0}) = (0.3, 0.15)$
  $(\Omega_{m,0}, \Omega_{\Lambda,0}) = (0.3, 0.3)$
  $(\Omega_{m,0}, \Omega_{\Lambda,0}) = (0.3, 1.0)$

- **Loitering, Big Bounce and Recollapse**
  Some real wilde ones !

  $(\Omega_{m,0}, \Omega_{\Lambda,0}) = (0.5, 2.0)$
  $(\Omega_{m,0}, \Omega_{\Lambda,0}) = (0.3, 2.0)$

$(\Omega_{m,0}, \Omega_{\Lambda,0}) = (0.2, 2.0)$
$(\Omega_{m,0}, \Omega_{\Lambda,0}) = (0.2, 3.0)$
$(\Omega_{m,0}, \Omega_{\Lambda,0}) = (2.5, 0.1)$
$(\Omega_{m,0}, \Omega_{\Lambda,0}) = (0.3, -0.3)$
$(\Omega_{m,0}, \Omega_{\Lambda,0}) = (0.1, -0.3)$

Note: in the case of *Big Bounce* or a *Recollapse* universe you need to be a bit more careful. The integral of expression eqn. (9) may not allow values of a lower than some finite value a_{min} (Big Bounce) or values of a higher than a finite value a_{max}. You can understand this from the fact that for a Universe with

$$\Omega_0 \;=\; \Omega_{m,0} \;+\; \Omega_{\Lambda,0} \;>\; 1 \tag{12}$$

an awkward situation may occur:

$$\frac{H^2}{H_0^2} \;=\; \frac{\Omega_{m,0}}{a^3} \;+\; \frac{1-\Omega_{m,0}-\Omega_{\Lambda,0}}{a^2} \;+\; \Omega_{\Lambda,0} \;<\; 0. \tag{13}$$

In other words, for finite values $a < a_{min}$ (for Big Bounce) or $a > a_{max}$ (for Recollapse),

$$H^2(a) \;<\; 0\;. \tag{14}$$

Big Bounce
Big Bounce occurs in a regime in which the combination of Ω_m and Ω_Λ conspire to:

$$\Omega_{\Lambda,0} \;>\; \begin{cases} 4\Omega_{m,0}\left[\cosh\left\{\frac{1}{3}\cosh^{-1}\left(\Omega_{m,0}^{-1}-1\right)\right\}\right] & \Omega_{m,0} < 0.5 \\ \\ 4\Omega_{m,0}\left[\cos\left\{\frac{1}{3}\cos^{-1}\left(\Omega_{m,0}^{-1}-1\right)\right\}\right] & \Omega_{m,0} \geq 0.5 \end{cases} \tag{15}$$

As this is not a real possibility (imaginary expansion), it means that such a Universe has to bounce: started from $a \to \infty$ it contracts down to the minimum value at a_{min}, and then re-expands again toward $a \to \infty$. In practice, the solution is found by computing the integral,

$$H_0\,(t-t_{min}) \;=\; \pm\int_{a_{min}}^{a} \frac{dx}{\sqrt{\Omega_{r,0}x^{-2} \,+\, \Omega_{m,0}x^{-1} \,+\, (1-\Omega_0) \,+\, \Omega_{\Lambda,0}x^2}}\;, \tag{16}$$

where the $+$ sign is the expanding part $t > t_{min}$, and the $-$ part corresponds to the contracting phase, in the time before the bounce $t < t_{min}$. Note that this concerns an eternally accelerating Universe, thus also $\ddot{a}(a_{min}) > 0$.

The value of the expansion factor a_{min} for a bouncing Universe, you should determine *numerically* by determining the root x of the cubic equation (the analytical expressions are unappetizing),

$$\Omega_{m,0} + (1 - \Omega_{m,0} - \Omega_{\Lambda,0})\, x \,+\, \Omega_{\Lambda_0,0}\, x^3 \,=\, 0 \tag{17}$$

Note to assure yourself that you take the solution $a_{min} > 0$ with acceleration

$$\frac{\ddot{a}}{a\, H_0^2} \,=\, -\frac{1}{2}\frac{\Omega_{m,0}}{a^3} \,+\, \Omega_{\Lambda,0} \,>\, 0 \tag{18}$$

for the *Big Bounce* universe.

Recollapse
A similar situation occurs for the universe with *Recollapse*. In that case the Universe reaches a maximum expansion factor a_{max} at which $H(t_{max}) = 0$. You can determine this also by determining the root a_{max} from the cubic equation

$$\Omega_{m,0} + (1 - \Omega_{m,0} - \Omega_{\Lambda,0})\, x \,+\, \Omega_{\Lambda_0,0}\, x^3 \,=\, 0 \tag{19}$$

for which the universe is eternally decelerating,

$$\frac{\ddot{a}}{a\, H_0^2} \,=\, -\frac{1}{2}\frac{\Omega_{m,0}}{a^3} \,+\, \Omega_{\Lambda,0} \,<\, 0\,. \tag{20}$$

Of course, in this case there are no solutions with $a > a_{max}$, and you should compute the solution from the integral

$$H_0\,(t - t_{max}) \,=\, \pm \int_{a_{max}}^{a} \frac{dx}{\sqrt{\Omega_{r,0} x^{-2} \,+\, \Omega_{m,0} x^{-1} \,+\, (1 - \Omega_0) \,+\, \Omega_{\Lambda,0} x^2}}\,, \tag{21}$$

where the $-$-sign corresponds to the expanding part before the Universe reaches its maximum expansion at a_{max} and the $+$-sign to the contracting/collapsing part $t > t_{max}$.