Astrophysical Hydrodynamics – Assignment 8

Saikat Chatterjee: room 192, saikat@astro.rug.nl, phone: 8689

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Turbulence: Accretion disks

Many astrophysical situations turn out to be described best by the configuration of an (accretion) disk, e.g. protoplanetary disks, neutron stars, black holes etc. The vertical extent of the disk is often given in terms of the (local) scaleheight $H(r)$.

1. **α model of global turbulence**

   Often, the disk is accreting onto the parent body through viscous forces induced by turbulence. The turbulent kinematic viscosity ν_T can be obtained from the geometry/configuration of the system under consideration,

   $$\nu_T \sim v_T \cdot L, \tag{1}$$

   where v_T is the typical velocity (actually the velocity variation w.r.t. the mean flow velocity) of the eddies created by the turbulence and L is the typical size of the largest eddies. However, one of the problems of accretion disk theory is that we do not know the exact details of this turbulent behaviour in accretion disks. This means, for example, that we do not know what the typical velocity is and so we have to make a few educated guesses to be able to come up with a model at all.

   1. Our first good idea is to assume that the turbulent velocity is less than the speed of sound in the disk. In this case, v_T can be written as a fraction of the local speed of sound, i.e. $v_T = \alpha c_s$ (Shakura and Sunyaev 1973) where $\alpha < 1$. Thinking of the disk's geometry, what is a good estimate for L? What will be Shakura and Sunyaev's approximation for the turbulent viscosity?

   Due to the turbulent viscosity (as with any viscosity) energy will be dissipated and eventually converted into heat. The actual diffusion, however, mainly occurs on the smallest scales. This means that the largest eddies in the turbulent flow will not dissipate (most of) their energy themselves. They will pass on their energy to smaller eddies, that will pass it on to again smaller ones, until we reach the smallest scale eddies. These eddies will dissipate and their energy will be passed on to the medium. However, this process does imply that the total amount of energy that will be dissipated is determined by the largest eddies; they initially contain the energy that will be passed on and eventually dissipated. We can therefore make an order of magnitude estimate of the amount of energy that is dissipated by turbulence, as characterized by the velocity and size scales v_T and L.

   2. How does the energy dissipation scale with α?

2. **Viscous spreading**

   Over time, a disk (or a ring of a disk) will spread due to viscosity. The heat that is dissipated will be radiated away and thus the ring will lose energy and go into an orbit closer to the object. Ultimately this leads to accretion on the central object

   In such a disk we have, apart from the orbital velocity v_θ, also a small radial flow v_r, caused by the effects of viscosity. We shall assume that $v_\theta \gg c_s$ such that we can neglect pressure forces. Alternatively, one can state that the disk should be thin, i.e. $H \ll R$.

Since this is in principle a two-dimensional problem, the continuity and Navier-Stokes equations can be integrated over z with surface densities $\Sigma = \int \rho dz$ so they become

$$\begin{aligned} \dot{\Sigma} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\Sigma v_r\right) &= 0 \text{ Continuity} \quad (2) \\ \Sigma\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_r v_\theta}{r}\right) &= \nu\left(\Delta v_\theta - \frac{v_\theta}{r^2}\right) \text{ Navier-Stokes} \quad (3) \end{aligned}$$

where we should be careful in treating the viscous terms, since the viscosity is dependent on r. This can be rearranged to

$$\dot{\Sigma} = -\frac{1}{r}\frac{\partial}{\partial r}\left(\left(\frac{d}{dr}\left(r v_\theta\right)\right)^{-1}\frac{\partial}{\partial r}\left(\nu\Sigma r^3\frac{d}{dr}\left(\frac{v_\theta}{r}\right)\right)\right) \quad (4)$$

3. In the case of a Keplerian disk v_θ is given by $v_\theta(r) = \beta r^{-\frac{1}{2}}$ for a certain β. Give an expression for $\dot{\Sigma}$ for a Keplerian disk in terms of Σ, ν and r.
4. **Optional:** Consider a ring of mass m which is initially located at $r = r_0$, i.e.

$$\Sigma(r, t_0) = \frac{m}{2\pi r_0}\delta(r - r_0) \quad (5)$$

The solution to the equation in question 3 for $t > 0$ is

$$\Sigma(x, \tau) = \frac{m}{\pi r_0^2 \tau x^{\frac{1}{4}}} e^{-\frac{1+x^2}{\tau}} I_{\frac{1}{4}}\left(\frac{2x}{\tau}\right) \quad (6)$$

with the dimensionless variables $x = \frac{r}{r_0}$ and $\tau = 12\frac{\nu t}{r_0^2}$. $I_{\frac{1}{4}}$ is the modified Bessel function of fractional order.

Use your favorite plotting tool to plot Σ as function of x at various (dimensionless) times τ (e.g. $\tau = \{0.03, 0.1, 3.0, 10\}$) so you will get an idea of what's happening. Use $m = 1$ and $r_0 = 1$.

Describe what happens to the mass (where does it go and why) and to the angular momentum of the system.