Tutorial II
Newtonian Cosmology; Hubble Expansion

Exercise I:

Newtonian Cosmology

In 1934 – i.e. way after Friedman derived his equations- Milne and McCrea showed that relations of the 'Friedman' form can be derived using **non-relativistic Newtonian dynamics**.

- Write down the field equation for the gravitational force in the non-relativistic limit.
- Imagine you are a particle moving outside a spherically symmetric mass concentration of radius R with a total mass M and a density profile $\rho(r)$. What two essential simplifications can you invoke to derive your equation of motion ?
- Write down the equation of motion (ie. the equation for your acceleration). In addition, derive the corresponding energy equation (conservation of energy).
- We go one step further, and assume you are embedded within the spherically symmetric mass concentration. Imagine you are at a radius r, what will be your equation of motion ?

Subsequently, the situation becomes even more benevolent: we find ourselves in a homogeneous and isotropic medium.

- Write down the equation of motion and the energy equation.
- What three qualitative different situations can you distinguish on the basis of the energy E of a shell ?
- Take a shell of initial radius $r_{1,i}$ and another shell of initial radius $r_{2,i}$, in how far does their evolution differ (or not) ? (assume that there are no non-radial motions). What does this imply for the evolution $r(t)$ for any shell in the mass distribution ?
- What does the latter imply for the evolution of the density $\rho(t)$.

In principle, we are now all set to solve the equation of motion of the system, as a function of E. In fact, it is possible to derive the full solution for any spherically symmetric - not even homogeneous - mass distribution. This is the socalled *Spherical Model*. It would be a good exercise to do so ... however, we go for the real work, solving the equation of motion for a matter-dominated FRW Universe.

Exercise II:
Solutions Matter-dominated FRW Universes

The general Friedman-Robertson-Walker-Lemître equation for a Universe including a non-zero cosmological constant is

$$\begin{aligned} \ddot{a} &= -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) a \;+\; \frac{\Lambda}{3}\, a \\ \dot{a}^2 &= \frac{8\pi G}{3}\,\rho a^2 \;-\; \frac{kc^2}{R_0^2} \;+\; \frac{\Lambda}{3}\, a^2 \end{aligned} \tag{1}$$

We are going to investigate the simple situation of a matter-dominated Universe. Assuming matter can be assumed to be "cosmic dust" (true for dark matter, and baryonic matter on large scales) pressure can be ignored, ie. $p = 0$. In a matter-dominated Universe the cosmological constant is zero, $\Lambda = 0$.

- On the basis of the full FRW equations above, derive the following FRW equations for a matter-dominated FRW Universe

$$\begin{aligned} \ddot{a} &= -\frac{1}{2}\,\Omega_0 H_0^2\,\frac{1}{a^2} \\ \dot{a}^2 &= \Omega_0 H_0^2\,\frac{1}{a} \;-\; H_0^2(\Omega_0 - 1) \end{aligned} \tag{2}$$

- Assume $k = 0$. What does this imply for Ω_0 ? Solve this equation for this situation, ie. derive the expansion factor $a(t)$. Note that this solution is known as the **Einstein-de Sitter Universe**.

Regretfully, for a general matter-dominated Universe you will not succeed in finding a direct solution. To be able to solve the FRW equations we need to resort to a parameterized solution. Introduce the parameter Φ, the socalled **development angle**.

$$\frac{d}{d\Phi} \equiv \frac{1}{H_0\,\sqrt{\Omega_0 - 1}}\, a\,\frac{d}{dt} \qquad \Omega_0 > 1 \tag{3}$$

$$\frac{d}{d\Phi} \equiv \frac{1}{H_0\,\sqrt{1 - \Omega_0}}\, a\,\frac{d}{dt} \qquad \Omega_0 < 1 \tag{4}$$

- Show that the FRW equation (of motion) in terms of the development angle becomes

$$\begin{aligned} \frac{d^2a}{d\Phi^2} &= \frac{1}{2}\frac{\Omega_0}{\Omega_0 - 1} - a\,, \qquad & \frac{dt}{d\Phi} &= \frac{a}{H_0\sqrt{\Omega_0 - 1}} \qquad & \Omega_0 > 1 \\ \frac{d^2a}{d\Phi^2} &= \frac{1}{2}\frac{\Omega_0}{1 - \Omega_0} + a\,, \qquad & \frac{dt}{d\Phi} &= \frac{a}{H_0\sqrt{1 - \Omega_0}} \qquad & \Omega_0 < 1 \end{aligned} \tag{5}$$

- Solve these second-order differential equations by using the solution ansatz

$$a(\Phi) = c_1\, e^{b_1 \Phi} + c_2\, e^{b_2 \Phi} + \frac{1}{2}\frac{\Omega_0}{\Omega_0 - 1} \tag{6}$$

After settling the values of b_1 and b_2, determine the values of c_1 and c_2 from the initial condition $a(t = 0) = a(\Phi = 0) = 0$). Show that you obtain the following set of solutions:

- for a high-density $\Omega_0 > 1$ Universe

$$\begin{aligned} a(\Phi) &= \frac{1}{2}\frac{\Omega_0}{\Omega_0 - 1}\,(1 - \cos\Phi) \\ H_0\,t(\Phi) &= \frac{1}{2}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}\,(\Phi - \sin\Phi) \end{aligned} \tag{7}$$

- and for a low-density $\Omega_0 < 1$ Universe

$$\begin{aligned} a(\Phi) &= \frac{1}{2}\frac{\Omega_0}{1 - \Omega_0}\,(\cosh\Phi - 1) \\ H_0\,t(\Phi) &= \frac{1}{2}\frac{\Omega_0}{(1 - \Omega_0)^{3/2}}\,(\sinh\Phi - \Phi) \end{aligned} \tag{8}$$

- Make a graph of solutions $a(t)$ (i.e. expansion factor a vs. time t) for the three classes of solution: Einstein-de Sitter Universe, Open Universe ($\Omega_0 < 1$) and Closed Universe ($\Omega_0 > 1$)

Exercise III:
Hubble Expansion and Bounded Objects

We have seen that galaxies are participating in the uniform Hubble expansion. Question is why we ourselves do not expand along. If this were so, we would not notice anything like expansion. Assume a Hubble parameter of $H_0 = 71\,\mathrm{km/s/Mpc}$. As a thought experiment compute

- the expected Hubble expansion rate between your toes and the tip of your head.
- the expected Hubble expansion rate between the core of the Earth and ourselves ?
- What is the reason behind the Hubble expansion being insignificant under these circumstances ? Suggestion: compute the gravitational binding energy/escape velocity at the surface of the Earth and compare to $v = Hr$.
- Repeat the same exercise for Planet Earth wrt. the Sun and Dwarf Planet Pluto wrt. the Sun. Subsequently, consider the Sun and the Galaxy. Next, consider the Local Group (mainly M31 and the Galaxy). Then, consider the Local Group, or the Galaxy, wrt. the Local Supercluster dominated by the Virgo Cluster. Thus, what is your conclusion with respect to the scale at which the Hubble expansion becomes noticeable ? Note that you are expected to look up the relevant numbers yourself !

Exercise IV (Computer Task):
Hubble Expansion and Anisotropic Velocities

Kinematically speaking, the isotropic and uniform Hubble expansion of the Universe is a rather special circumstance. We may appreciate this when looking at the general flow of a fluid around a position $\mathbf{r}_0$. The kth component $v_k(\mathbf{r}$ of the velocity at location $\mathbf{r}$ is given by

$$v_k(\mathbf{r}) \,=\, v_k(\mathbf{r}_0) \,+\, \frac{1}{3}\nabla\cdot\mathbf{v}\,(r_k - r_{k,0}) \,+\, \sum_j \sigma_{kj}(r_j - r_{j,0}) \,+\, \sum_j \omega_{kj}(r_j - r_{j,0}) \tag{9}$$

in which the divergence $\nabla\cdot\mathbf{v}$ encapsulates the expansion or contraction of a volume element, the shear σ_{ij} its shape deformation and ω_{ij} the vorticity,

$$\begin{aligned}
\nabla\cdot\mathbf{v} \quad &\equiv \quad \left(\frac{\partial v_1}{\partial r_1} + \frac{\partial v_2}{\partial r_2} + \frac{\partial v_3}{\partial r_3}\right) \\
\sigma_{ij} \quad &\equiv \quad \left(\frac{\partial v_i}{\partial r_j} + \frac{\partial v_j}{\partial r_i}\right) - \frac{1}{3}(\nabla\cdot\mathbf{v})\,\delta_{ij} \\
\omega_{ij} \quad &\equiv \quad \left(\frac{\partial v_i}{\partial r_j} - \frac{\partial v_j}{\partial r_i}\right)
\end{aligned} \tag{10}$$

The Hubble expansion is unique in that it does not have any anisotropic terms, both shear and vorticity are equal to zero. In other words, the Hubble parameter is equal to

$$H(t) \,=\, \frac{1}{3}\nabla\cdot\mathbf{v} \tag{11}$$

In the lecture the Hubble expansion illustrated by means of a two-dimensional cartoon involving hundred randomly distributed points within a square. In this computer task you will need to follow up on this experiment for the generic case involving both an anisotropic shear term and a vorticity term. You may need whatever computer program (Matlab, Python, IDL) you feel most at ease with.

- Distribute $N = 1000$ points randomly within a box of size 100×100. This defines timestep t_1. The particles have an initial location $\vec{r}_j(t_1) = (r_{1,j}, r_{2,i})$, $j = (1, \ldots, N)$.

We are going to follow the evolution of the initial particle distribution at 2 subsequent timesteps t_2 and t_3. The initial timestep is t_1. At t_1 the particles have an initial position $\vec{r}_j(t_1) = (r_{1,j}, r_{2,j})$, $j = (1, \ldots, N)$. Imagine we are at a central position $\vec{s}$, from where we observe the displacement of the surrounding Universe.

In a time interval $\Delta t = (t - t_1)$ each particle m gets displaced by an amount $\Delta\vec{r}_m(t)$,

$$\vec{r}_m(t) \quad = \quad \vec{r}_m(t_1) \,+\, \Delta\vec{r}_m(t) \tag{12}$$

where the displacement is a product of the deformation $\vec{\mathcal{D}}$ with the time interval Δt,

$$\Delta\vec{r}(t) \;=\; \vec{\mathcal{D}}\,\Delta t\,, \tag{13}$$

The deformation is the sum of an expansion/contraction H, shear σ and vorticity ω term.

Assume we have a particle m with initial coordinates $\vec{r}(t_1) = (r_1, r_2)$. then the k-th coordinate of its deformation $\vec{\mathcal{D}}$ is equal to

$$\mathcal{D}_k(t) \;=\; H\,(r_k - s_k) \,+\, \sigma_{kj}(r_j - s_j) \,+\, \omega_{kj}(r_j - s_j)\,. \tag{14}$$

where we use the Einstein summation convention !

For our experiment the resulting location $\vec{r}(t) = (r_1(t), r_2(t))$ at time t of the particle with initial location $\vec{r}(t_1) = (r_1, r_2)$ is

$$r_k(t) \;=\; r_k \,+\, \mathcal{H}(r_k - s_k) \,+ \Sigma_{kj}(r_j - s_j) \,+\, \Omega_{kj}(r_j - s_j)\,, \tag{15}$$

where $\mathcal{H} \equiv H\Delta t$, $\Sigma_{kj} \equiv \sigma_{kj}\Delta t$ and $\Omega_{kj} \equiv \omega_{kj}\Delta t$. Thus, for the initial time

$$t_1: \qquad \mathcal{H} = 0, \quad \Sigma_{kj} = 0, \quad \Omega_{kj} = 0 \tag{16}$$

Note that strictly speaking this expression is only valid for small displacements (small timesteps): keep it moderate ... (but not too small either, otherwise it is not too clarifying.

For the traceless shear tensor we have the following conditions:

$$\Sigma_{ij} \;=\; \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \tag{17}$$

for which $\sigma_{11} + \sigma_{22} = 0$ (so that Σ_{ij} is effectively specified by 2 numbers) while the vorticity tensor is specified via one number, ω,

$$\Omega_{ij} \;=\; \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \tag{18}$$

Generate 6 different configurations:

- pure expansion only: $\Sigma_{ij} = \Omega_{ij} = 0$
- pure shear only: $\mathcal{H} = \Omega_{ij} = 0$
- pure vorticity only: $\mathcal{H} = \Sigma_{ij} = 0$

- expansion + shear: $\Omega_{ij} = 0$
- expansion + vorticity: $\Sigma_{ij} = 0$
- expansion + shear + vorticity

For each timestep plot particle distribution within the central box of size 50×50. You should decide yourself on the values for Σ_{ij}, Ω_{ij} and H. You have some freedom in choice, but do not assume values which are too radical.

- Repeat the same, yet with $\vec{s} = (75, 75)$ as centre of the plotbox of size 50×50.