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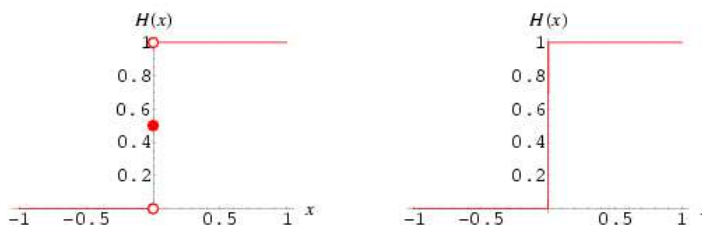
Created, developed, and  
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## Heaviside Step Function

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The Heaviside step function is a mathematical function denoted  $H(x)$ , or sometimes  $\theta(x)$  or  $u(x)$  (Abramowitz and Stegun 1972, p. 1020), and also known as the "unit step function." The term "Heaviside step function" and its symbol can represent either a [piecewise constant function](#) or a [generalized function](#).



When defined as a piecewise constant function, the Heaviside step function is given by

$$H(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x = 0 \\ 1 & x > 0 \end{cases} \quad (1)$$

(Abramowitz and Stegun 1972, p. 1020). The plot above shows this function (left figure), and and how it would appear if displayed on an oscilloscope (right figure).

When defined as a [generalized function](#), it can be defined as a function  $\theta(x)$  such that

$$\int \theta(x) \phi'(x) dx = -\phi(0) \quad (2)$$

for  $\phi'(x)$  the derivative of a sufficiently smooth function  $\phi(x)$  that decays sufficiently quickly (Kanwal 1998).

*Mathematica* represents the Heaviside generalized function as `HeavisideTheta`, while using `UnitStep` to represent the piecewise function `Piecewise[{{1, x >= 0}}`] (which, it should be noted, adopts the convention  $H(0) = 1$  instead of the conventional definition  $H(0) = 1/2$ ).

The shorthand notation

$$H_c(x) \equiv H(x - c) \quad (3)$$

is sometimes also used.

The Heaviside step function is related to the [boxcar function](#) by

$$\Pi(x) = H\left(x + \frac{1}{2}\right) - H\left(x - \frac{1}{2}\right) \quad (4)$$

and can be defined in terms of the [sgn](#) function by

$$H(x) = \frac{1}{2} [1 + \text{sgn}(x)]. \quad (5)$$

The [derivative](#) of the step function is given by

$$\frac{d}{dx} H(x) = \delta(x), \quad (6)$$

where  $\delta(x)$  is the [delta function](#) (Bracewell 1994, p. 94).

The Heaviside step function is related to the [ramp function](#)  $R(x)$  by

$$R(x) = x H(x), \quad (7)$$

and to the derivative of  $R(x)$  by

$$\frac{d}{dx} R(x) = H(x). \quad (8)$$


The two are also connected through

$$R(x) = H(x) * H(x), \quad (9)$$

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where  $*$  denotes [convolution](#).

Bracewell (1999) gives many identities, some of which include the following. Letting  $*$  denote the [convolution](#),

$$H(x) * f(x) = \int_{-\infty}^{\infty} f(x') dx' \tag{10}$$

$$H(t) * H(t) = \int_{-\infty}^{\infty} H(u) H(t-u) du \tag{11}$$

$$= H(0) \int_0^{\infty} H(t-u) du \tag{12}$$

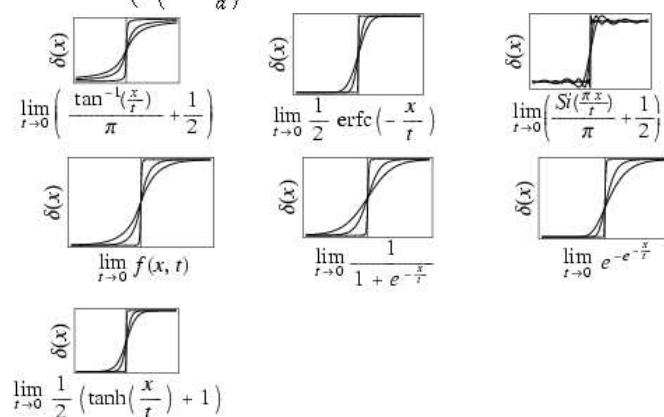
$$= H(0) H(t) \int_0^{\infty} du \tag{13}$$

$$= t H(t). \tag{14}$$

In addition,

$$H(\alpha x + b) = H\left(x + \frac{b}{\alpha}\right) H(\alpha) + H\left(-x - \frac{b}{\alpha}\right) H(-\alpha) \tag{15}$$

$$= \begin{cases} H\left(x + \frac{b}{\alpha}\right) & \alpha > 0 \\ H\left(-x - \frac{b}{\alpha}\right) & \alpha < 0. \end{cases} \tag{16}$$



The Heaviside step function can be defined by the following limits,

$$H(x) = \lim_{t \rightarrow 0} \left[ \frac{1}{2} + \frac{1}{\pi} \tan^{-1} \left( \frac{x}{t} \right) \right] \tag{17}$$

$$= \frac{1}{\sqrt{\pi}} \lim_{t \rightarrow 0} \int_{-x}^{\infty} t^{-1} e^{-u^2/t^2} du \tag{18}$$

$$= \frac{1}{2} \lim_{t \rightarrow 0} \operatorname{erfc} \left( -\frac{x}{t} \right) \tag{19}$$

$$= \frac{1}{\pi} \lim_{t \rightarrow 0} \int_{-\infty}^x t^{-1} \operatorname{sinc} \left( \frac{u}{t} \right) du = \frac{1}{\pi} \lim_{t \rightarrow 0} \int_{-\infty}^x \frac{1}{u} \sin \left( \frac{u}{t} \right) du \tag{20}$$

$$= \frac{1}{2} + \frac{1}{\pi} \lim_{t \rightarrow 0} \operatorname{si} \left( \frac{\pi x}{t} \right) \tag{21}$$

$$= \lim_{t \rightarrow 0} \begin{cases} \frac{1}{2} e^{x/t} & \text{for } x \leq 0 \\ 1 - \frac{1}{2} e^{-x/t} & \text{for } x \geq 0 \end{cases} \tag{22}$$

$$= \lim_{t \rightarrow 0} \frac{1}{1 + e^{-x/t}} \tag{23}$$

$$= \lim_{t \rightarrow 0} e^{-e^{-x/t}} \tag{24}$$

$$= \frac{1}{2} \lim_{t \rightarrow 0} \left[ 1 + \tanh \left( \frac{x}{t} \right) \right] \tag{25}$$

$$= \lim_{t \rightarrow 0} \int_{-\infty}^x t^{-1} \Lambda \left( \frac{x - \frac{1}{2} t}{t} \right) dx, \tag{26}$$

where  $\operatorname{erfc}(x)$  is the [erfc function](#),  $\operatorname{si}(x)$  is the [sine integral](#),  $\operatorname{sinc}(x)$  is the [sinc function](#), and  $\Lambda(x)$  is the one-argument [triangle function](#). The first four of these are illustrated above for  $t = 0.2, 0.1,$  and  $0.01$ .

Of course, any monotonic function with constant unequal horizontal asymptotes is a Heaviside step function under appropriate scaling and possible reflection. The [Fourier transform](#) of the Heaviside step function is given by

$$\mathcal{F}[H(x)] = \int_{-\infty}^{\infty} e^{-2\pi i k x} H(x) dx = \frac{1}{2} \left[ \delta(k) - \frac{i}{\pi k} \right], \tag{27}$$

where  $\delta(k)$  is the [delta function](#).

**SEE ALSO:** [Absolute Value](#), [Boxcar Function](#), [Delta Function](#), [Fourier Transform--Heaviside Step Function](#), [Ramp Function](#), [Rectangle Function](#), [Sgn](#), [Sigmoid Function](#), [Square Wave](#), [Triangle Function](#). [[Pages Linking Here](#)]

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