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Convolution Theorem

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Let $f(t)$ and $g(t)$ be arbitrary functions of time t with Fourier transforms. Take

$$f(t) = \mathcal{F}_\nu^{-1}[F(\nu)](t) = \int_{-\infty}^{\infty} F(\nu) e^{2\pi i \nu t}\, d\nu \qquad (1)$$

$$g(t) = \mathcal{F}_\nu^{-1}[G(\nu)](t) = \int_{-\infty}^{\infty} G(\nu) e^{2\pi i \nu t}\, d\nu, \qquad (2)$$

where $\mathcal{F}_\nu^{-1}(t)$ denotes the inverse Fourier transform (where the transform pair is defined to have constants $A = 1$ and $B = -2\pi$). Then the convolution is

$$f * g \equiv \int_{-\infty}^{\infty} g(t') f(t - t')\, dt' \qquad (3)$$

$$= \int_{-\infty}^{\infty} g(t') \left[\int_{-\infty}^{\infty} F(\nu) e^{2\pi i \nu (t - t')}\, d\nu \right] dt'. \qquad (4)$$

Interchange the order of integration,

$$f * g = \int_{-\infty}^{\infty} F(\nu) \left[\int_{-\infty}^{\infty} g(t') e^{-2\pi i \nu t'}\, dt' \right] e^{2\pi i \nu t}\, d\nu \qquad (5)$$

$$= \int_{-\infty}^{\infty} F(\nu) G(\nu) e^{2\pi i \nu t}\, d\nu \qquad (6)$$

$$= \mathcal{F}_\nu^{-1}[F(\nu) G(\nu)](t). \qquad (7)$$

So, applying a Fourier transform to each side, we have

$$\mathcal{F}[f * g] = \mathcal{F}[f] \mathcal{F}[g]. \qquad (8)$$

The convolution theorem also takes the alternate forms

$$\mathcal{F}[f g] = \mathcal{F}[f] * \mathcal{F}[g] \qquad (9)$$

$$\mathcal{F}^{-1}(\mathcal{F}[f] \mathcal{F}[g]) = f * g \qquad (10)$$

$$\mathcal{F}^{-1}(\mathcal{F}[f] * \mathcal{F}[g]) = f g. \qquad (11)$$

SEE ALSO: Autocorrelation, Convolution, Fourier Transform, Wiener-Khinchin Theorem. [Pages Linking Here]

REFERENCES:

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