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Convolution

A convolution is an integral that expresses the amount of overlap of one function g as it is shifted over another function f. It therefore "blends" one function with another. For example, in synthesis imaging, the measured dirty map is a convolution of the "true" CLEAN map with the dirty beam (the Fourier transform of the sampling distribution). The convolution is sometimes also known by its German name, *faltung* ("folding").

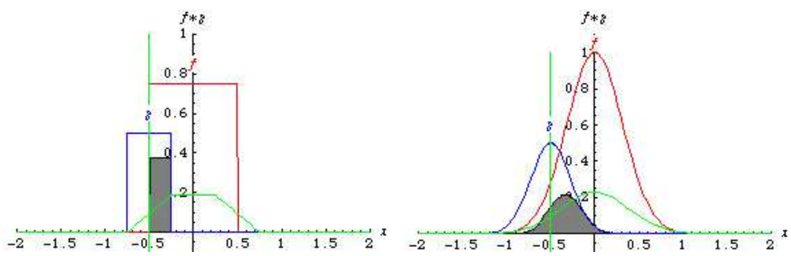
Abstractly, a convolution is defined as a product of functions f and g that are objects in the algebra of Schwartz functions in $\mathbb{R}^n$. Convolution of two functions f and g over a finite range $[0, t]$ is given by

$$f * g \equiv \int_0^t f(\tau)\, g(t-\tau)\, d\tau, \qquad (1)$$

where the symbol $f*g$ (occasionally also written as $f \otimes g$) denotes convolution of f and g. Convolution is more often taken over an infinite range,

$$f * g \equiv \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau = \int_{-\infty}^{\infty} g(\tau)\, f(t-\tau)\, d\tau \qquad (2)$$

(Bracewell 1999, p. 25).



The animations above graphically illustrate the convolution of two rectangle functions (left) and two Gaussians (right). In the plots, the green curve shows the convolution of the blue and red curves as a function of t, the position indicated by the vertical green line. The gray region indicates the product $g(\tau)\, f(t-\tau)$ as a function of t, so its area as a function of t is precisely the convolution.

The convolution of two rectangle functions $f = \Pi_{t_1,t_2}(t)$ and $g = \Pi_{u_1,u_2}(t)$ has the particularly simple form

$$\begin{aligned} f * g = {} & [(t-t_1-u_1)\,\Pi(t-t_1-u_1) - (t-t_2-u_1)\,\Pi \\ & (t-t_2-u_1) - (t-t_1-u_2)\,\Pi(t-t_1-u_2) + (t-t_2-u_2)\,\Pi(t-t_2-u_2)]. \end{aligned} \qquad (3)$$

Even more amazingly, the convolution of two Gaussians $f = e^{-(t-\mu_1)^2/(2\sigma_1^2)} / (\sigma_1 \sqrt{2\pi})$ and $g = e^{-(t-\mu_2)^2/(2\sigma_2^2)} / (\sigma_2 \sqrt{2\pi})$ is another Gaussian

$$f * g = \frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}}\, e^{-[t-(\mu_1+\mu_2)]^2/[2(\sigma_1^2+\sigma_2^2)]}. \qquad (4)$$

Let f, g, and h be arbitrary functions and a a constant. Convolution satisfies the properties

$$f * g = g * f \qquad (5)$$
$$f * (g * h) = (f * g) * h \qquad (6)$$
$$f * (g + h) = (f * g) + (f * h) \qquad (7)$$

(Bracewell 1999, p. 27), as well as

$$a(f * g) = (a f) * g = f * (a g). \qquad (8)$$

Taking the derivative of a convolution gives

$$\frac{d}{dx}(f * g) = \frac{df}{dx} * g + f * \frac{dg}{dx}. \qquad (9)$$

The area under a convolution is the product of areas under the factors,

$$\int_{-\infty}^{\infty} (f * g)\, dx = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(u)\, g(x-u)\, du\right] dx \qquad (10)$$
$$= \int_{-\infty}^{\infty} f(u) \left[\int_{-\infty}^{\infty} g(x-u)\, dx\right] du \qquad (11)$$

$$= \left[\int_{-\infty}^{\infty} f(u)\,du\right]\left[\int_{-\infty}^{\infty} g(x)\,dx\right]. \quad (12)$$

The horizontal function centroids add

$$\langle x(f*g)\rangle = \langle x f\rangle + \langle x g\rangle, \quad (13)$$

as do the variances

$$\langle x^2 (f*g)\rangle = \langle x^2 f\rangle + \langle x^2 g\rangle, \quad (14)$$

where

$$\langle x^n f\rangle \equiv \frac{\int_{-\infty}^{\infty} x^n f(x)\,dx}{\int_{-\infty}^{\infty} f(x)\,dx}. \quad (15)$$

There is also a definition of the convolution which arises in probability theory and is given by

$$F(t)*G(t) = \int F(t-x)\,dG(x), \quad (16)$$

where $\int F(t-x)\,dG(x)$ is a Stieltjes integral.

SEE ALSO: Autocorrelation, Cauchy Product, Convolution Theorem, Cross-Correlation, Recurrence Plot, Wiener-Khinchin Theorem. [Pages Linking Here]

REFERENCES:

Bracewell, R. "Convolution" and "Two-Dimensional Convolution." Ch. 3 in *The Fourier Transform and Its Applications, 3rd ed.* New York: McGraw-Hill, pp. 25-50 and 243-244, 1999.

Hirschman, I. I. and Widder, D. V. *The Convolution Transform.* Princeton, NJ: Princeton University Press, 1955.

Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 464-465, 1953.

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Convolution and Deconvolution Using the FFT." §13.1 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 531-537, 1992.

Weisstein, E. W. "Books about Convolution." http://www.ericweisstein.com/encyclopedias/books/Convolution.html.

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