Astrophysical Hydrodynamics – Assignment 2

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In this assignment we will start with a quick review of thermodynamics and after that we go into Euler equation, vorticity equation and Kelvin's circulation theorem.

1 Thermodynamics: Exact differentials, Maxwell relations

From the previous assignment we know that if $\boldsymbol{F}$ is a conservative vector field then, $\boldsymbol{F}\cdot d\boldsymbol{r} = \boldsymbol{\nabla}\phi\cdot d\boldsymbol{r} = d\phi$, is an exact differential. In component notation, $F_x dx + F_y dy + F_z dz = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = d\phi$. Remember in first law of thermodynamics, internal energy dU is an exact differential but heat $đQ$ and work done $đW$ aren't. They depend on the path, not only the initial and final points. But using tempertaure T as integrating factor we convert inexact differential $đQ$ to entropy $dS = \frac{đQ}{T}$.

1. Show from Stokes' theorem for two variables (also known as Green's theorem in plane), that if $dz = Mdx + Ndy$ has to be an exact differential the following condition is obeyed:

$$\left(\frac{\partial M}{\partial y}\right)_x = \left(\frac{\partial N}{\partial x}\right)_y. \tag{1}$$

2. From the four thermodynamic potentials U, H, F, G (internal energy, enthalpy, Helmholtz and Gibbs free energy respectively) derive four Maxwell relations of thermodynamics:

$$\begin{aligned} \left(\frac{\partial T}{\partial V}\right)_S &= -\left(\frac{\partial P}{\partial S}\right)_V, & \left(\frac{\partial T}{\partial P}\right)_S &= \left(\frac{\partial V}{\partial S}\right)_P, \\ \left(\frac{\partial S}{\partial V}\right)_T &= \left(\frac{\partial P}{\partial T}\right)_V, & -\left(\frac{\partial S}{\partial P}\right)_T &= \left(\frac{\partial V}{\partial T}\right)_P. \end{aligned} \tag{2}$$

2 Momentum conservation: the Euler equation

The Euler equation describes the rate of change of the momentum of a fluid with time. Remember that the time derivative of the momentum has units of force. For an inviscid (frictionless) fluid of volume V the Euler equation has three components:

- External volume forces $\boldsymbol{f}$ (force per unit mass, i.e. units of acceleration) affect every fluid volume element dV in the volume V. By multiplying with the density and integrating over the whole volume we get the momentum rate of change due to volume forces:

$$\int_V \rho \boldsymbol{f} dV\,. \tag{3}$$

- The pressure of the fluid also contributes to the rate of change of momentum of the fluid, because pressure is force per unit area and hence momentum transfer per unit time through a unit area element. This adds a term to the Euler equation that goes like

$$-\int_S p\boldsymbol{n} dS\,. \tag{4}$$

- Finally, there is momentum of fluid elements moving into and out of the volume V, i.e. the momentum flux:

$$-\int_S (\rho \boldsymbol{u})\boldsymbol{u}\cdot\boldsymbol{n}dS\,. \tag{5}$$

Together these form the Euler equation:

$$\frac{d}{dt}\int_V \rho\boldsymbol{u}dV = \int_V \rho\boldsymbol{f}dV - \int_S p\boldsymbol{n}dS - \int_S (\rho\boldsymbol{u})\boldsymbol{u}\cdot\boldsymbol{n}dS\,. \tag{6}$$

1. Derive the following form of the Euler equation:

$$\rho\frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla}p + \rho\boldsymbol{f}\,. \tag{7}$$

3 Advection

In this problem we consider infinitesimal line elements and see how they are transported along with the flow of a fluid.

1. At time t, neighbouring fluid particles A and B are at position vectors $\boldsymbol{r}$ and $\boldsymbol{r}+\boldsymbol{dl}$, respectively. At time $t+\delta t$, particle A is at $\boldsymbol{r}+\delta t\boldsymbol{u}(\boldsymbol{r})$, where $\boldsymbol{u}(\boldsymbol{r})$ is the velocity field of the fluid. Similarly, particle B is at $\boldsymbol{r}+\boldsymbol{dl}+\delta t\boldsymbol{u}(\boldsymbol{r}+\boldsymbol{dl})$. Use this to show that the time evolution of the line element $\boldsymbol{dl}$ which joins A and B is given by

$$\frac{D}{Dt}\boldsymbol{dl} = (\boldsymbol{dl}\cdot\boldsymbol{\nabla})\boldsymbol{u}\,. \tag{8}$$

   Line elements are generally stretched and rotated by the flow (similarly area elements and volume elements). This often leads to amazing patterns. Equation 8 describes how an infinitesimal line element is transported in a fluid. This mode of transportation of a quantity or in this case a vector is called advection, which is a subcategory of convection (convection also includes the separate process of dissipation).

4 Inviscid isentropic fluid flow

In an isentropic fluid the enthalpy per unit mass is given by,

$$dh \equiv Tds + vdp = vdp = \frac{dp}{\rho} \tag{9}$$

1. Assume a conservative force $\boldsymbol{f}$. Derive the vorticity equation for isentropic fluids from the Euler equation 7:

$$\frac{\partial\boldsymbol{\omega}}{\partial t} = \boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{\omega})\,. \tag{10}$$

   Hint: use the result $\boldsymbol{\nabla}(\frac{\boldsymbol{u}^2}{2}) = \boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{u}) + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}$

2. From thermodynamics we get,

$$\frac{Du}{Dt} = T\frac{Ds}{Dt} - p\frac{Dv}{Dt}. \tag{11}$$

   Now using the following equations for an ideal gas,

$$pv = RT, \qquad u = C_vT, \qquad C_p - C_v = R \tag{12}$$

   prove, if an air parcel has initial temperature and pressure T_0 and p_0 respectively, then an isentropic motion leads to the final temperature and pressure T and p such that following relation is true:

$$T = T_0\Big(\frac{p}{p_0}\Big)^{R/C_p} \tag{13}$$

5 Inviscid non-isentropic fluid flow

For a non isentropic fluid flow ($ds \neq 0$) under a conservative force field $\boldsymbol{f}$, the Euler equation becomes,

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{\omega}) + \frac{1}{\rho^2}(\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla} p). \tag{14}$$

1. Show that any moving particle carries with it a constant value of the product $(\boldsymbol{\omega} \cdot \boldsymbol{\nabla} s/\rho)$. In other words,

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega} \cdot \boldsymbol{\nabla} s}{\rho}\right) = 0 \tag{15}$$

2. Now we will derive the rate of change of entropy of an ideal gas during diabatic heating. First, using the same procedure as for the isentropic ideal gas flow, show that:

$$\frac{Ds}{Dt} = C_p \frac{D \ln \theta}{Dt} \tag{16}$$

where $1/\theta = p^{(R/C_p)}/T$. Now use $đQ = T ds$ and prove,

$$C_p \frac{D\theta}{Dt} = \frac{\theta}{T}\dot{Q} \tag{17}$$

The quantity θ is called potential temperature in the context of atmospheric physics. It shows that during an adiabatic flow of ideal gas, potential temperature is conserved which gives back the relationship (13) we proved before.

References

1. Zemansky M.W. and Dittman R.H., Heat and Thermodynamics
2. Landau L.D. and Lifshitz E. M., Course of Theoretical Physics, Vol. 6: Fluid Mechanics