Astrophysical Hydrodynamics – Assignment 5

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This assignment and the next one are on gravity waves, capillary waves and sound waves. They are actually part of same assignment set and based on the lectures on Sound waves.

1 Sound Waves

We will calculate the energy of sound in an adiabatic fluid.

1. Assume that ρ is the density of the fluid, e the internal energy per unit mass and v the speed of the fluid. Give a general expression for the total energy E of a unit volume of the fluid.

2. Use the thermodynamic relation $de = Tds - pdV$ and the assumption of an isentropic fluid to calculate an expression for $\frac{\partial}{\partial \rho}(\rho e)$ and $\frac{\partial^2}{\partial \rho^2}(\rho e)$

   (hint: note that the enthalpy $h = e + \frac{p}{\rho}$ and sound speed $c_s = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s}$).

3. Assume a small perturbation in the density, $\rho = \rho_0 + \delta\rho$, and the internal energy $e = e_0 + \delta e$. Do a Taylor expansion up to the 2nd order of ρe to ρ around ρ_0 and write the equation from (1) with the results from (2) to show that:

$$E \approx \rho_0 e_0 + h_0 \delta\rho + \frac{1}{2} c_s^2 \frac{\delta\rho^2}{\rho_0} + \frac{1}{2}\rho_0 v^2. \tag{1}$$

   Assume that $\delta\rho v^2$ can be neglected (it is a 3rd order term).

4. Why are the first term $\rho_0 e_0$ and the second term $h_0\delta\rho$ not relevant to our problem (of determining the total energy of the entire wave)? The energy per unit volume of fluid can then be written as

$$E = \frac{1}{2} c_s^2 \frac{\delta\rho^2}{\rho_0} + \frac{1}{2}\rho_0 v^2. \tag{2}$$

2 Gravity Waves

In this situation we assume a potential flow and that the vertical displacement ζ is very small. This give rise to the equations

$$\nabla^2 \Phi = 0 \quad \text{Poisson equation} \tag{3}$$

$$\left(\frac{\partial \Phi}{\partial z} + \frac{1}{g}\frac{\partial^2 \Phi}{\partial t^2}\right)_{z=\zeta\approx 0} = 0 \quad \text{boundary condition} \tag{4}$$

1. The depth of the fluid is h and take the surface at $z = 0$. Which new boundary condition should we impose on the fluid (and the potential)?

2. We (still) expect a simple periodic function in time as our solution:

$$\Phi = f(z)\cos(kx - \omega t). \tag{5}$$

Use Poisson's equation to find the general solution for $f(z)$

3. Using the boundary condition in (1), show that

$$\Phi = A\cosh(kz + kh)\cos(kx - \omega t) \tag{6}$$

for arbitrary A.

4. Use the old boundary condition in Eq. (4) to show that the relation between k and ω is

$$\omega^2 = gk\tanh(kh) \tag{7}$$

5. Calculate the velocity of propagation of the wave $U = \frac{\partial\omega}{\partial k}$ and show that for the limiting case that $\lambda \ll h$ this is just

$$U = \frac{1}{2}\sqrt{\frac{g}{k}} \tag{8}$$

3 Waves in an expanding fluid

We will consider the equations of fluid dynamics in an expanding background, e.g. the universe.

The equations of motion of a perfect fluid are:

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}) = 0 \qquad \text{Continuity Equation,} \tag{9}$$

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v} + \frac{1}{\rho}\boldsymbol{\nabla}p + \boldsymbol{\nabla}\Phi = 0 \qquad \text{Euler Equation,} \tag{10}$$

$$\nabla^2\Phi = 4\pi G\rho \qquad \text{Poisson's Equation.} \tag{11}$$

We again assume an adiabatic fluid, with no spatial variations in the equation of state, so the sound speed c is given by

$$c^2 = \frac{p}{\rho} \tag{12}$$

We consider small perturbations

$$\rho = \rho_0 + \rho_1, \quad p = p_0 + p_1, \quad \mathbf{v} = \mathbf{v_0} + \mathbf{v_1}, \quad \Phi = \Phi_0 + \Phi_1 \tag{13}$$

1. **Expansion**

We have picked an arbitrary origin for our coordinate system and we write the expansion of the fluid as

$$\rho_0 = \rho_{00}\left(\frac{R_0}{R}\right)^3 \tag{14}$$

$$\mathbf{v}_0 = \frac{\dot{R}}{R}\mathbf{r} = H\mathbf{r} \tag{15}$$

where $R(t)$ is governed by the Friedmann equations in case of the Universe.

Show that the perturbed solution of equations (10) - (12) are then given by

$$\dot{\rho}_1 + 3\frac{\dot{R}}{R}\rho_1 + \frac{\dot{R}}{R}(\mathbf{r}\cdot\boldsymbol{\nabla})\,\rho_1 + \rho_0\boldsymbol{\nabla}\cdot\mathbf{v}_1 = 0 \tag{16}$$

$$\dot{\mathbf{v_1}} + \frac{\dot{R}}{R}\mathbf{v}_1 + \frac{\dot{R}}{R}(\mathbf{r}\cdot\boldsymbol{\nabla})\,\mathbf{v}_1 = -\frac{1}{\rho_0}\boldsymbol{\nabla}p_1 - \boldsymbol{\nabla}\Phi_1 \tag{17}$$

$$\nabla^2\Phi_1 = 4\pi G\rho_1 \tag{18}$$

2. **Expanding Wave**

   Since equations (17) to (19) are homogeneous we expect the perturbations to be (a superposition of) plane-wave solutions. Take as *ansatz*

$$\rho_1(r,t) = \rho_1(t) e^{\frac{i\mathbf{r}\cdot\mathbf{q}}{R}} \tag{19}$$

   and likewise for $\mathbf{v_1}$ and Φ_1. $\mathbf{q}/R(t)$ replaces the usual wavenumber $\mathbf{k}$ because we anticipate that the waves will be stretched by the expansion.

   Show that equations (17) to (19) can be written as

$$\dot{\rho}_1 + \frac{3\dot{R}}{R}\rho_1 + \frac{i\rho_0}{R}\mathbf{q}\cdot\mathbf{v}_1 = 0 \tag{20}$$

$$\dot{\mathbf{v}}_1 + \frac{\dot{R}}{R}\mathbf{v}_1 = -\frac{ic^2}{\rho_0 R}\rho_1\mathbf{q} - \boldsymbol{\nabla}\Phi_1 \tag{21}$$

$$\boldsymbol{\nabla}\Phi = -4\pi i G\rho_1 R\frac{\mathbf{q}}{q^2} \tag{22}$$

References

1. Landau L.D. and Lifshitz E. M., Course of Theoretical Physics, Vol. 6: Fluid Mechanics