– Astrophysical Hydrodynamics –
Assignment 5

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1 Bow Shock

We will determine the shock front between two binary stars with masses M_1 and M_2, rotating in circular orbits around their common barycenter. Star 1 produces a wind with speed u_1 that emits mass (from the star) at a rate $\dot{M}_1 = dM_1/dt$ and likewise for star 2. The winds will collide in a so called 'bow shock'. The surface of the bow shock is defined by the momentum flux condition

$$\rho_1 u_{1\perp}^2 = \rho_2 u_{2\perp}^2 \tag{1}$$

where the $\perp$ denotes that the velocities perpendicular to the shock front's surface are considered.

Here we assume a 'cold flow', i.e. the pressure can be ignored. This is a fine approximation for O stars. For completeness, the Rankine-Hugoniot Jump conditions are:

$$\rho_1 u_{1\perp} = \rho_2 u_{2\perp} \tag{2}$$

$$\rho_1 u_{1\perp}^2 + P_1 = \rho_2 u_{2\perp}^2 + P_2 \tag{3}$$

$$\frac{u_{1\perp}^2}{2} + \frac{\gamma P_1}{(\gamma-1)\rho_1} = \frac{u_{2\perp}^2}{2} + \frac{\gamma P_2}{(\gamma-1)\rho_2} \tag{4}$$

These are (respectively) the conditions for mass-flux conservation (2), momentum-flux conservation (3) and energy-flux conservation (4). In this exercise, however, we will only use the second equation, and in a simplified form (as mentioned, without the pressure term).

1. **Rotational Forces**

   Choose a coordinate system in which star 1 is fixed at the center and star 2 orbits at fixed distance D around star 1. This must be a rotating coordinate system that moves in the opposite direction to star 1; see the top images in figure 1. Due to this, there will be two fictitious rotational forces, the centrifugal and Coriolis forces, that cause accelerations (in the rotating system) given respectively by:

   $$-\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) \ . \tag{5}$$

   and

   $$-2\boldsymbol{\Omega} \times \boldsymbol{u} \tag{6}$$

   a. Consider a fluid element in the wind from star 1. Derive two expressions for the displacement $\delta r = \frac{1}{2}at^2$ due to the accelerations of the two rotational forces, as a function of only r, u and Ω.

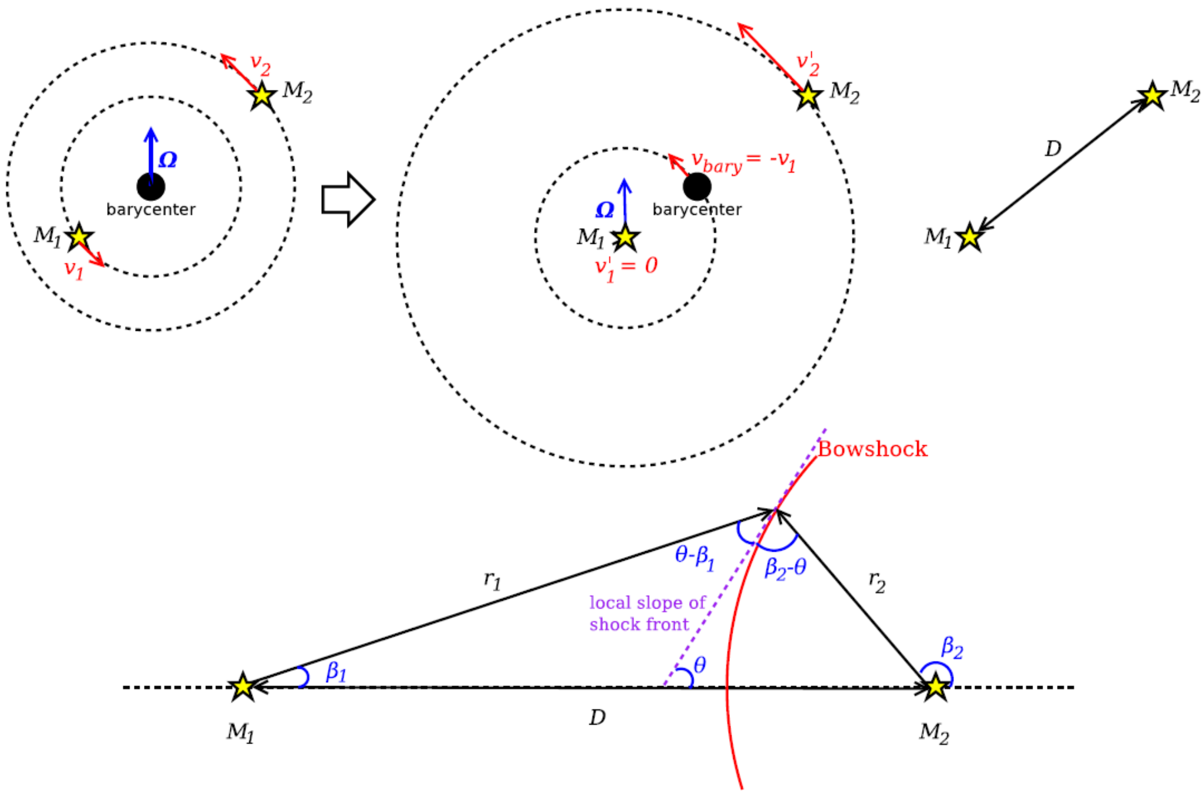

Figure 1: The top-central image depicts a rotating coordinate system with star 1 at the center. $\mathbf{\Omega}$ points out of the paper, towards the reader. At the bottom a sketch of the situation for question 2 is drawn.

b. We can approximate the wind as being a purely radial flow (i.e. we can ignore the rotational forces) if the displacement due to rotational forces is very small compared to the total displacement r from the initial position (the star's position). What condition on r does this imply?

c. If we assume $u_1 = u_2 = 2000$ km s^{-1}, $M_1 = M_2 = 10M_\odot$, and $D = 10R_*$ where $R_* = 3R_\odot$ is the individual stellar radius, is the condition fullfilled? Express your (numerical) estimate in units of D, for which you can use that for a binary system we have that

$$\Omega = \sqrt{\frac{G(M_1 + M_2)}{D^3}}\,. \tag{7}$$

d. Argue that the same relations hold for the wind from star 2, i.e. that we can also approximate wind 2 as a radial flow.

2. **Slope of the Bow Shock**

Ignore rotational effects and show that the local slope of the surface of the bow shock obeys

$$\frac{dy}{dx} = \frac{r_2\sqrt{R}\sin\beta_1 + r_1 \sin\beta_2}{r_2\sqrt{R}\cos\beta_1 + r_1\cos\beta_2} \tag{8}$$

where r_i is the distance vector from star i to the bow shock front, and β_i is the angle that $\boldsymbol{r}_i$ makes with respect to the line through both stars. At the bottom of figure 1 is a sketch of the situation. The constant is given by

$$R = \frac{\dot{M}_1 u_1}{\dot{M}_2 u_2}\,. \tag{9}$$

Hint 1: rewrite equation 1 to get rid of the $u_{i\perp}$'s and ρ_i's. Hint 2: what conservation law can we use to get rid of the ρ_i's? Hint 3: what is $\frac{dy}{dx}$ as function of θ?

3. **Position of the Bow Shock**

Assume $\dot{M}_1/\dot{M}_2 = 10$ and $u_1 = u_2$. Where does the shock front cross the imaginary line (M1,M2) between the two stars, in terms of D? Hint: what should happen to $\frac{dy}{dx}$ (or $\frac{dx}{dy}$) on this line between the two stars?

2 The isothermal normal shock

For shock waves in an adiabatic gas, the pressure and density are related by a polytropic gas law of the form

$$P = P_0 \left(\frac{\rho}{\rho_0} \right)^{\gamma} . \tag{10}$$

In this assignment we look at a special case: that of an isothermal gas where the temperature on both sides of the shock is the same:

$$T_1 = T_2 = T . \tag{11}$$

Formally this corresponds to $\gamma = 1$ as the ideal gas law gives $P = \rho RT/\mu$.

One can express the gas pressure in terms of the (now constant) isothermal sound speed c_s:

$$P(\rho) = \rho c_s^2 , \quad \text{where} \quad c_s \equiv \sqrt{RT/\mu} . \tag{12}$$

An isothermal gas can arise in astrophysics when the gas on both sides of the shock is immersed in a strong radiation field that imposes its temperature on the gas, acting as a thermostat. Then something happens akin to what is illustrated in the figure below. The gas first encounters a real shock in which the temperature sharply rises. This shock is immediately followed by a transition layer where the excess thermal energy of the gas is radiated away. This stops when the gas attains the original (upstream) temperature. In this assignment we collapse this transition layer to zero thickness[1].

Consider a normal shock where the velocity is perpendicular to the shock front. The strength of the shock can be characterised by the isothermal Mach number

$$\mathcal{M} = u/c_s . \tag{13}$$

The table below gives the values of the flow parameters on both sides of the shock in the up- and downstream region.

Quantity	**upstream**	**downstream**
Velocity	u_1	u_2
Density	ρ_1	ρ_2
Mach number	$\mathcal{M}_1 = u_1/c_s$	$\mathcal{M}_2 = u_2/c_s$

1. As in any shock, the mass flux and the momentum flux must be the same on both sides of the shock so that no mass or momentum accumulates in the (infinitely thin) shock. Show that, in the case of a normal isothermal shock, mass- and momentum conservation imply the following jump conditions:

$$\rho_1 \mathcal{M}_1 = \rho_2 \mathcal{M}_2 \qquad \text{(mass conservation)} ; \tag{14}$$

$$\rho_1(\mathcal{M}_1^2 + 1) = \rho_2(\mathcal{M}_2^2 + 1) \quad \text{(momentum conservation)} . \tag{15}$$

2. Derive that these two jump conditions mean that the downstream Mach number $\mathcal{M}_2$ and the upstream Mach number $\mathcal{M}_1$ satisfy the relation

$$\mathcal{M}_1 \mathcal{M}_2^2 - (\mathcal{M}_1^2 + 1)\mathcal{M}_2 + \mathcal{M}_1 = 0 . \tag{16}$$

[1] One can show that for a steady shock, with $\partial/\partial t = 0$ for all relevant quantities, that simplification is not even needed. The results you will obtain in this assignment are then valid for a transition layer of arbitrary thickness! The reason is that the 'flux in = flux out' principle, on which the whole calculation is based, now holds for a volume bounded by any two surfaces that are positioned perpendicular to the flow: you can not store mass, momentum or energy in the volume between the two surfaces. If you did, the density and the velocity would no longer satisfy $\partial\rho/\partial t = 0$, $\partial\boldsymbol{u}/\partial t = 0$ within that volume! The obvious choice for the placement of these two surfaces is one immediately ahead of the shock, and one immediately behind the transition layer where the temperature returns to the upstream value.

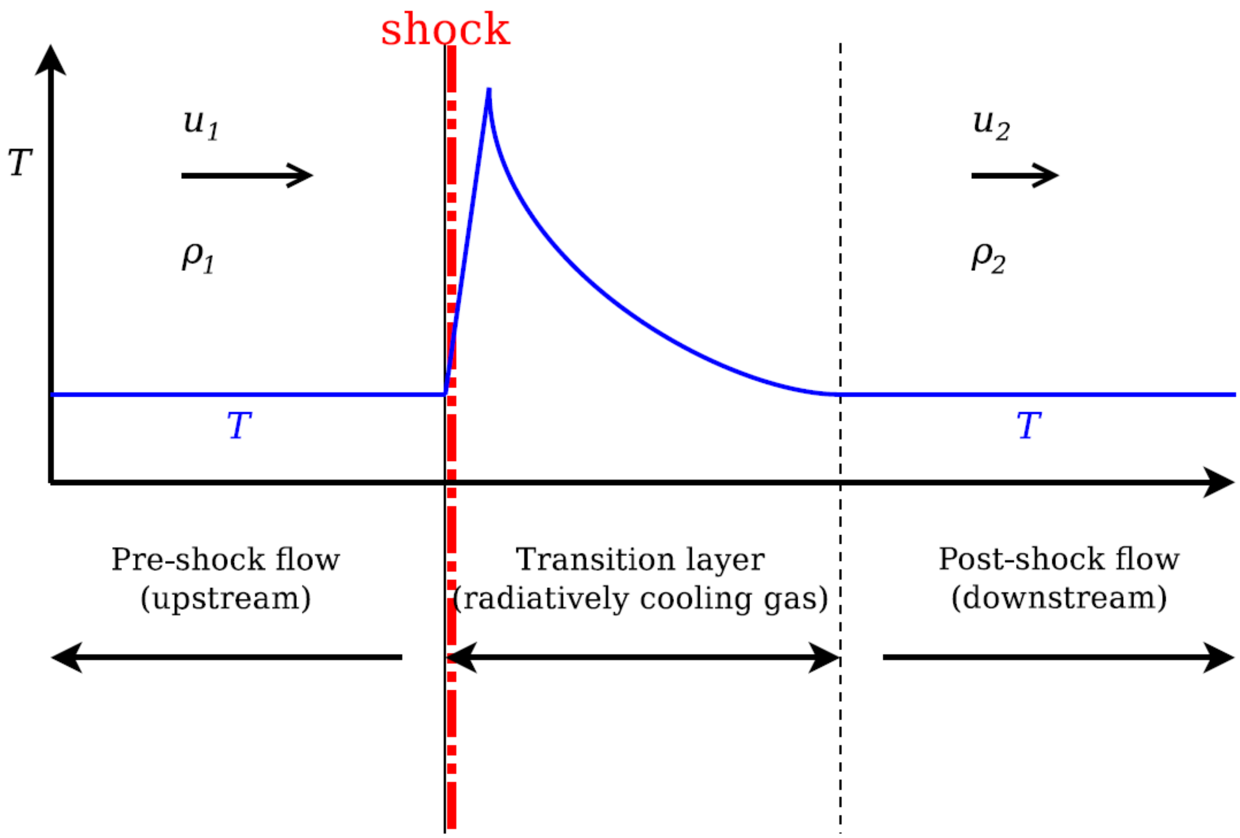

Figure 2: A schematic representation of the behaviour of the gas temperature T (blue curve) in an isothermal shock. In this figure, the gas flows from left to right. First the incoming gas encounters a true shock, where (as in any shock) the temperature, density and pressure rise sharply. Then the excess thermal energy per particle is radiated away as the gas cools in a transition layer behind the shock. The cooling stops when the temperature returns to the pre-shock value. The downstream state you are asked to calculate in this assignment corresponds to the state of the gas behind this transition layer.

3. Derive that there are two solutions, one trivial solution and one for a true isothermal shock:

$$\mathcal{M}_2 = \mathcal{M}_1 \quad \text{(the trivial `no shock' solution)}\,; \tag{17}$$

$$\mathcal{M}_2 = 1/\mathcal{M}_1 \quad \text{(the isothermal shock jump condition)}\,. \tag{18}$$

   Give the compression $r = \rho_2/\rho_1 = u_1/u_2$ in the shock solution. This dimensionless parameter determines the relation between upstream and downstream quantities u, ρ and P.

4. In the case treated in the lectures, we need three conservation laws (conservation of mass, momentum and energy, expressed in terms of their respective fluxes) to find the final relation linking the upstream state and the downstream state of the gas: the Rankine-Hugoniot relations.

   Give a *physical* argument why in the isothermal case the two laws of mass- and momentum conservation are sufficient to totally determine the downstream state, given the upstream state of the gas.

5. In an isothermal gas, the energy flux equals

$$\rho u \left[\frac{u^2}{2} + c_s^2 \ln\left(\frac{\rho}{\rho_0} \right) \right] \,, \tag{19}$$

   with ρ_0 an arbitrarily chosen reference density. It is convenient for this particular problem to take that density to be equal to the upstream density:

$$\rho_0 = \rho_1\,. \tag{20}$$

   Show, by rewriting the flux so that all but one ρ_i's (or all if you convert the last ρ_i into a pressure P_i) and all u_i's are gone, that this energy flux is not the same on both sides of the shock if $\mathcal{M}_1 \neq \mathcal{M}_2$. This means that due to the interaction of the gas with the radiation the energy per unit mass of the flow is no longer conserved!