LINEAR PHASE FILTERS

This linear phase filter description can be generalised into a formalism for four type of FIR filters:

Type 1: symmetric sequence of odd length

Type 2: symmetric sequence of even length

Type 3: anti-symmetric sequence of odd length

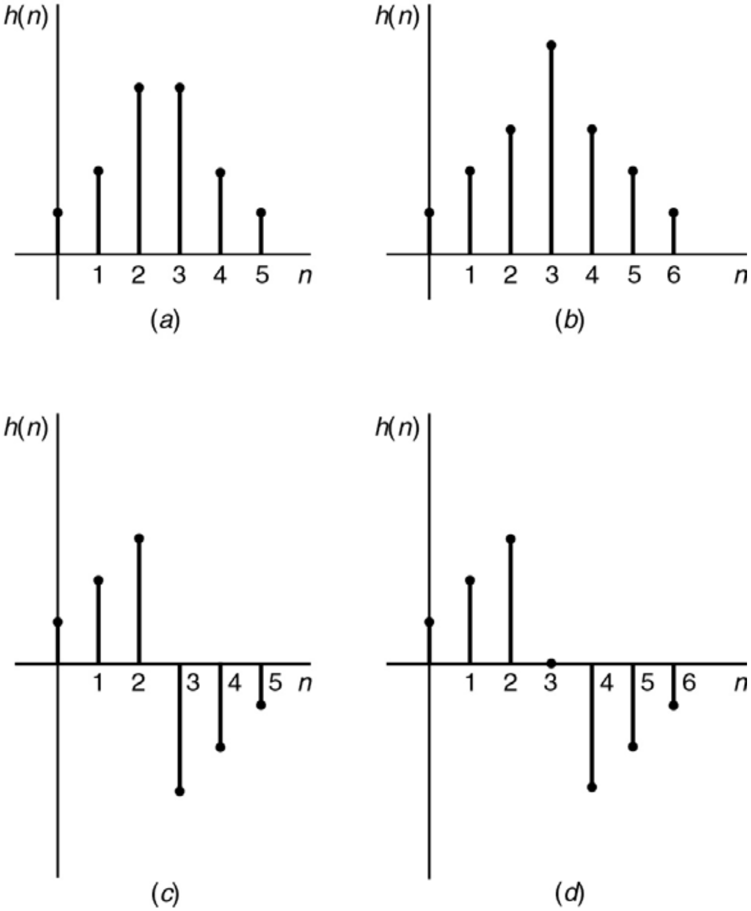
Type 4: anti-symmetric sequence of even length

LINEAR PHASE FILTERS

- ➲ There are four possible scenarios: filter length even or odd, and impulse response is either symmetric or antisymmetric

FIR II: even length, symmetric

FIR I: odd length, symmetric



FIR IV: even length, antisymmetric

FIR III: odd length, antisymmetric

Note for this case that h[M/2]=0

FIR I AND FIR II TYPES

- For symmetric coefficients, we can show that the frequency responses are of the following form:
- FIR II (M is odd, the sequence is symmetric and of even length)

$$\boxed{H(\omega) = 2e^{-j\underset{\alpha}{\frac{M}{2}}\omega}\overbrace{\left(\sum_{i=0}^{(M-1)/2} h[i]\cos\left(\frac{M}{2}-i\right)\omega\right)}^{G(\omega)}}$$

  - Note that this of the form $H(\omega)=e^{-j\alpha\omega}G(\omega)$, where $\alpha=M/2$, and $G(\omega)$ is the real quantity (the summation term)➔ Output is delayed by M/2 samples!
- FIR I (M is even, sequence is symmetric and of odd length)

$$\boxed{H(\omega) = e^{-j\frac{M}{2}\omega}\left(h[\frac{M}{2}] + 2\sum_{i=1}^{M/2} h[i]\cos\left(\frac{M}{2}-i\right)\omega\right)}$$

  - Again, this system has linear phase (the quantity inside the parenthesis is a real quantity) and the phase delay is M/2 samples.

FIR III AND FIR IV TYPES

- For antisymmetric sequences, we have h[n]=-h[M-n], which gives us ***sin*** terms in the summation expression:
- FIR IV (M is odd, the sequence is antisymmetric and of even length)

$$H(\omega) = 2e^{j\left[-\frac{M}{2}\omega+\frac{\pi}{2}\right]}\left(\sum_{i=0}^{(M-1)/2} h[i]\sin\left(\frac{M}{2}-i\right)\omega\right)$$

- FIR III (M is even, the sequence is antisymmetric and of odd length)

$$H(\omega) = 2e^{j\left[-\frac{M}{2}\omega+\frac{\pi}{2}\right]}\left(\sum_{i=1}^{M/2-1} h[i]\sin\left(\frac{M}{2}-i\right)\omega\right)$$

  - In both cases, the phase response is of the form θ(ω)=-(M/2) ω + π/2, hence generalized linear phase. Again, in all of theses cases, the filter output is delayed by M/2 samples. Also, for all cases, if G(ω)<0, an additional π term is added to the phase, which causes the samples to be flipped.

LINEAR PHASE FILTERS

Consider the length 3 FIR filter: $\boxed{h[n]=[\alpha_0,\alpha_1,\alpha_2]}$

the corresponding transfer function is: $\boxed{H(e^{j\omega})=\alpha_0+\alpha_1 e^{-j\omega}+\alpha_0 e^{-j2\omega}}$

or: $\boxed{H(e^{j\omega})=\alpha_0(1+e^{-j2\omega})+\alpha_1 e^{-j\omega}=\alpha_0(e^{j\omega}+e^{-j\omega})e^{-j\omega}+\alpha_1 e^{-j\omega}}$

which is a **Type I** FIR filter function: $\boxed{H(e^{j\omega})=(2\alpha_0\cos\omega+\alpha_1)e^{-j\omega}}$

the magnitude function and phase are:

$$\boxed{\left|H(e^{j\omega})\right|=\left|2\alpha_0\cos\omega+\alpha_1\right|,\quad \theta(\omega)=-\omega+\beta}$$

$$\boxed{\begin{array}{lll}\beta=0 & \text{if} & 2\alpha_0\cos\omega+\alpha_1>0\\ \beta=\pi & \text{if} & 2\alpha_0\cos\omega+\alpha_1<0\end{array}}$$

LINEAR PHASE FILTERS

To make a HP filter we want to stop low frequencies ($|H|=0$ at $\omega=0.1$) and pass high frequencies ($|H|= 1$ at $\omega=0.4$). So we need to solve:

$$\boxed{\begin{aligned}2\alpha_0\cos(0.1)+\alpha_1&=0\\2\alpha_0\cos(0.4)+\alpha_1&=1\end{aligned}}\Longrightarrow\boxed{\begin{aligned}\alpha_0&=-6.76195\\\alpha_1&=13.456335\end{aligned}}$$

using of the difference equation

$$\boxed{\begin{aligned}y[n]&=h[0]x[n]+h[1]x[n-1]+h[3]x[n-2]\\&=\alpha_0x[n]+\alpha_1x[n-1]+\alpha_0x[n-2]\end{aligned}}$$

we can write the corresponding FIR filter as:

$$\boxed{\begin{aligned}&y[n]=-6.76195(x[n]+x[n-2])+13.456335x[n-1]\\&\text{with}\quad x[n]=\{\cos(0.1n)+\cos(0.4n)\}\mu[n]\end{aligned}}$$

LINEAR PHASE FILTERS

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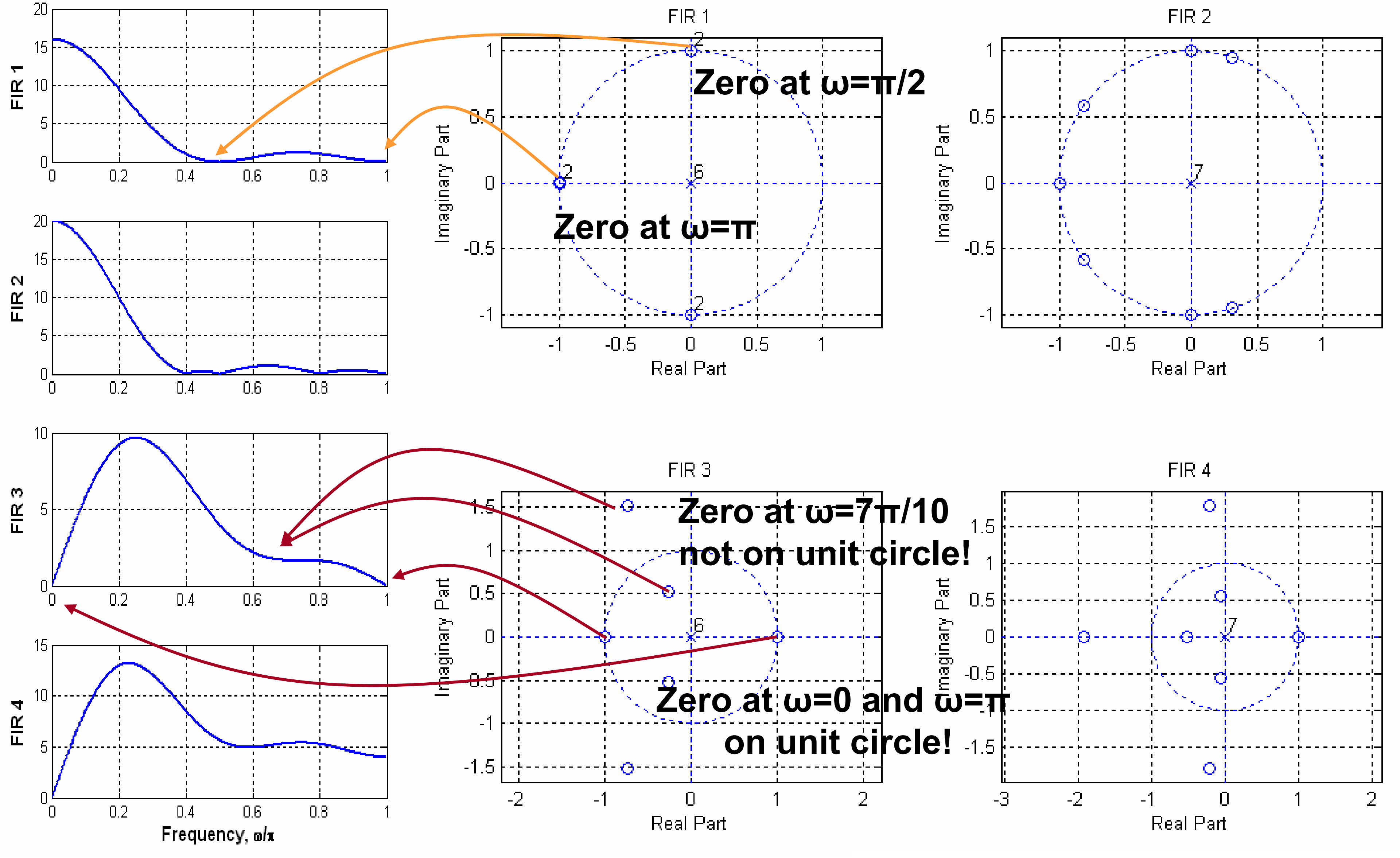
AN EXAMPLE – MATLAB DEMO

```
h1=[1 2 3 4 3 2 1]; % FIR 1
h2=[1 2 3 4 4 3 2 1]; % FIR 2
h3=[-1 -2 -3 0 3 2 1]; %FIR 3
h4=[-1 -2 -3 -4 4 3 2 1] ; % FIR 4


[H1 w]=freqz(h1, 1, 512); [H2 w]=freqz(h2, 1, 512);
[H3 w]=freqz(h3, 1, 512); [H4 w]=freqz(h4, 1, 512);

% Plot the magnitude and phase responses
% in angular frequency from 0 to pi
subplot(421); plot(w/pi, abs(H1));grid; ylabel('FIR 1')
subplot(422); plot(w/pi, unwrap(angle(H1)));grid;
subplot(423); plot(w/pi, abs(H2));grid; ylabel('FIR 2')
subplot(424); plot(w/pi, unwrap(angle(H2)));grid;
subplot(425); plot(w/pi, abs(H3));grid; ylabel('FIR 3')
subplot(426); plot(w/pi, unwrap(angle(H3)));grid;
subplot(427); plot(w/pi, abs(H4));grid
xlabel('Frequency, \omega/\pi'); ylabel('FIR 4')
subplot(428);  plot(w/pi, unwrap(angle(H4)));grid
xlabel('Frequency, \omega/\pi')
```

```
%Plot the zero - pole plots
figure
subplot(221)
zplane(h1,1); grid
title('FIR 1')
subplot(222)
zplane(h2,1);grid
title('FIR 2')
subplot(223)
zplane(h3,1);grid
title('FIR 3')
subplot(224)
zplane(h4,1);grid
title('FIR 4')
```

lin_phase_demo2.m

ZEROS & POLES OF AN FIR FILTER EXAMPLE



Zero at ω=π/2

Zero at ω=π

Zero at ω=7π/10 not on unit circle!

Zero at ω=0 and ω=π on unit circle!

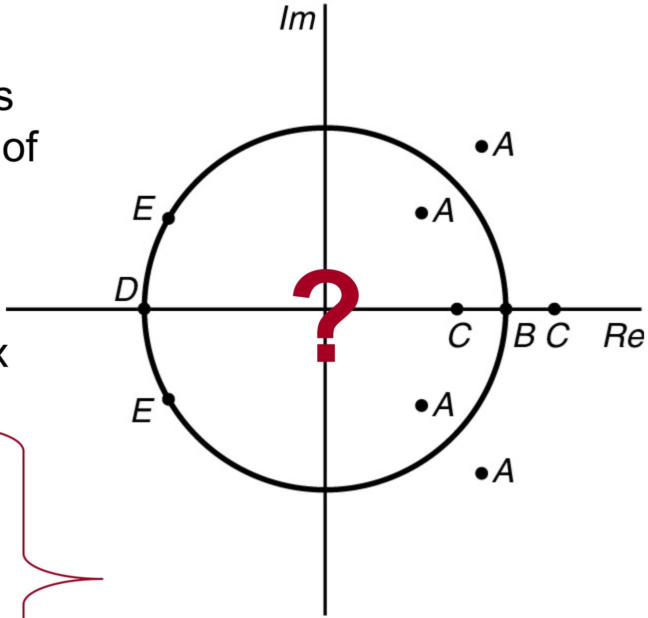
ZERO LOCATIONS OF LINEAR PHASE FILTERS

- For linear phase filters, the impulse response satisfies

$$h[n]=\pm h[M-n]\Rightarrow H(z)=\pm\sum_{n=0}^{M}h(M-n)z^{-n} \qquad H(z)=\pm z^{-M}H(z^{-1})$$

  - We can make the following observations as facts

1. If z_0 is a zero of H(z), so too is $1/z_0=z_0^{-1}$
2. Real zeros that are not on the unit circle, always occur in pairs such as $(1-\alpha z^{-1})$ and $(1-\alpha^{-1}z^{-1})$, which is a direct consequence of Fact 1.
3. If the zero is complex, $z_0=\alpha e^{j\theta}$, then its conjugate $\alpha e^{-j\theta}$ is also zero. But by the above statements, their symmetric counterparts $\alpha^{-1}e^{j\theta}$ and $\alpha^{-1}e^{-j\theta}$ must also be zero. So, complex zeros occur in quadruplets
4. If M is odd (filter length is even), and symmetric (that is, h[n]=h[M-n]), then H(z) must have a zero at z=-1
5. If the filter is antisymmetric, that is, h[n]=-h[M-n], then H(z) must have a zero at z=+1 for both odd and even M
6. If the filter is symmetric and h[n]=-h[M-n], and M is even, then H(z) must have a zero at z=-1



Why…? (p. 376)

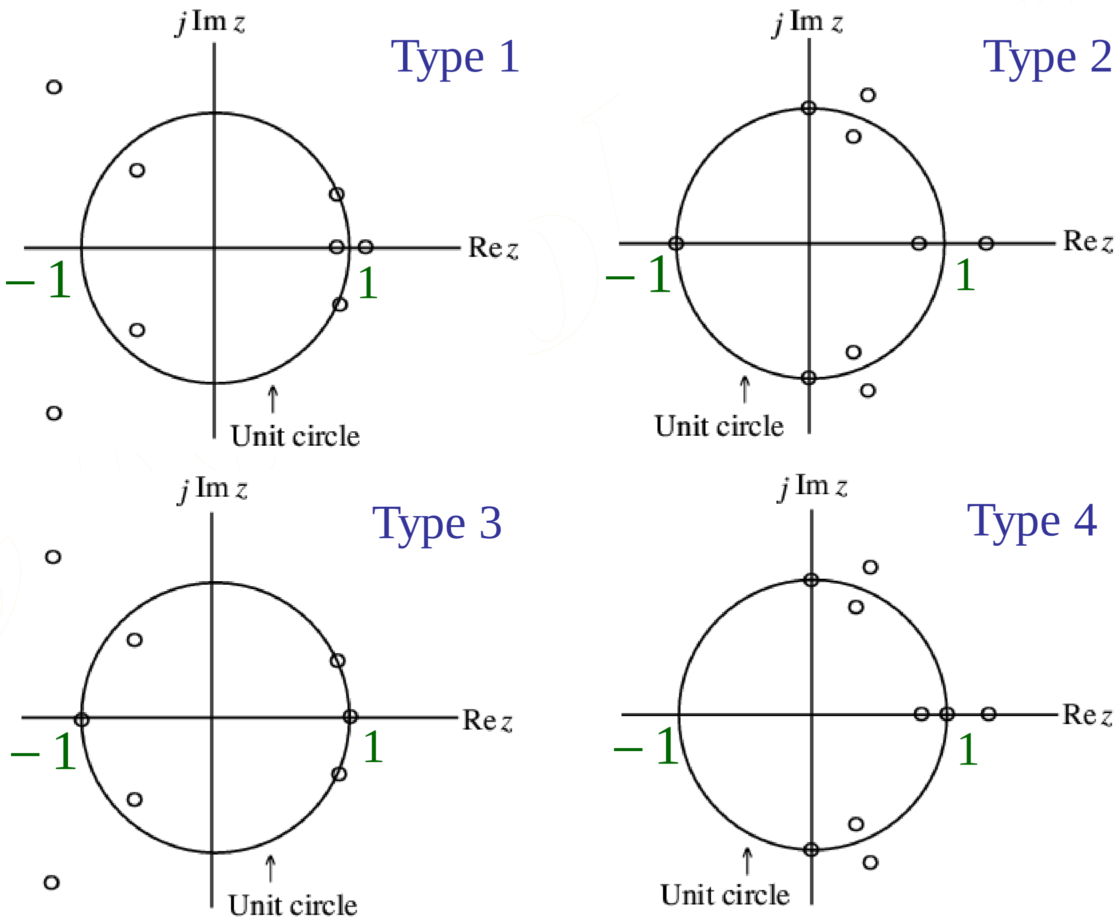
LINEAR PHASE FILTER ZERO LOCATIONS

Type 1 FIR filter: Either an even number or no zeros at z = 1 and z=-1

Type 2 FIR filter: Either an even number or no zeros at z = 1, and an odd number of zeros at z=-1

Type 3 FIR filter: An odd number of zeros at z = 1 and and z=-1

Type 4 FIR filter: An odd number of zeros at z = 1, and either an even number or no zeros at z=-1



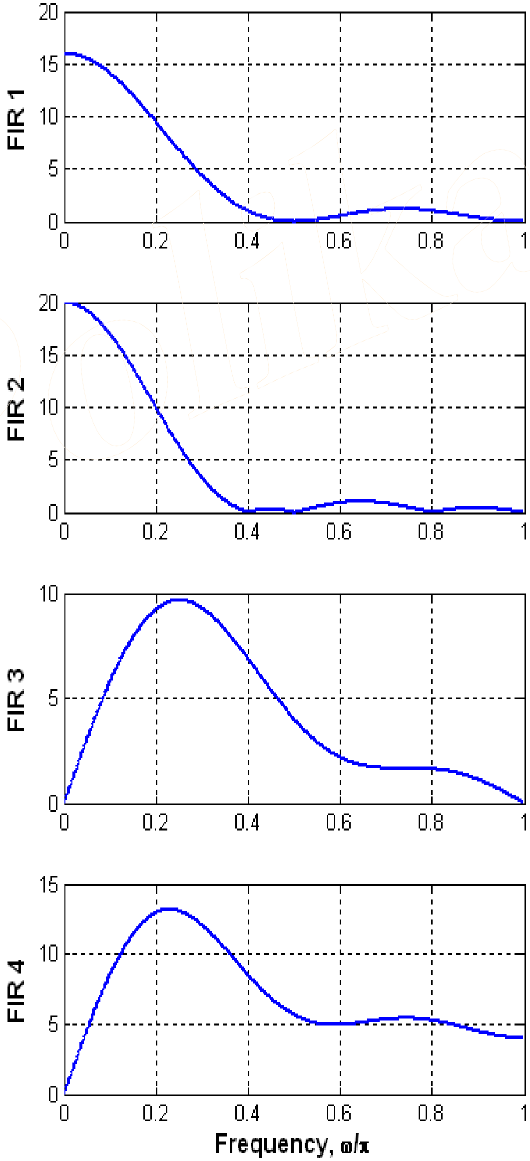
The presence of zeros at z=±1 leads to some limitations on the use of these linear-phase transfer functions for designing frequency-selective filters

A Type 2 FIR filter cannot be used to design a highpass filter since it always has a zero at z=-1

A Type 3 FIR filter has zeros at both z = 1 and z=-1, and hence cannot be used to design either a lowpass or a highpass or a bandstop filter

A Type 4 FIR filter is not appropriate to design a lowpass filter due to the presence of a zero at z = 1

Type 1 FIR filter has no such restrictions and can be used to design almost any type of filter



Type 1 FIR Filter: Either an even number or no zeros at z=1 and z=-1

Type 2 FIR Filter: Either an even number or no zeros at z=1 and an odd number of zeros at z=-1

Type 3 FIR Filter: An odd number of zeros at z=1 and z=-1

Type 4 FIR Filter: An odd number of zeros at z=1 and either an even number or no zeros at z=-1

So: Type 2 cannot be used as a highpass filter, type 4 not as a lowpass, and type 3 not as lowpass or highpass. No restrictions for type 1.

SIMPLE FIR FILTERS

- Two-point (M=1→ 1st order) moving average filter: simplest FIR filter
  - h[n]= (½) [1 1] = [½ ½] → h[n]= ½ (δ[n]+δ[n-1])
    → $H(z) = (½)(1+z^{-1}) = (z+1)/(2z)$
  - Notice that H(z) has a zero at z=-1, and a pole at z=0.
    - Remember: for stable systems (and FIR filters are always stable), the frequency response can be obtained by substituting $z=e^{j\omega}$. z=-1 ←→ ω=π
    - The zero at ω=π (suppress high frequency components π), coupled with the pole at z=0 (amplify zero frequency), makes this a lowpass filter

$$H(\omega) = H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}} = \frac{z+1}{2z}\Big|_{z=e^{j\omega}} = \frac{e^{j\omega}+1}{2e^{j\omega}}$$

$$\omega = 0 \Rightarrow H(e^{j0}) = 1$$

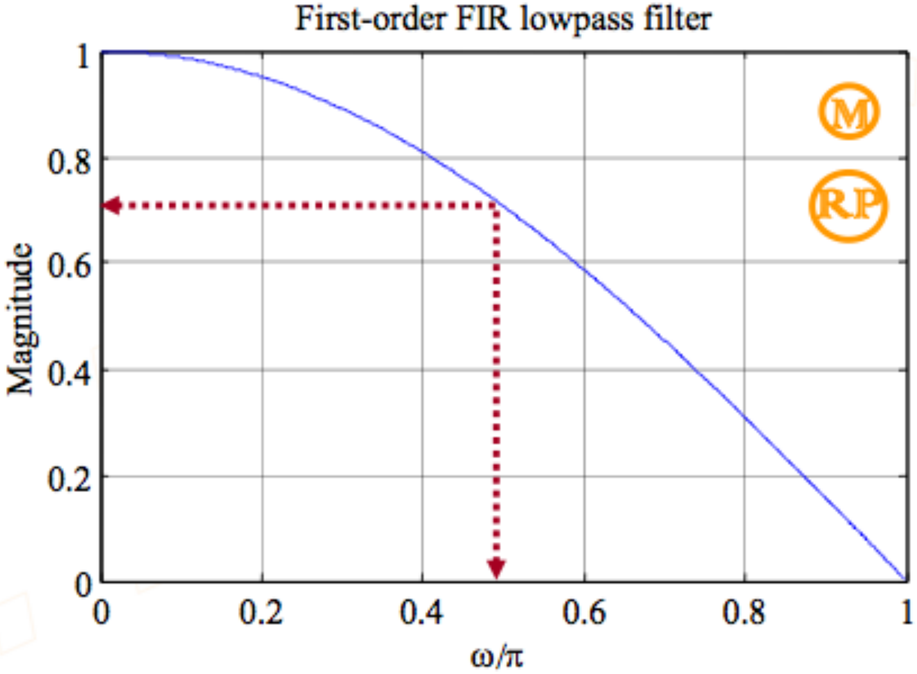
$$\omega = \pi \Rightarrow H(e^{j\pi}) = 0$$

How would this frequency response look when plotted as a function of ω?

SIMPLE FIR FILTERS

$$H(e^{j\omega}) = \frac{e^{j\omega}+1}{2e^{j\omega}} = e^{-j\omega/2}\cos(\omega/2)$$

Monotonically decreasing function of ω
Hence a low pass filter.



System Gain at some frequency ω:

$$\boxed{G(\omega) = 20\log_{10}|H(\omega)|}$$

The frequency ω$_c$ at which $|H(\omega_c)| = \frac{1}{\sqrt{2}}|H(0)|$ is of special interest:

3-dB cutoff frequency

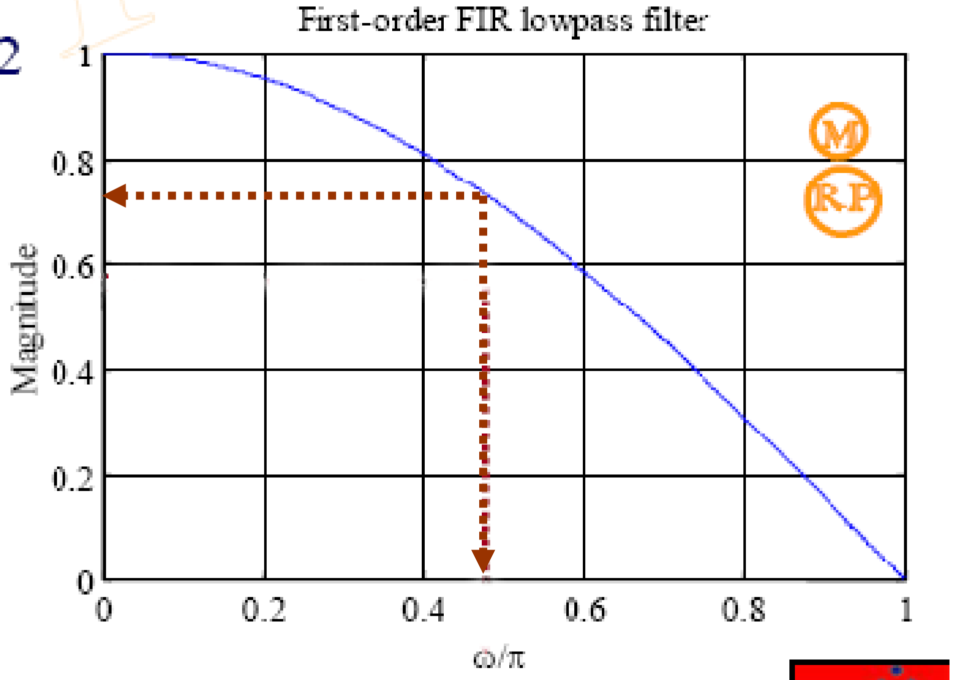
$$|G(\omega_c)| = 20\log_{10}|H(\omega)|\Big|_{\omega=\omega_c} = 20\log_{10}\frac{1}{\sqrt{2}}|H(0)|$$

$$= 20\log_{10}\overbrace{|H(0)|}^{1} - 20\log_{10}\left(\sqrt{2}\right) = 0 - 3.0103 \cong -3.0\text{dB}$$

CUTOFF-FREQUENCY

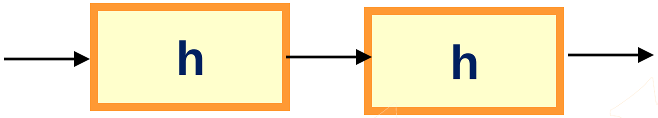
- For realizable filters, the *cutoff frequency* is the frequency at which the system gain reaches its 0.707 multiple of its peak value.
- This gain represents the frequency at which the signal power is half of its peak power! (why?)
- For a lowpass filter, the gain at the cut-off frequency is 3dB less than its gain at zero frequency (or 0.707 of its zero frequency amplitude, or half the power of its power at zero frequency).
- For the first order filter, this occurs at $\omega_c = \pi/2$

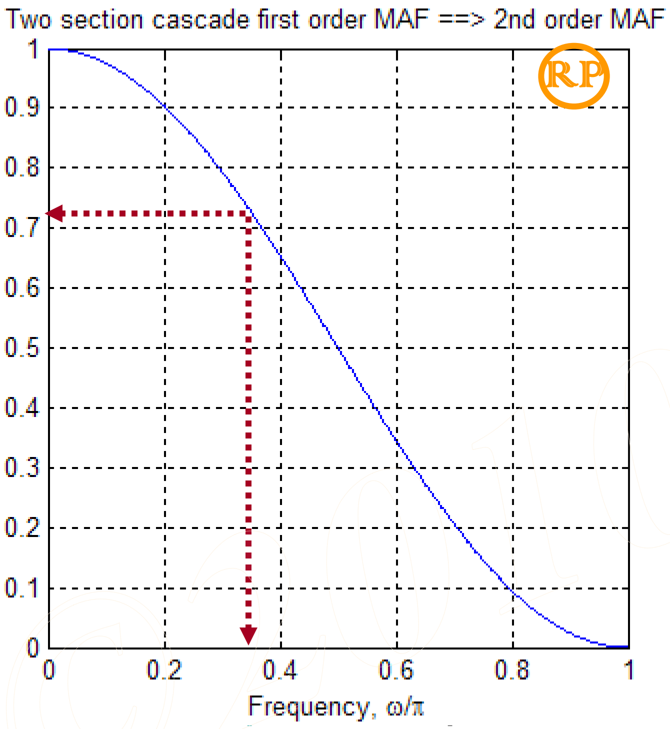
$$H(e^{j\omega}) = \frac{e^{j\omega}+1}{2e^{j\omega}} = e^{-j\omega/2}\cos(\omega/2)$$

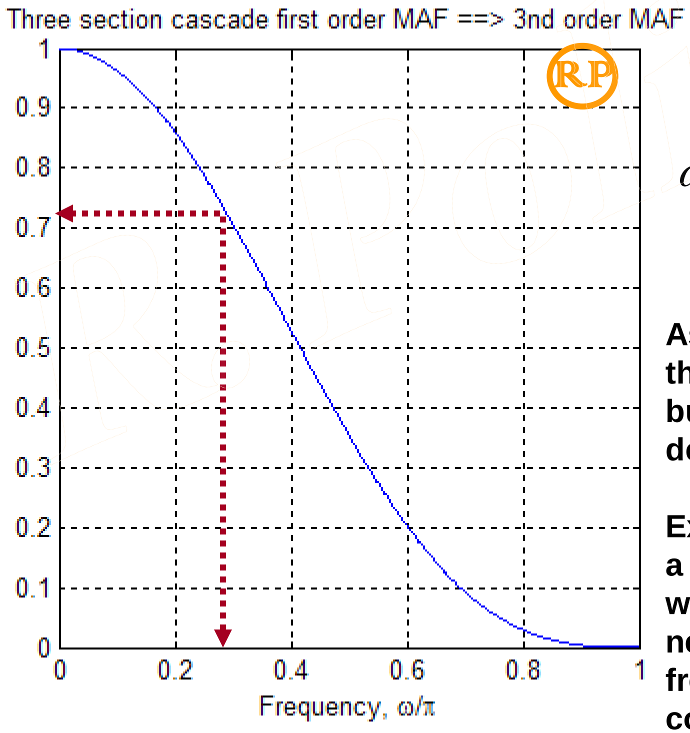


CASCADED FIR FILTERS

- Now consider a second order MAF, M=3







$$\omega_c = 2\cos^{-1}\left(2^{-\left(\frac{1}{2M}\right)}\right)$$

As the order M increases, the filter becomes sharper but the passband also decreases!

Ex: If we want a LPF with a cutoff frequency of .12π, what order filter do we need? What linear frequency does this correspond to?

```
subplot(121)
w=linspace(0, pi, 512);
H2=exp(-j*w/2).*cos(w/2).*exp(-j*w/2).*cos(w/2);
plot(w/pi, abs(H2)); grid
xlabel('Frequency, \omega/\pi')
```

```
%(three  sections)
subplot(122)
H3=exp(-j*w/2).*cos(w/2).*exp(-j*w/2).*cos(w/2).*exp(j*w/2).*cos(w/2);
plot(w/pi, abs(H3)); grid
xlabel('Frequency, \omega/\pi')
```

HOW ABOUT HIGH PASS FILTERS?

- A high pass filter can easily be obtained from a lowpass filter by

LPF → $H(z) = (½)(1+z^{-1}) = (z+1)/(2z)$ → Replace z with –z to obtain the HPF $H_1(z)$

-

Notice that $H_1(z)$ has a zero at z=1, and a pole at z=0.

$$H_1(z) = \frac{1}{2}\left(1 - z^{-1}\right) = \frac{z-1}{2z}$$

- Therefore, the frequency response has a zero at ω=0, corresponding to z=1
- The zero at ω=0 (suppress low frequency components 0), makes this a highpass filter

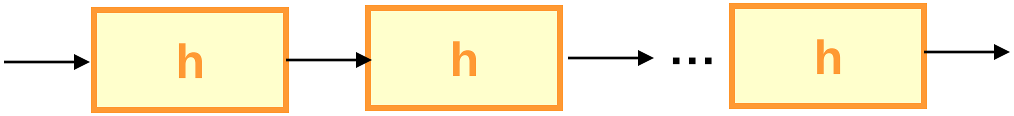
$$\omega = 0 \;\Rightarrow H(e^{j0}) = 0$$

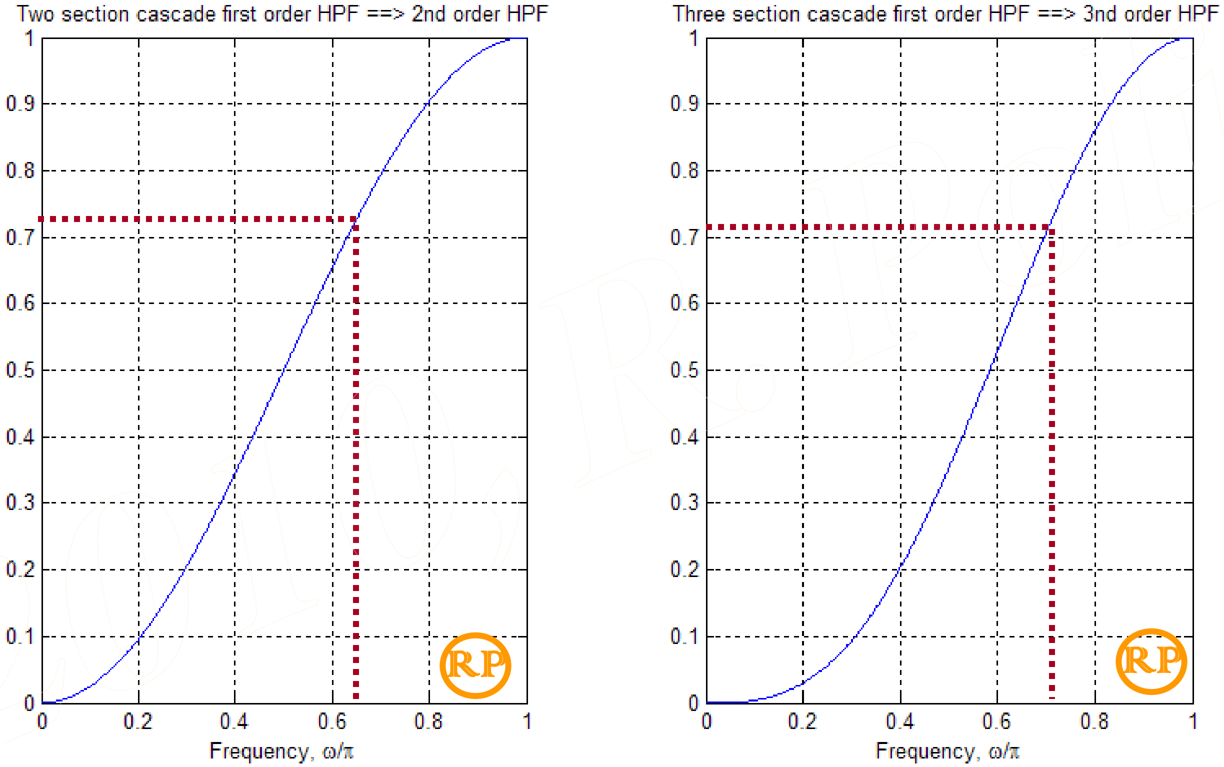
$$\omega = \pi \;\Rightarrow H(e^{j\pi}) = 1$$

How would this frequency response look when plotted as a function of ω?

ω/π

CASCADING HPF





```
subplot(121)

w=linspace(0, pi, 512);
H2=j*exp(-j*(w/2)).*sin(w/2)*j.*exp(-j*(w/2)).*sin(w/2);
plot(w/pi, abs(H2)); grid
xlabel('Frequency, \omega/\pi')
```

```
%(three  sections)
subplot(122)
H3=j*exp(-j*(w/2)).*sin(w/2)*j.*exp(-j*(w/2)).*sin(w/2)*j.*exp(-
j*(w/2)).*sin(w/2);
plot(w/pi, abs(H3)); grid
xlabel('Frequency, \omega/\pi')
```

HPF

- Alternately, a higher-order highpass filter of the form

$$H_1(z) = \frac{1}{M}\sum_{n=0}^{M-1}(-1)^n z^{-n}$$

is obtained by replacing *z* with ***-z*** in the transfer function of a moving average filter

Recall that the moving average filter had a transfer function of:

$$H(z) = \frac{1}{M}\sum_{n=0}^{M-1} z^{-n} = \frac{1-z^{-M}}{M\left(1-z^{-1}\right)} : \frac{z^M - 1}{M[z^{M-1}(z-1)]}$$

IIR LPF FILTERS

- A first-order causal lowpass IIR digital filter has a transfer function given by

$$\boxed{H_{LP}(z) = \frac{1-\alpha}{2}\left(\frac{1+z^{-1}}{1-\alpha z^{-1}}\right) = \frac{1-\alpha}{2}\left(\frac{z+1}{z-\alpha}\right)}$$

where $|\alpha| < 1$ for stability

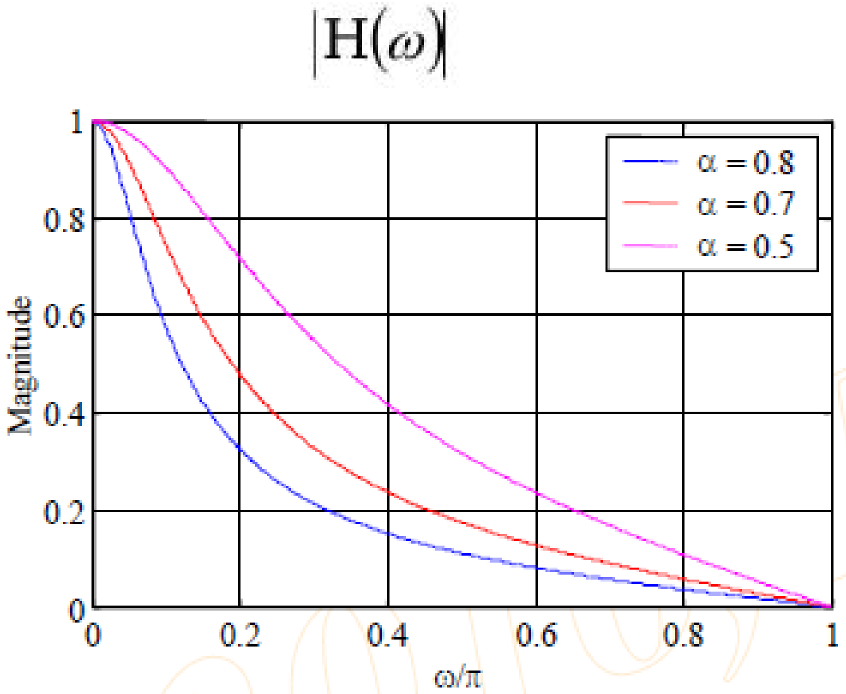
Normalization term that ensures that the gain at ω = 0 is 0 dB (magnitude of 1)

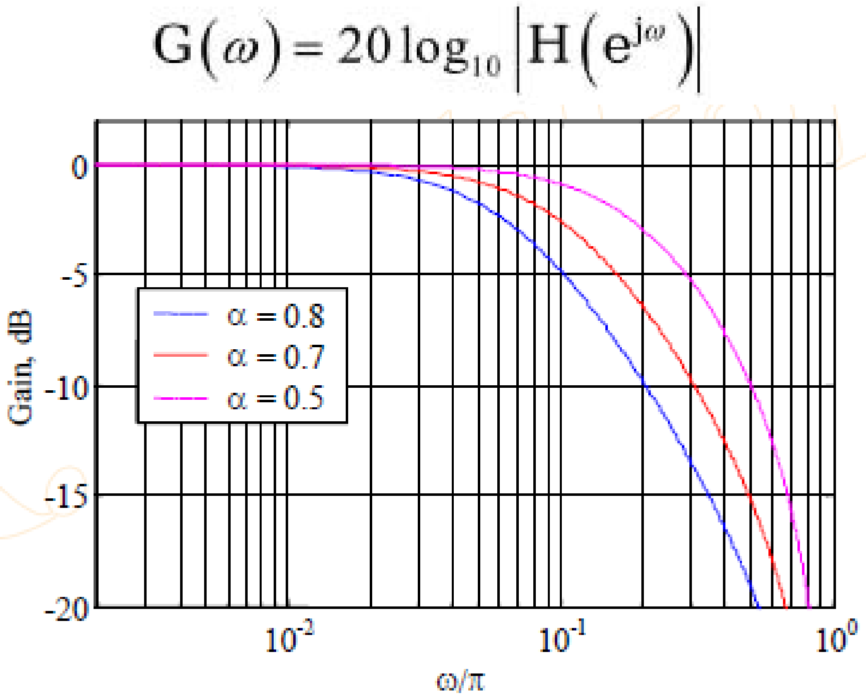
  - The above transfer function has a zero at $z=-1$ i.e., at $\omega = \pi$ which is in the stopband
  - $H_{LP}(z)$ has a real pole at $z = \alpha$
  - As ω increases from 0 to π, the magnitude of the zero vector decreases from a value of 1 to 0, whereas, for a positive value of α, the magnitude of the pole vector increases from a value of $1-\alpha$ to $1+\alpha$
  - The maximum value of the magnitude function is 1 at $\omega = 0$, and the minimum value is 0 at $\omega = \pi$

$$|H_{LP}(e^{j0})| = 1, \qquad |H_{LP}(e^{j\pi})| = 0 \quad \Longrightarrow$$

LPF

IIR LPF FILTERS


$|H(\omega)|$


$G(\omega) = 20\log_{10}\left|H\left(e^{j\omega}\right)\right|$

3-dB cutoff frequency

$$\cos\omega_C = \frac{2\alpha}{1+\alpha^2}$$

$$\alpha = \frac{1-\sin\omega_C}{\cos\omega_C}$$

To find 3-db cutoff frequency, solve for ω_c in $|H(\omega_c)|^2 = ½$

Find α corresponding to a given 3-db cutoff frequency, ω_c

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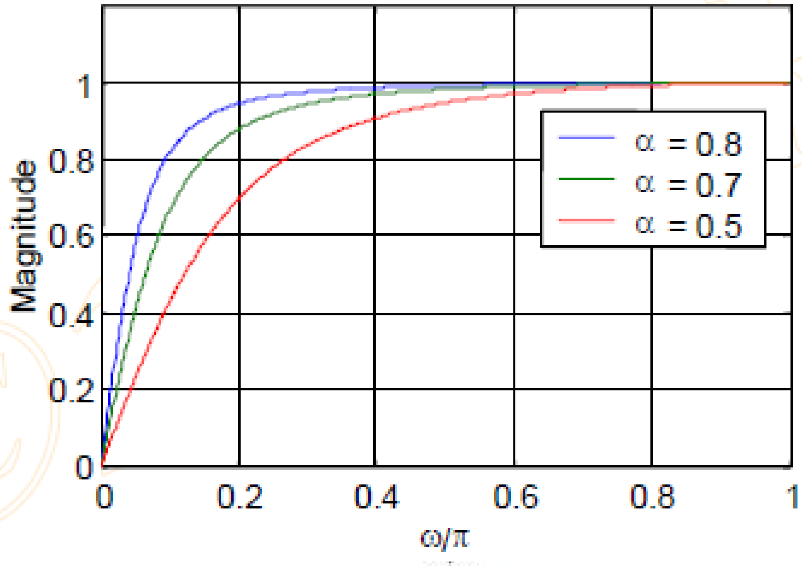
IIR HPF FILTERS

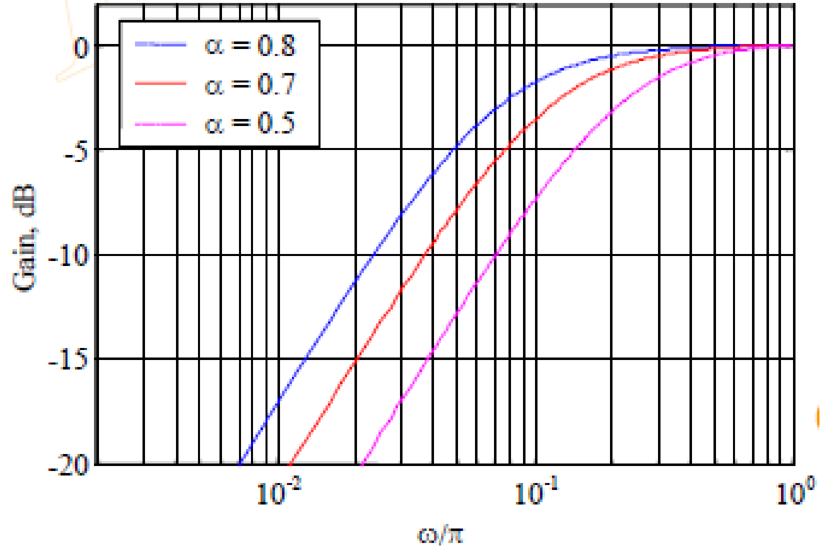
- A first-order causal highpass IIR digital filter has a transfer function given by where $|\alpha| < 1$ for stability

$$H_{HP}(z) = \frac{1+\alpha}{2}\left(\frac{1-z^{-1}}{1-\alpha z^{-1}}\right) = \frac{1+\alpha}{2}\left(\frac{z-1}{z-\alpha}\right)$$

  - Note that one can obtain a HPF simply by replacing ***z*** with a ***–z*** from LPF. The above transfer function is slightly different, however, provides a better HPF.

- This transfer function has a zero at $z = 1 \Rightarrow \omega = 0$ which is in the stopband





For a given 3-dB cutoff frequency, the corresponding α is:

$$\alpha = \frac{(1-\sin\omega_C)}{\cos\omega_C}$$

EXAMPLE

- W can easily design a first order IIR HPF, with say a cutoff frequency of 0.8π: Sin(0.8 π)=0.587785, Cos(0.8 π)=-0.80902➔ α=-0.5095245

$$H_{HP}(z) = \frac{1+\alpha}{2}\left(\frac{1-z^{-1}}{1-\alpha z^{-1}}\right) = 0.245238\left(\frac{1-z^{-1}}{1+0.5095245\,z^{-1}}\right)$$

- Higher order filters can also be easily designed simply by cascading several first order filters. Note that for each cascaded filter, the overall impulse response is the convolution of individual impulse responses, h'[n]=(h[n]*h[n]*h[n]*…*h[n]), or the overall frequency response is H'(ω)= H(ω). H(ω)… H(ω)

BANDPASS IIR FILTERS

- A general 2nd order bandpass IIR filter transfer function is

$$\boxed{H_{BP}(z) = \frac{1-\alpha}{2}\left(\frac{1-z^{-2}}{1-\beta(1+\alpha)\ z^{-1}+\alpha\, z^{-2}}\right)}$$

$|\alpha| < 1$ and $|\beta| < 1$ for stability

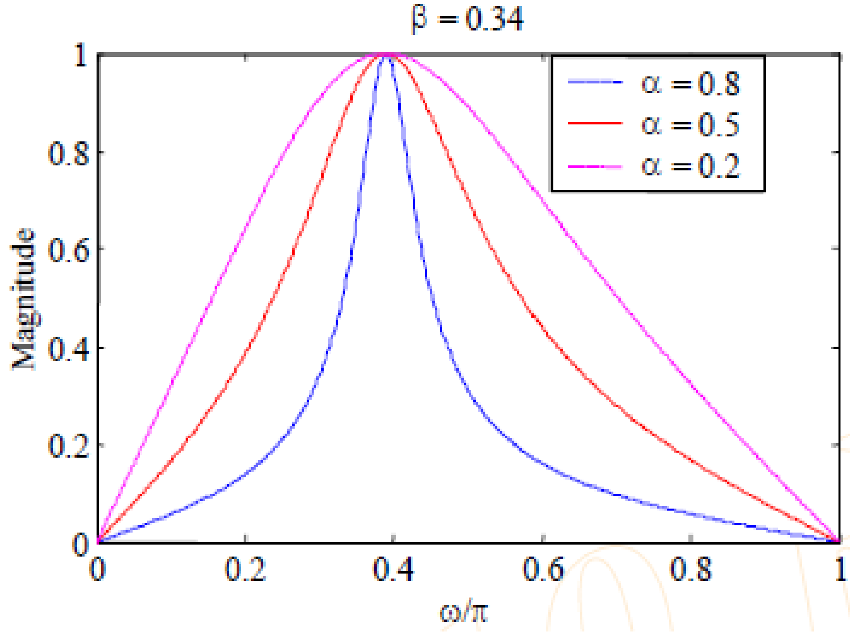
  - This function has a zero both at z=-1 and z=1, that is both at $\omega = 0$ and $\omega = \pi$
  - It obtains its maximum value at $\omega = \omega_0$, called the ***center frequency***, given by

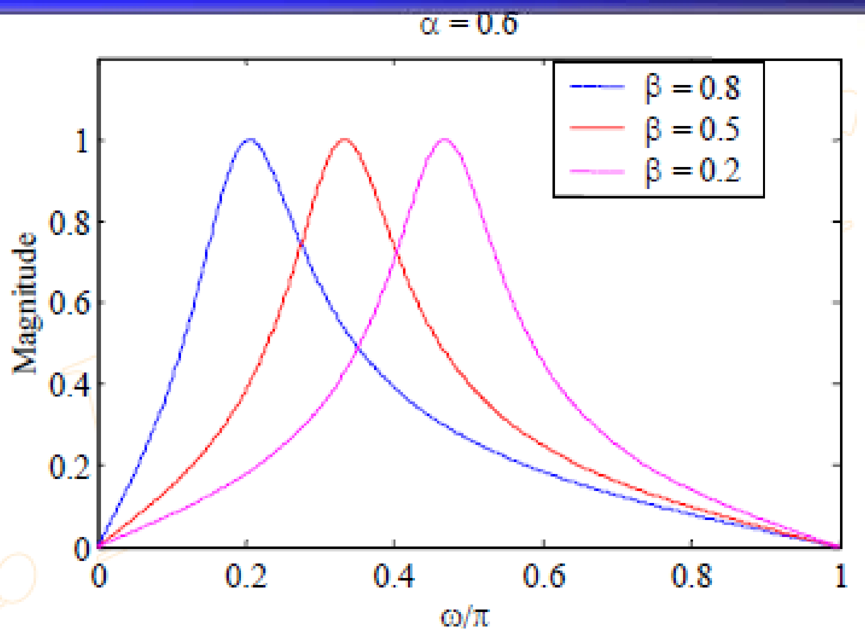
$$\boxed{\omega_o = \cos^{-1}(\beta)}$$

  - This filter has two cutoff frequencies, ω_{c1} and ω_{c2}, where $|H_{BP}(\omega)|^2 = ½$. These frequencies are also called ***3-dB cutoff frequencies***.
  - The difference between the two cutoff frequencies is called the ***3-dB bandwidth***, given by

$$\boxed{B_w = \omega_{c2} - \omega_{c1} = \cos^{-1}\left(\frac{2\alpha}{1+\alpha^2}\right), \quad \omega_{c2} > \omega_{c1}}$$

BANDPASS IIR FILTERS





For example, to design a 2nd order BPF with a center frequency of 0.4π and 3-dB bandwidth of 0.1π ➔ $\beta = \cos(\omega_0) = \cos(0.4\pi) = 0.30901$, and $2\alpha/(1+\alpha^2)=\cos(B_w)$ ➔ $\alpha_1 = 1.37638$ and $\alpha_2=0.72654$. Note that only the second one provides a stable transfer function:

$$H_{BP}(z) = \frac{1-\alpha}{2}\left(\frac{1-z^{-2}}{1-\beta(1+\alpha)\,z^{-1}+\alpha\,z^{-2}}\right)$$

⇨

$$\xcancel{H_{BP}(z) = -0.18819\,\frac{1-z^{-2}}{1-0.7343424z^{-1}+1.37638z^{-2}}}$$

$$H''_{BP}(z) = 0.13673\,\frac{1-z^{-2}}{1-0.533531z^{-1}+0.72654253z^{-2}}$$

BANDSTOP IIR FILTERS

- A general 2nd order bandstop IIR filter transfer function is

$$\boxed{H_{BS}(z) = \frac{1+\alpha}{2}\left(\frac{1-2\beta z^{-1}+z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}}\right)}$$

|α| < 1 and |β| < 1 for stability

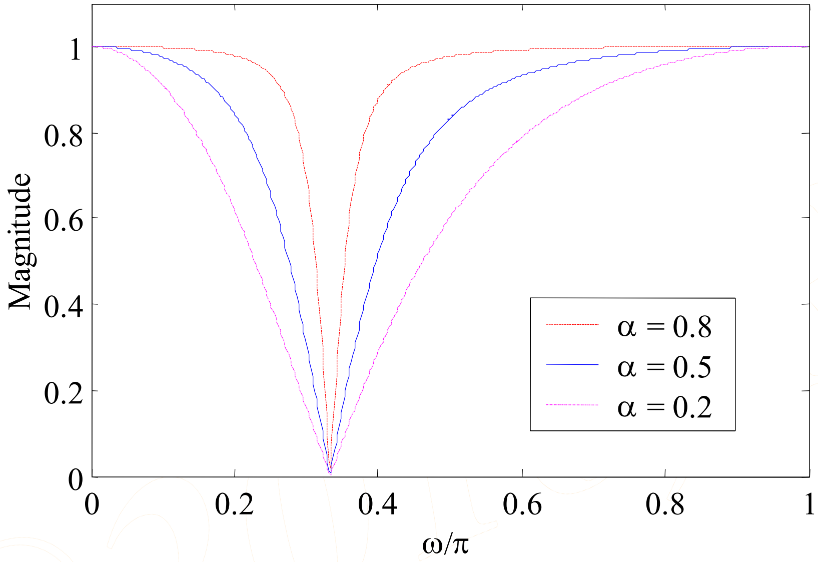
  - This function achieves its maximum value of unity at at z=±1, i.e, at $\omega = 0$ and $\omega = \pi$
  - It has a zero at $\omega = \omega_0$, called the ***notch frequency***, given by

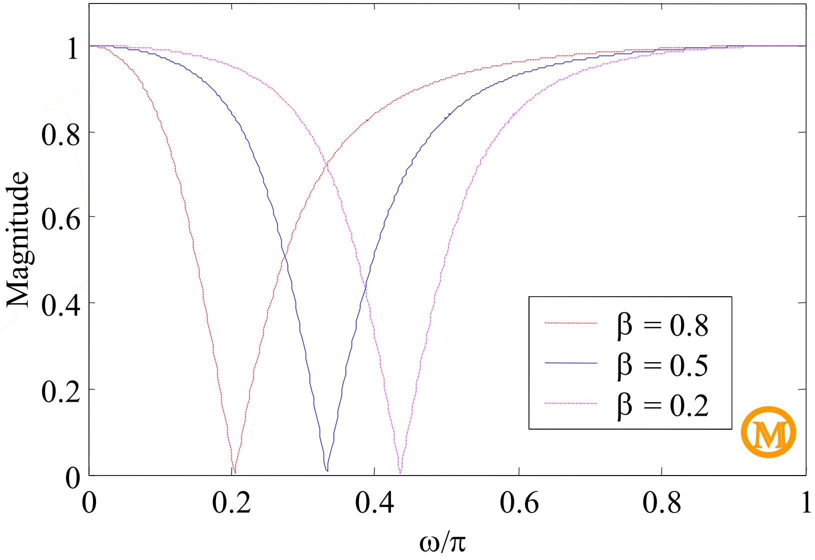
$$\boxed{\omega_o = \cos^{-1}(\beta)}$$

  - Therefore, this filter is also called the ***notch filter***.
  - Similar to bandpass, there are two values, ω_{c1} and ω_{c2}, called the 3-dB cutoff frequencies where the frequency response magnitude reaches ½, i.e., $|H_{BS}(\omega)|^2 = ½$.
  - The difference between these two frequencies is again called the 3-dB bandwidth, given by

$$\boxed{B_w = \omega_{c2} - \omega_{c1} = \cos^{-1}\left(\frac{2\alpha}{1+\alpha^2}\right), \quad \omega_{c2} < \omega_{c1}}$$

BANDSTOP IIR FILTERS





HIGHER ORDER IIR FILTERS

- Again, note that any of these filters can be designed to be of higher order, which typically provides sharper and narrower transition bands
- Simply cascade a basic filter structure as many times necessary to achieve the higher order filter.
- For example, to cascade K-first order LPFs

$$\boxed{H_{LP}(z)=\frac{1-\alpha}{2}\left(\frac{1+z^{-1}}{1-\alpha z^{-1}}\right)} \Rightarrow \boxed{G_{LP}(z)=\left(\frac{1-\alpha}{2}\cdot\frac{1+z^{-1}}{1-\alpha z^{-1}}\right)^{K}}$$

  - It can be shown that

$$\alpha=\frac{1+(1-C)\cos\omega_c-\sin\omega_c\sqrt{2C-C^2}}{1-C+\cos\omega_c} \qquad C=2^{(K-1)/K}$$

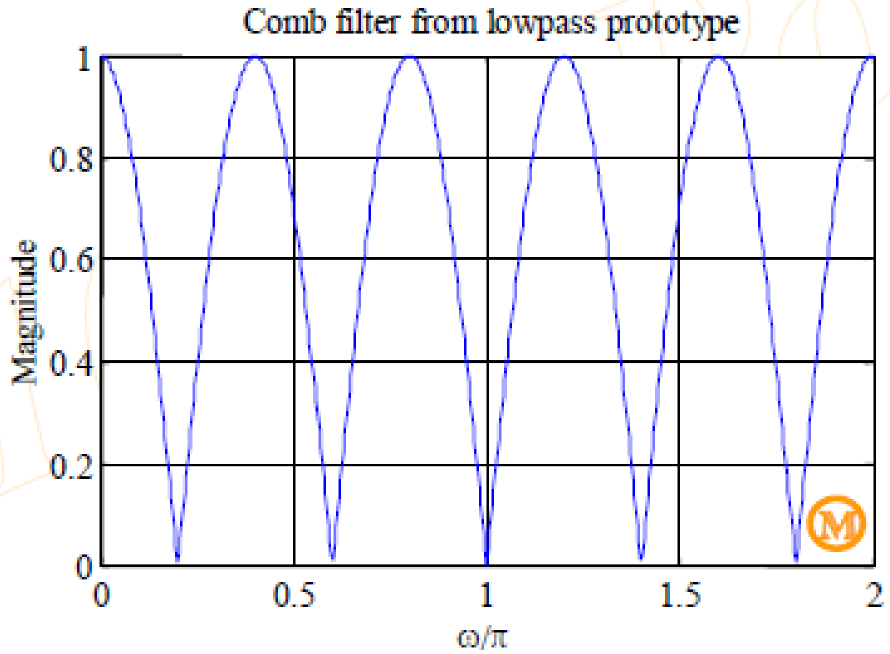
COMB FILTERS

- Note that LPF, HPF, BPF, BSF all have a single passband and/or single stopband. Many applications require several such frequency regions, which can be provided by a **comb filter**:
- A comb filter typically has a frequency response that is a periodic function of ω, with a period of 2π/L, L indicating the number of passband/stopbands.
- If H(z) is a filter with a single passband and/or a single stopband, a comb filter can be easily generated from it by replacing each delay in its realization with L delays resulting in a structure with a transfer function given by $\mathbf{H_{comb}(z) = H(z^L)}$
  - If |H(ω)| exhibits a peak at ω_p, then $H_{comb}(\omega)$ will exhibit L peaks at $\boldsymbol{\omega_p k/L}$, $\mathbf{0 \le \boldsymbol{k} \le \boldsymbol{L}}$-1, in the frequency range $0 \le \omega \le 2\pi$
  - Likewise, if |H(ω)|has a notch at ω_0, then $H_{comb}(\omega)$ will have L notches at $\boldsymbol{\omega_0 k/L}$, $\mathbf{0 \le \boldsymbol{k} \le \boldsymbol{L}}$ in the frequency range $0 \le \omega \le 2$
  - A comb filter can be generated from either an FIR or an IIR prototype filter

COMB FILTERS

- Starting from a lowpass transfer function:

$$H(z) = \frac{1}{2}(1 + z^{-1}) \Rightarrow H_{comb}(z) = H(z^{L}) = \frac{1}{2}(1 + z^{-L})$$



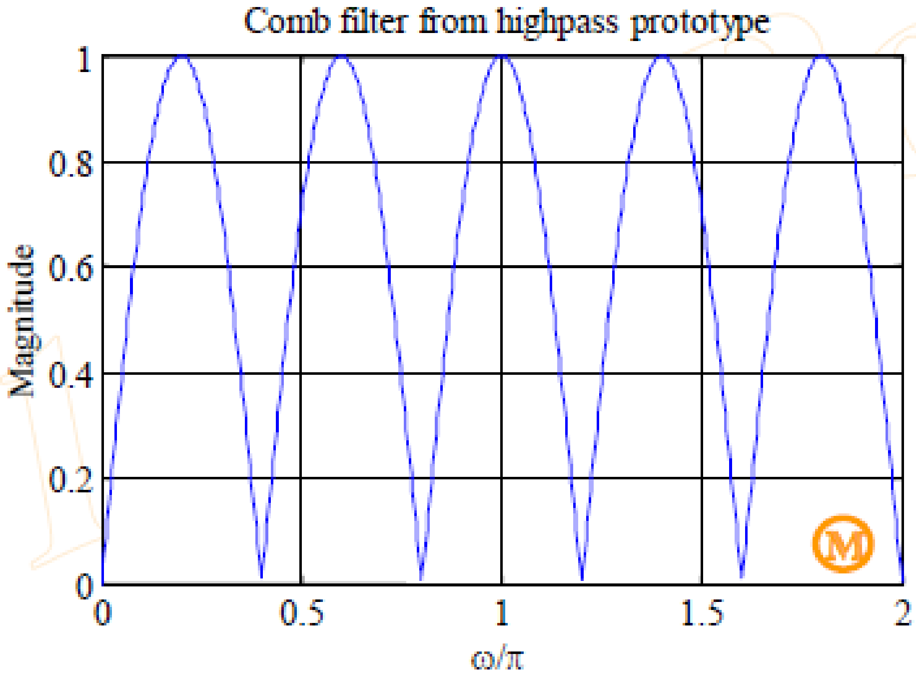
(L=5) L *notches* are at ω=(2k+1)π/L and L *peaks* are at ω=2πk/L

$$0 \leq \omega < 2\pi$$

COMB FILTERS

- Starting from a highpass transfer function

$$\boxed{H(z)=\frac{1}{2}(1-z^{-1}) \Rightarrow H_{comb}(z)=H(z^{L})=\frac{1}{2}(1-z^{-L})}$$



(L=5) L *peaks* are at ω=(2k+1)π/L and L *notches* are at ω=2πk/L

MINIMUM & MAXIMUM PHASE FILTERS

- Without further details, we provide the following properties and definitions:
  - It can be shown that a causal stable transfer function with all zeros outside the unit circle has an excess phase compared to a causal transfer function with identical magnitude but having all zeros inside the unit circle
  - A causal stable transfer function with all zeros inside the unit circle is called a ***minimum-phase transfer function***
  - A causal stable transfer function with all zeros outside the unit circle is called a ***maximum-phase transfer function.***
  - Typically, we are interested in minimum phase transfer functions.
    - So try to design your filters with zeros inside the unit circle

ALL PASS FILTERS

- A filter that passes all frequencies equally is called an allpass filter.
  - Any nonminimum-phase transfer function can be expressed as the product of a minimum-phase transfer function and a stable ***allpass transfer function***, where an allpass transfer function has a unit magnitude for all frequencies.
  - All pass transfer functions are typically IIR whose magnitude response satisfy
    1. **$|H_A(\omega)|^2=1$ for all ω**
    2. **For every zero at $z=\alpha e^{j\theta}$, there must be a pole at $z=\alpha^{-1}e^{-j\theta}$ or vice versa, so that zeros and poles cancel each other out.**
  - Recall that all poles of a causal stable transfer function must lie inside the unit circle. It then follows that all of the zeros of a causal, stable, allpass filter must lie outside of the unit circle, in a ***mirror image symmetry*** formation.

ALL PASS FILTERS

- A general APF, then, has a transfer function

$$H_A \; z = \pm \frac{d_M + d_{M-1} z^{-1} + \cdots + d_1 z^{-M+1} + z^{-M}}{\underbrace{1 + d_1 z^{-1} + \cdots + d_{M-1} z^{-M+1} + d_M z^{-M}}_{D_M \; z}} = \pm \frac{z^{-M} D_M \;\; z^{-1}}{D_M \;\; z}$$

Mirror image polynomials

- An APF has a factored transfer function of the form

$$\boxed{A_M(z) = \pm \prod_{i=1}^{M} \left(\frac{-\lambda_i^{*} + z^{-1}}{1 - \lambda_i z^{-1}} \right)}$$

- Some properties:
  - For stability, all poles must be inside the unit circle, i.e., $|\lambda_i| < 1, 1 \le i \le M$
  - Then, all zeros, must necessarily be outside of the unit circle, with mirror image symmetry with respect to poles
  - Note that $|H_A(z)| = 1$, only for $|z| = 1$

All Pass Filters

We can show that the magnitude funtion is 1 everywhere:

$$A_M(z)A_M(z^{-1}) = \frac{z^{-M}D_M(z^{-1})}{D_M(z)}\frac{z^{M}D_M(z)}{D_M(z^{-1})} = 1$$

or:

$$\left|A_M(e^{j\omega})\right|^2 = A_M(z)A_M(z^{-1}) = 1$$

Note that the phase of an all pass filter function is decreasing monotonically with frequency. This holds for any arbitrary causal stable all pass filter function.

Consider the first-order all pass function:

$$A_M(e^{j\omega}) = \frac{-\lambda^* + e^{-j\omega}}{1-\lambda e^{-j\omega}}$$

All Pass Filters

Substituting $\lambda = re^{j\phi}$ gives: $\boxed{A_M(e^{j\omega}) = \frac{-\lambda^* + e^{-j\omega}}{1-\lambda e^{-j\omega}} = e^{-j\omega}\left[\frac{1-re^{j(\omega-\varphi)}}{1-re^{-j(\omega-\varphi)}}\right]}$

We can now derive an expression for the phase function:

$$\boxed{\theta_c(\omega) = -\omega - 2\tan^{-1}\left[\frac{r\sin(\omega-\varphi)}{1-r\cos(\omega-\varphi)}\right]}$$

→ $\boxed{\frac{d\theta_c(\omega)}{d\omega} = -\frac{(1-r^2)}{\left|1+re^{-j(\omega-\varphi)}\right|^2}}$ for stability: $|r| < 1$

→ $\boxed{\frac{d\theta_c(\omega)}{d\omega} < 0, \quad 0 \le \omega \le \pi}$ θ_c monotonically decreasing

All Pass Filters

The group delay function is: $$\boxed{\theta_c(\omega) = -\int_0^{\omega} \tau_g(\varphi)\,d\varphi + \theta_c(0)}$$

$\boxed{A_M(e^{j0}) = 1 \rightarrow \theta_c(0) = 0 \rightarrow \theta_c(\omega) < 0}$: the phase function is negative

1. A causal stable real coefficient all pass filter is a lossless bounded (LBR) real transfer function
2. The magnitude function is $$\boxed{\left|A_M(z)\right| = \begin{cases} <1 & \text{for} \quad |z| > 1 \\ =1 & \text{for} \quad |z| = 1 \\ >1 & \text{for} \quad |z| < 1 \end{cases}}$$
3. An M-th order stable real all pass transfer function satisfies the equation: $$\boxed{\int_0^{\pi} \tau_g(\omega)\,d\omega = M\pi}$$

AN EXAMPLE

$$H_A \; z = \frac{-0.2+0.18z^{-1}+0.4z^{-2}+z^{-3}}{1+0.4z^{-1}+0.18z^{-2}-0.2z^{-3}}$$

```
%All pass example

b=[-0.2  0.18 0.4 1];
a=[1 0.4 0.18 -0.2];

[H w]=freqz(b,a,1024);
subplot(211)
plot(w/pi, abs(H));
title('Magnitude response of the all pass filter')
grid
subplot(212)
plot(w/pi, angle(H));
title('Phase response of the all pass filter')
grid
xlabel('Normalized angular frequency, \omega/\pi')

figure
zplane(b,a)
```