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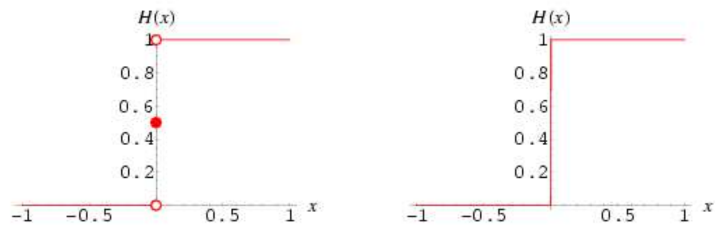
Heaviside Step Function

The Heaviside step function is a mathematical function denoted $H(x)$, or sometimes $\theta(x)$ or $u(x)$ (Abramowitz and Stegun 1972, p. 1020), and also known as the "unit step function." The term "Heaviside step function" and its symbol can represent either a piecewise constant function or a generalized function.



When defined as as a piecewise constant function, the Heaviside step function is given by

$$H(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x = 0 \\ 1 & x > 0 \end{cases} \quad (1)$$

(Abramowitz and Stegun 1972, p. 1020). The plot above shows this function (left figure), and and how it would appear if displayed on an oscilloscope (right figure).

When defined as a generalized function, it can be defined as a function $\theta(x)$ such that

$$\int \theta(x)\,\phi'(x)\,dx = -\phi(0) \quad (2)$$

for $\phi'(x)$ the derivative of a sufficiently smooth function $\phi(x)$ that decays sufficiently quickly (Kanwal 1998).

Mathematica represents the Heaviside generalized function as `HeavisideTheta`, while using `UnitStep` to represent the piecewise function `Piecewise[{{1, x >= 0}}]` (which, it should be noted, adopts the convention $H(0) = 1$ instead of the conventional definition $H(0) = 1/2$).

The shorthand notation

$$H_c(x) \equiv H(x-c) \quad (3)$$

is sometimes also used.

The Heaviside step function is related to the boxcar function by

$$\Pi(x) = H\left(x+\tfrac{1}{2}\right) - H\left(x-\tfrac{1}{2}\right) \quad (4)$$

and can be defined in terms of the sgn function by

$$H(x) = \tfrac{1}{2}\left[1 + \operatorname{sgn}(x)\right]. \quad (5)$$

The derivative of the step function is given by

$$\frac{d}{dx} H(x) = \delta(x), \quad (6)$$

where $\delta(x)$ is the delta function (Bracewell 1994, p. 94).

The Heaviside step function is related to the ramp function $R(x)$ by

$$R(x) = x\,H(x), \quad (7)$$

and to the derivative of $R(x)$ by

$$\frac{d}{dx} R(x) = H(x). \quad (8)$$

The two are also connected through

$$R(x) = H(x) * H(x), \quad (9)$$

where $*$ denotes convolution.

Bracewell (1999) gives many identities, some of which include the following. Letting $*$ denote the convolution,

$$H(x) * f(x) = \int_{-\infty}^{x} f(x')\,d x' \qquad (10)$$

$$H(t) * H(t) = \int_{-\infty}^{\infty} H(u)\,H(t-u)\,d u \qquad (11)$$

$$= H(0) \int_{0}^{\infty} H(t-u)\,d u \qquad (12)$$

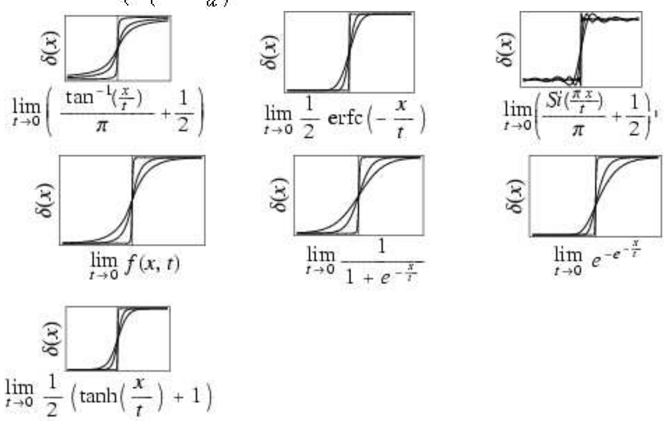
$$= H(0)\,H(t) \int_{0}^{t} d u \qquad (13)$$

$$= t\,H(t). \qquad (14)$$

In addition,

$$H(a x + b) = H\left(x + \frac{b}{a}\right) H(a) + H\left(-x - \frac{b}{a}\right) H(-a) \qquad (15)$$

$$= \begin{cases} H\left(x + \frac{b}{a}\right) & a > 0 \\ H\left(-x - \frac{b}{a}\right) & a < 0. \end{cases} \qquad (16)$$



The Heaviside step function can be defined by the following limits,

$$H(x) = \lim_{t\to 0}\left[\frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{x}{t}\right)\right] \qquad (17)$$

$$= \frac{1}{\sqrt{\pi}} \lim_{t\to 0} \int_{-x}^{\infty} t^{-1} e^{-u^2/t^2}\,d u \qquad (18)$$

$$= \frac{1}{2} \lim_{t\to 0} \operatorname{erfc}\left(-\frac{x}{t}\right) \qquad (19)$$

$$= \frac{1}{\pi} \lim_{t\to 0} \int_{-\infty}^{x} t^{-1} \operatorname{sinc}\left(\frac{u}{t}\right) d u = \frac{1}{\pi} \lim_{t\to 0} \int_{-\infty}^{x} \frac{1}{u} \sin\left(\frac{u}{t}\right) d u \qquad (20)$$

$$= \frac{1}{2} + \frac{1}{\pi} \lim_{t\to 0} \operatorname{si}\left(\frac{\pi x}{t}\right) \qquad (21)$$

$$= \lim_{t\to 0} \begin{cases} \frac{1}{2} e^{x/t} & \text{for } x \le 0 \\ 1 - \frac{1}{2} e^{-x/t} & \text{for } x \ge 0 \end{cases} \qquad (22)$$

$$= \lim_{t\to 0} \frac{1}{1 + e^{-x/t}} \qquad (23)$$

$$= \lim_{t\to 0} e^{-e^{-x/t}} \qquad (24)$$

$$= \frac{1}{2} \lim_{t\to 0} \left[1 + \tanh\left(\frac{x}{t}\right)\right] \qquad (25)$$

$$= \lim_{t\to 0} \int_{-\infty}^{x} t^{-1} \Lambda\left(\frac{x - \frac{1}{2} t}{t}\right) d x, \qquad (26)$$

where $\operatorname{erfc}(x)$ is the erfc function, $\operatorname{si}(x)$ is the sine integral, $\operatorname{sinc}(x)$ is the sinc function, and $\Lambda(x)$ is the one-argument triangle function. The first four of these are illustrated above for $t = 0.2$, 0.1, and 0.01.

Of course, any monotonic function with constant unequal horizontal asymptotes is a Heaviside step function under appropriate scaling and possible reflection. The Fourier transform of the Heaviside step function is given by

$$\mathcal{F}[H(x)] = \int_{-\infty}^{\infty} e^{-2\pi i k x} H(x)\,d x = \frac{1}{2}\left[\delta(k) - \frac{i}{\pi k}\right], \qquad (27)$$

where $\delta(k)$ is the delta function.

SEE ALSO: Absolute Value, Boxcar Function, Delta Function, Fourier Transform--Heaviside Step Function, Ramp Function, Rectangle Function, Sgn, Sigmoid Function, Square Wave, Triangle Function. [Pages Linking Here]

RELATED WOLFRAM SITES:

http://functions.wolfram.com/GeneralizedFunctions/UnitStep/,
http://functions.wolfram.com/GeneralizedFunctions/UnitStep2/

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