Delta Function

The delta function is a generalized function that can be defined as the limit of a class of delta sequences. The delta function is sometimes called "Dirac's delta function" or the "impulse symbol" (Bracewell 1999). It is implemented in *Mathematica* as `DiracDelta`[*x*].

Formally, δ is a linear functional from a space (commonly taken as a Schwartz space S or the space of all smooth functions of compact support D) of test functions f. The action of δ on f, commonly denoted $\delta[f]$ or $\langle\delta, f\rangle$, then gives the value at 0 of f for any function f. In engineering contexts, the functional nature of the delta function is often suppressed.

The delta function can be viewed as the derivative of the Heaviside step function,

$$\frac{d}{dx}[H(x)] = \delta(x) \qquad (1)$$

(Bracewell 1999, p. 94).

The delta function has the fundamental property that

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a) \qquad (2)$$

and, in fact,

$$\int_{a-\epsilon}^{a+\epsilon} f(x)\,\delta(x-a)\,dx = f(a) \qquad (3)$$

for $\epsilon > 0$.

Additional identities include

$$\delta(x-a) = 0 \qquad (4)$$

for $x \neq a$, as well as

$$\delta(a\,x) = \frac{1}{|a|}\,\delta(x) \qquad (5)$$

$$\delta(x^2 - a^2) = \frac{1}{2\,|a|}\,[\delta(x+a) + \delta(x-a)] \qquad (6)$$

More generally, the delta function of a function of x is given by

$$\delta[g(x)] = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|}, \qquad (7)$$

where the x_is are the roots of g. For example, examine

$$\delta(x^2 + x - 2) = \delta[(x-1)(x+2)]. \qquad (8)$$

Then $g'(x) = 2\,x + 1$, so $g'(x_1) = g'(1) = 3$ and $g'(x_2) = g'(-2) = -3$, giving

$$\delta(x^2 + x - 2) = \tfrac{1}{3}\,\delta(x-1) + \tfrac{1}{3}\,\delta(x+2). \qquad (9)$$

The fundamental equation that defines derivatives of the delta function $\delta(x)$ is

$$\int f(x)\,\delta^{(n)}(x)\,dx \equiv -\int \frac{\partial f}{\partial x}\,\delta^{(n-1)}(x)\,dx. \qquad (10)$$

Letting $f(x) = x\,g(x)$ in this definition, it follows that

$$\int x\,g(x)\,\delta'(x)\,dx = -\int \delta(x)\,\frac{\partial}{\partial x}[x\,g(x)]\,dx \qquad (11)$$

$$= -\int \delta(x)\,[g(x) + x\,g'(x)]\,dx \qquad (12)$$

$$= -\int g(x)\,\delta(x)\,dx, \qquad (13)$$

where the second term can be dropped since $\int x\,g'(x)\,\delta(x)\,dx = 0$, so (13) implies

$$x\,\delta'(x) = -\delta(x). \qquad (14)$$

In general, the same procedure gives

$$\int [x^n\,f(x)]\,\delta^{(n)}(x)\,dx = (-1)^n \int \frac{\partial^n[x^n\,f(x)]}{\partial x^n}\,\delta(x)\,dx, \qquad (15)$$

but since any power of x times $\delta(x)$ integrates to 0, it follows that only the constant term contributes. Therefore, all terms multiplied by derivatives of $f(x)$ vanish, leaving $n!\,f(x)$, so

$$\int [x^n f(x)]\,\delta^{(n)}(x)\,d\,x = (-1)^n\,n!\int f(x)\,\delta(x)\,d\,x, \tag{16}$$

which implies

$$x^n\,\delta^{(n)}(x) = (-1)^n\,n!\,\delta(x). \tag{17}$$

Other identities involving the derivative of the delta function include

$$\delta'(-x) = -\delta'(x) \tag{18}$$

$$\int_{-\infty}^{\infty} f(x)\,\delta'(x-a)\,d\,x = -f'(a) \tag{19}$$

$$(\delta' * f)(a) = \int_{-\infty}^{\infty} \delta'(a-x)\,f(x)\,d\,x = f'(a) \tag{20}$$

where $*$ denotes convolution,

$$\int_{-\infty}^{\infty} |\delta'(x)|\,d\,x = \infty, \tag{21}$$

and

$$x^2\,\delta'(x) = 0. \tag{22}$$

An integral identity involving $\delta(1/x)$ is given by

$$\int_{-1}^{1} \delta\left(\frac{1}{x}\right) d\,x = 0. \tag{23}$$

The delta function also obeys the so-called sifting property

$$\int f(x)\,\delta(x-x_0)\,d\,x = f(x_0) \tag{24}$$

(Bracewell 1999, pp. 74-75).

A Fourier series expansion of $\delta(x-a)$ gives

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \delta(x-a)\cos(n\,x)\,d\,x \tag{25}$$

$$= \frac{1}{\pi}\cos(n\,a) \tag{26}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \delta(x-a)\sin(n\,x)\,d\,x \tag{27}$$

$$= \frac{1}{\pi}\sin(n\,a), \tag{28}$$

so

$$\delta(x-a) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}[\cos(n\,a)\cos(n\,x) + \sin(n\,a)\sin(n\,x)] \tag{29}$$

(30)

$$= \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\cos[n(x-a)]. \tag{31}$$

The delta function is given as a Fourier transform as

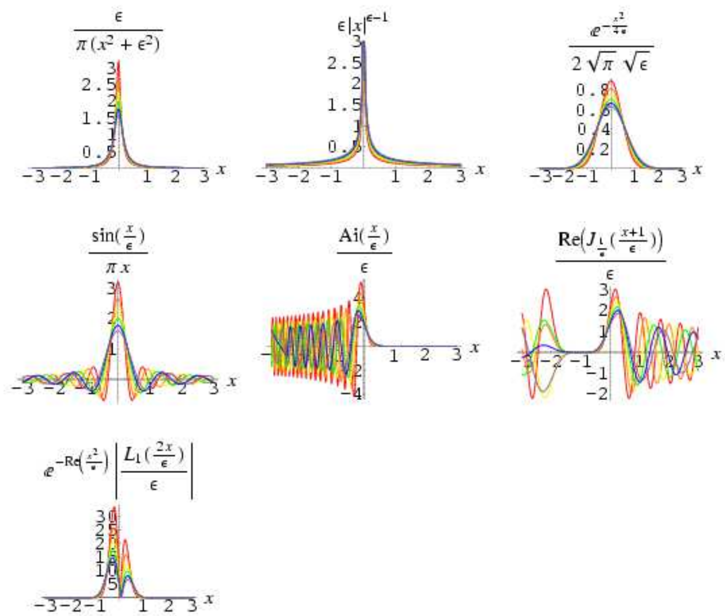
$$\delta(x) = \mathcal{F}_k[1](x) = \int_{-\infty}^{\infty} e^{-2\pi i k x}\,d\,k. \tag{32}$$

Similarly,

$$\mathcal{F}_x^{-1}[\delta(x)](k) = \int_{-\infty}^{\infty} \delta(x)\,e^{2\pi i k x}\,d\,x = 1 \tag{33}$$

(Bracewell 1999, p. 95). More generally, the Fourier transform of the delta function is

$$\mathcal{F}_x[\delta(x-x_0)](k) = \int_{-\infty}^{\infty} e^{-2\pi i k x}\,\delta(x-x_0)\,d\,x = e^{-2\pi i k x_0}. \tag{34}$$



The delta function can be defined as the following limits as $\epsilon \to 0$,

$$\delta(x) = \frac{1}{\pi}\lim_{\epsilon\to 0}\frac{\epsilon}{x^2+\epsilon^2}, \quad (35)$$

$$= \lim_{\epsilon\to 0}\epsilon|x|^{\epsilon-1} \quad (36)$$

$$= \lim_{\epsilon\to 0^+}\frac{1}{2\sqrt{\pi\epsilon}}e^{-x^2/(4\epsilon)} \quad (37)$$

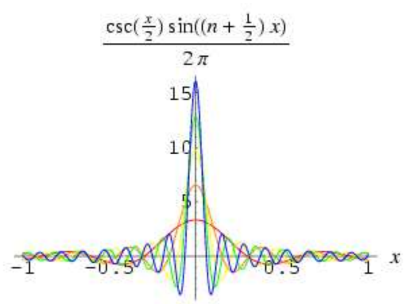
$$= \lim_{\epsilon\to 0}\frac{1}{\pi x}\sin\left(\frac{x}{\epsilon}\right) \quad (38)$$

$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\mathrm{Ai}\left(\frac{x}{\epsilon}\right) \quad (39)$$

$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}J_{1/\epsilon}\left(\frac{x+1}{\epsilon}\right) \quad (40)$$

$$= \lim_{\epsilon\to 0}\left|\frac{1}{\epsilon}e^{-x^2/\epsilon}L_n\left(\frac{2x}{\epsilon}\right)\right|, \quad (41)$$

where $\mathrm{Ai}(x)$ is an Airy function, $J_n(x)$ is a Bessel function of the first kind, and $L_n(x)$ is a Laguerre polynomial of arbitrary positive integer order.



The delta function can also be defined by the limit as $n \to \infty$

$$\delta(x) = \lim_{n\to\infty}\frac{1}{2\pi}\frac{\sin\left[\left(n+\frac{1}{2}\right)x\right]}{\sin\left(\frac{1}{2}x\right)}. \quad (42)$$

Delta functions can also be defined in two dimensions, so that in two-dimensional Cartesian coordinates

$$\delta^2(x,y) = \begin{cases}0 & x^2+y^2 \neq 0\\ \infty & x^2+y^2 = 0,\end{cases} \quad (43)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta^2(x,y)\,dx\,dy = 1 \quad (44)$$

$$\delta^2(ax,by) = \frac{1}{|ab|}\delta^2(x,y), \quad (45)$$

and

$$\delta^2(x,y) = \delta(x)\,\delta(y). \quad (46)$$

Similarly, in polar coordinates,

$$\delta^2(x, y) = \frac{\delta(r)}{\pi |r|} \quad (47)$$

(Bracewell 1999, p. 85).

In three-dimensional Cartesian coordinates

$$\delta^3(x, y, z) = \delta^3(\mathbf{x}) = \begin{cases} 0 & x^2 + y^2 + z^2 \neq 0 \\ \infty & x^2 + y^2 + z^2 = 0 \end{cases} \quad (48)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta^3(x, y, z)\, dx\, dy\, dz = 1 \quad (49)$$

and

$$\delta^3(x, y, z) = \delta(x)\,\delta(y)\,\delta(z). \quad (50)$$

in cylindrical coordinates (r, θ, z),

$$\delta^3(r, \theta, z) = \frac{\delta(r)\,\delta(z)}{\pi r}. \quad (51)$$

In spherical coordinates (r, θ, ϕ),

$$\delta^3(r, \theta, \phi) = \frac{\delta(r)}{2\pi r^2} \quad (52)$$

(Bracewell 1999, p. 85).

A series expansion in cylindrical coordinates gives

$$\delta^3(\mathbf{r_1} - \mathbf{r_2}) = \frac{1}{r_1}\,\delta(r_1 - r_2)\,\delta(\theta_1 - \theta_2)\,\delta(z_1 - z_2) \quad (53)$$

$$= \frac{1}{r_1}\,\delta(r_1 - r_2)\,\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{i m (\theta_1 - \theta_2)}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i k (z_1 - z_2)}\, dk. \quad (54)$$

The solution to some ordinary differential equations can be given in terms of derivatives of $\delta(x)$ (Kanwal 1998). For example, the differential equation

$$x(1 - x)\, y'' + (4 - 6x)\, y' - 6y = 0 \quad (55)$$

has classical solution

$$y(x) = \frac{C_1}{x^3} + \frac{x^2 - x - 1 + 2(x - 2)\ln(x - 1)}{x^3(x - 1)}\, C_2, \quad (56)$$

and distributional solution

$$y(x) = C_1\, \delta''(x) \quad (57)$$

(M. Trott, pers. comm., Jan. 19, 2006). Note that unlike classical solutions, a distributional solution to an nth-order ODE need not contain n independent constants.

SEE ALSO: Delta Sequence, Doublet Function, Fourier Transform--Delta Function, Generalized Function, Impulse Symbol, Poincaré-Bertrand Theorem, Shah Function, Sokhotsky's Formula. [Pages Linking Here]

RELATED WOLFRAM SITES: http://functions.wolfram.com/GeneralizedFunctions/DiracDelta/, http://functions.wolfram.com/GeneralizedFunctions/DiracDelta2/

REFERENCES:

Arfken, G. *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 481-485, 1985.

Bracewell, R. "The Impulse Symbol." Ch. 5 in *The Fourier Transform and Its Applications, 3rd ed.* New York: McGraw-Hill, pp. 69-97, 1999.

Dirac, P. A. M. *Quantum Mechanics, 4th ed.* London: Oxford University Press, 1958.

Gasiorowicz, S. *Quantum Physics.* New York: Wiley, pp. 491-494, 1974.

Kanwal, R. P. "Applications to Ordinary Differential Equations." Ch. 6 in *Generalized Functions, Theory and Technique, 2nd ed.* Boston, MA: Birkhäuser, pp. 291-255, 1998.

Papoulis, A. *Probability, Random Variables, and Stochastic Processes, 2nd ed.* New York: McGraw-Hill, pp. 97-98, 1984.

Spanier, J. and Oldham, K. B. "The Dirac Delta Function $\delta(x - a)$." Ch. 10 in *An Atlas of Functions.* Washington, DC: Hemisphere, pp. 79-82, 1987.

van der Pol, B. and Bremmer, H. *Operational Calculus Based on the Two-Sided Laplace Integral.* Cambridge, England: Cambridge University Press, 1955.

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