# Astrophysical Hydrodynamics – Assignment 7

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## 1 Incompressible viscous flow

Consider fluid between two infinite flat plates separated by a distance $L$ (direction of $y$). The lower plate is stationary while the upper one is moving with a velocity $U$ parallel to itself (direction of $x$). The plates are also maintained at different temperatures. Neglect gravity and assume a steady flow in the x-direction, with $u = u(y), T = T(y)$. The governing equations are,

$$\frac{dp}{dx} - \mu \frac{d^2 u}{dy^2} = 0 \tag{1}$$

$$k \frac{d^2 T}{dy^2} + \mu \left( \frac{du}{dy} \right)^2 = 0 \tag{2}$$

1. **Velocity profile**

   If we assume that pressure gradient is imposed externally so that $dp/dx$ is a known constant, show that the velocity profile is,

   $$u(y) = \frac{y}{L} U - \frac{L^2}{2\mu} \frac{dp}{dx} \frac{y}{L} \left( 1 - \frac{y}{L} \right) \tag{3}$$

   [Hint: you have to use the boundary conditions: $u = 0$ at $y = 0$ and $u = U$ at $y = L$.]

2. **Couette flow**

   If no pressure gradient is imposed on the flow then, $\frac{dp}{dx} = 0$. Show that temperature profile $T(y)$ in this case turns out to be:

   $$\frac{(T - T_0)}{(T_1 - T_0)} = \frac{y}{L} + \frac{\mu}{2k} \frac{U^2}{(T_1 - T_0)} \frac{y}{L} \left( 1 - \frac{y}{L} \right) \tag{4}$$

   and maximum temperature occurs at

   $$y_m = \frac{L}{2} + \frac{k(T_1 - T_0)}{\mu U^2} L \tag{5}$$

   where $T = T_0$ at $y = 0$ and $T = T_1$ at $y = L$.

3. **Poiseuille flow**

   If the velocity of the upper plate is zero ($U = 0$) then the flow is driven solely by the pressure gradient. Show that maximum velocity in this case occurs at the center, $y = L/2$ and it is

   $$u_m = \frac{L^2}{8\mu} \frac{dp}{dx} \tag{6}$$

   If mean velocity is defined by volume flow per unit area, show that in this case it is given by,

   $$\bar{U} = \frac{1}{L} \int_0^L u(y) dy = \frac{2}{3} u_m \tag{7}$$

## 2 Taylor-Couette Flow

In this exercise we will calculate the (steady state) flow of a fluid between two rotating (hollow) cylinders. The system is cylindrically symmetric and can be approximated as infinite in length. Therefore the best coordinate system is cylindrical, $r$,$\phi$,$z$. Assume the two cylinders have radii $R_1$ and $R_2$ with $R_2 > R_1$ and angular velocities $\Omega_1$ and $\Omega_2$. Symmetry arguments lead us conclude that

$$\begin{aligned} v_z &= 0 &(8)\\ v_r &= 0 &(9)\\ v_\phi &= v_\phi(r) \equiv \Omega r &(10)\\ P &= P(r) &(11) \end{aligned}$$

The cylindrical Navier-Stokes equations are written as (Landau & Lifshitz p. 48)

$$\begin{aligned} \frac{\partial v_r}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\, v_r - \frac{v_\phi^2}{r} &= -\frac{1}{\rho}\frac{\partial P}{\partial r} + \nu\left(\Delta v_r - \frac{2}{r^2}\frac{\partial v_\phi}{\partial \phi} - \frac{v_r}{r^2}\right) &(12)\\ \frac{\partial v_\phi}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\, v_\phi + \frac{v_\phi v_r}{r} &= -\frac{1}{\rho r}\frac{\partial P}{\partial \phi} + \nu\left(\Delta v_\phi - \frac{2}{r^2}\frac{\partial v_r}{\partial \phi} - \frac{v_\phi}{r^2}\right) &(13)\\ \frac{\partial v_z}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\, v_z &= -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\Delta v_z &(14) \end{aligned}$$

with $\nu$ the kinematic viscosity ($\nu = \eta/\rho$).

1. Show that under the above cylindrical symmetry assumptions the Navier-Stokes equations reduce to

$$\begin{aligned} \frac{\partial P}{\partial r} &= \frac{\rho v_\phi^2}{r} &(15)\\ \frac{\partial^2 v_\phi}{\partial r^2} + \frac{1}{r}\frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r^2} &= 0 &(16)\\ &&(17) \end{aligned}$$

2. Solve the latter equation by assuming a power law ($v_\phi = Cr^n$) and show that the general solution for the velocity obeys

$$v_\phi = \Omega r = Ar + \frac{B}{r}. \quad (18)$$

   $A$ and $B$ are given by boundary conditions, what are the boundary conditions? Derive expressions for $A$ and $B$.

Such flow becomes unstable to the formation of Taylor vortices if $\Omega_1 \gg \Omega_2$.

> For low angular velocities the flow is steady and purely azimuthal. This basic state is known as *circular Couette flow*, after Maurice Marie Alfred Couette who used this experimental device as a means to measure viscosity. Sir Geoffrey Ingram Taylor investigated the stability of the Couette flow in a ground-breaking paper which has been a cornerstone in the development of hydrodynamic stability theory.
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> Taylor showed that when the angular velocity of the inner cylinder is increased above a certain threshold, Couette flow becomes unstable and a secondary steady state characterized by axisymmetric toroidal vortices, known as *Taylor vortex flow*, emerges. Subsequently increasing the angular speed of the cylinder the system undergoes a progression of instabilities which lead to states with greater spatio-temporal complexity, with the next state being called as *wavy vortex flow*. If the two cylinders rotate in opposite sense then *spiral vortex flow* arises. Beyond a certain Reynolds number there is the onset of turbulence.

We will examine a few cases of (in)stability in this flow in what follows.

3. The centrifugal force $F_C$ acting on the fluid particles is balanced by the pressure force $F_P$. Calculate the angular momentum $\mu$ of a fluid particle at position $r$ and show that the centrifugal force $F_C = \Omega^2 r$ can be written as

$$F_C(r) = \frac{\mu(r)^2}{mr^3}. \tag{19}$$

Hint: $\mu(r) = |\mathbf{p} \times \mathbf{r}|$.

4. Consider a particle initially at radial postion $r = r_0$ is displaced by a very small fraction to $r' = r_0 + \delta r > r_0$. If we assume angular momentum conservation, the centrifugal force of the displaced particle is

$$F_C = \frac{\mu(r_0)^2}{mr'^3} \equiv \frac{\mu_0^2}{mr'^3}, \tag{20}$$

but the pressure force is the same as for the other particles at that distance given by

$$F_P = \frac{\mu(r')^2}{mr'^3} \equiv \frac{\mu'^2}{mr'^3}, \tag{21}$$

Stability occurs when the forces on a the displaced particle will push it back, i.e. $F_P > F_C$. Therefore stability occurs when $\mu'^2 > \mu_0^2$.

Use a Taylor expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)(x-a)^n \tag{22}$$

of $\mu(r')$ around $r_0$ to show that the criterion can be written as

$$\mu \frac{\partial \mu}{\partial r} > 0 \tag{23}$$

Note that $(r' - r_0)$ is a positive difference.

5. Evaluate this expression to

$$\left(\Omega_2 R_2^2 - \Omega_1 R_1^2\right) \Omega > 0 \tag{24}$$

($\Omega = v_\phi / r$).

Hint 1: evaluate $\partial_r \Omega$ and $\partial_r \mu$.

Hint 2: always-positive terms are always positive.

6. Argue for each of the following cases whether the flow is stable or instable.

    1. The cylinders rotate in a different direction
    2. The cylinders rotate in the same direction
        (a) The center cylinder is stationary
        (b) The outer cylinder is stationary