FIR and IIR Transfer Functions

the input – output relation of an LTI system is:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

the output in the z – domain is:

$$Y(z) = H(z).X(z)$$

Where

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

so we can write the transfer function in the familiar form:

$$H(z) = \frac{Y(z)}{X(z)}$$

and to solve $H(z)$ (transfer function of the impulse response $h[n]$) we can apply the rules as discussed before.

FIR and IIR Transfer Functions

recall the LTI FIR input – output relation: $$y[n] = \sum_{k=N_1}^{N_2} h[k]x[n-k]$$

where the impulse response $h[n]$ is defined for $N_1 \le n \le N_2$

the transfer function is then given by: $$H(z) = \sum_{n=N_1}^{N_2} h[n]z^{-n}$$

for a causal FIR filter, $0 \le N_1 \le N_2$. Note that all poles are at the origin of the z – plane and that the ROC is the entire z – plane except the origin.

FIR and IIR Transfer Functions

recall the LTI IIR difference equation:
$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

applying the z – transform to both sides of the difference equation and making use of the linearity and time shifting properties of the z – transform gives:

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

leading to the generalised form:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Frequency Response

- Recall that the output of an LTI system in the frequency domain is: $\boxed{H(\omega)=Y(\omega)/X(\omega)}$
  - where H(ω) is called the ***frequency response*** of the system, which relates the input and the output of an LTI system in the frequency domain. The frequency response can also be represented in terms of CCLDE coefficients:

$$\boxed{H(\omega)=\frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}=\frac{b_0+b_1e^{-j\omega}+b_2e^{-j\omega 2}+\cdots+b_M e^{-j\omega M}}{a_0+a_1e^{-j\omega}+a_2e^{-j\omega 2}+\cdots+a_N e^{-j\omega N}}}$$

- A generalization of the freq. response is the transfer func., computed in the z-domain
  - The function H(z), which is the z-transform of the impulse response h[n] of the LTI system, is called the ***transfer function*** or the ***system function*** $\boxed{H(z)=Y(z)/X(z)}$
  - Using the CCLDE coefficients

$$\boxed{H(z)=\frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}=z^{(N-M)}\frac{\sum_{k=0}^{M} b_k z^{M-k}}{\sum_{k=0}^{N} a_k z^{N-k}}=\frac{b_0}{a_0}\cdot\frac{\prod_{k=1}^{M}(1-\zeta_k z^{-1})}{\prod_{k=1}^{N}(1-p_k z^{-1})}=\frac{b_0}{a_0}z^{(N-M)}\frac{\prod_{k=1}^{M}(z-\zeta_k)}{\prod_{k=1}^{N}(z-p_k)}}$$

CCLDE coefficients

Zeros & poles

Zero & pole factors

Frequency Response ←→ Transfer Function

- If the ROC of the transfer function $H(z)$ includes the unit circle, then the frequency response $H(\omega)$ of the LTI digital filter can be obtained simply as follows: $\boxed{H(e^{j\omega}) = H(\omega) = H(z)\big|_{z=e^{j\omega}}}$

- Assuming that the DTFT exists, starting with the factored z-transform, we can write the frequency response of a typical LTI system as

$$H(z) = \frac{b_0}{a_0} z^{(N-M)} \frac{\prod_{k=1}^{M}(z-\zeta_k)}{\prod_{k=1}^{N}(z-p_k)} \quad \Rightarrow \quad H(e^{j\omega}) = \frac{b_0}{a_0} e^{j\omega(N-M)} \frac{\prod_{k=1}^{M}(e^{j\omega}-\zeta_k)}{\prod_{k=1}^{N}(e^{j\omega}-p_k)}$$

From which we can obtain the magnitude and phase response:

$$\left|H(e^{j\omega})\right| = \left|\frac{b_0}{a_0}\right| \frac{\prod_{k=1}^{M}\left|e^{j\omega}-\zeta_k\right|}{\prod_{k=1}^{N}\left|e^{j\omega}-p_k\right|}$$

$$\arg H(e^{j\omega}) = \arg(b_0/a_0) + \omega(N-M) + \sum_{k=1}^{M}\arg(e^{j\omega}-\zeta_k) - \sum_{k=1}^{N}\arg(e^{j\omega}-p_k)$$

Interpretation of the Frequency Response

➲ Take a close look at the magnitude and phase responses in terms of zeros and poles:

$$\left|H(e^{j\omega})\right| = \left|\frac{b_0}{a_0}\right| \frac{\prod_{k=1}^{M}\left|e^{j\omega} - \zeta_k\right|}{\prod_{k=1}^{N}\left|e^{j\omega} - p_k\right|}$$

- **The magnitude response |H(ω)| at a specific value of ω is given by the *<u>product of the magnitudes of all zero vectors divided by the product of the magnitudes of all pole vectors</u>***

$$\arg H(e^{j\omega}) = \arg(b_0 / a_0) + \omega(N - M) + \sum_{k=1}^{M} \arg(e^{j\omega} - \zeta_k) - \sum_{k=1}^{N} \arg(e^{j\omega} - p_k)$$

- **The phase response at a specific value of ω is obtained by adding the phase of the term b_0/a_0 and the linear-phase term ω(N-M) to the *<u>sum of the angles of the zero vectors minus the angles of the pole vectors</u>***

So What Does This All Mean?

- An approximate plot of the magnitude and phase responses of the transfer function of an LTI digital filter can be developed by examining the pole and zero locations
- Now, the frequency response has the smallest magnitude around $\omega = \zeta$, and the largest magnitude around $\omega = p$.
  - Of course, at $\omega = p$, the response is infinitely large, and at $\omega = \zeta$, the response is zero
- Therefore:
  - **To highly attenuate signal components in a specified frequency range, we need to place zeros very close to or on the unit circle in this range**
  - **Likewise, to highly emphasize signal components in a specified frequency range, we need to place poles very close to or on the unit circle in this range**

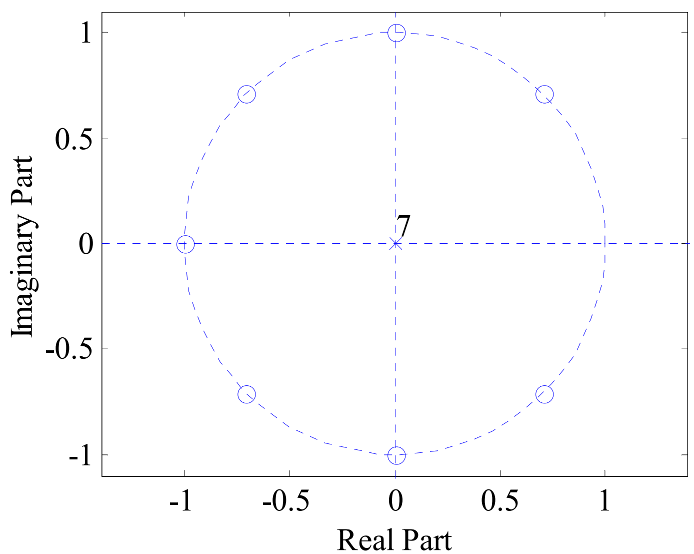
An Example

- Consider the M-point moving-average FIR filter with an impulse response

$$h[n] = \begin{cases} 1/M, & 0 \le n \le M-1 \\ 0, & \text{otherwise} \end{cases}$$

$$H(z) = \frac{1}{M}\sum_{n=0}^{M-1} z^{-n} = \frac{1-z^{-M}}{M\left(1-z^{-1}\right)} = \frac{z^{M}-1}{M\left[z^{M}\left(z-1\right)\right]}$$

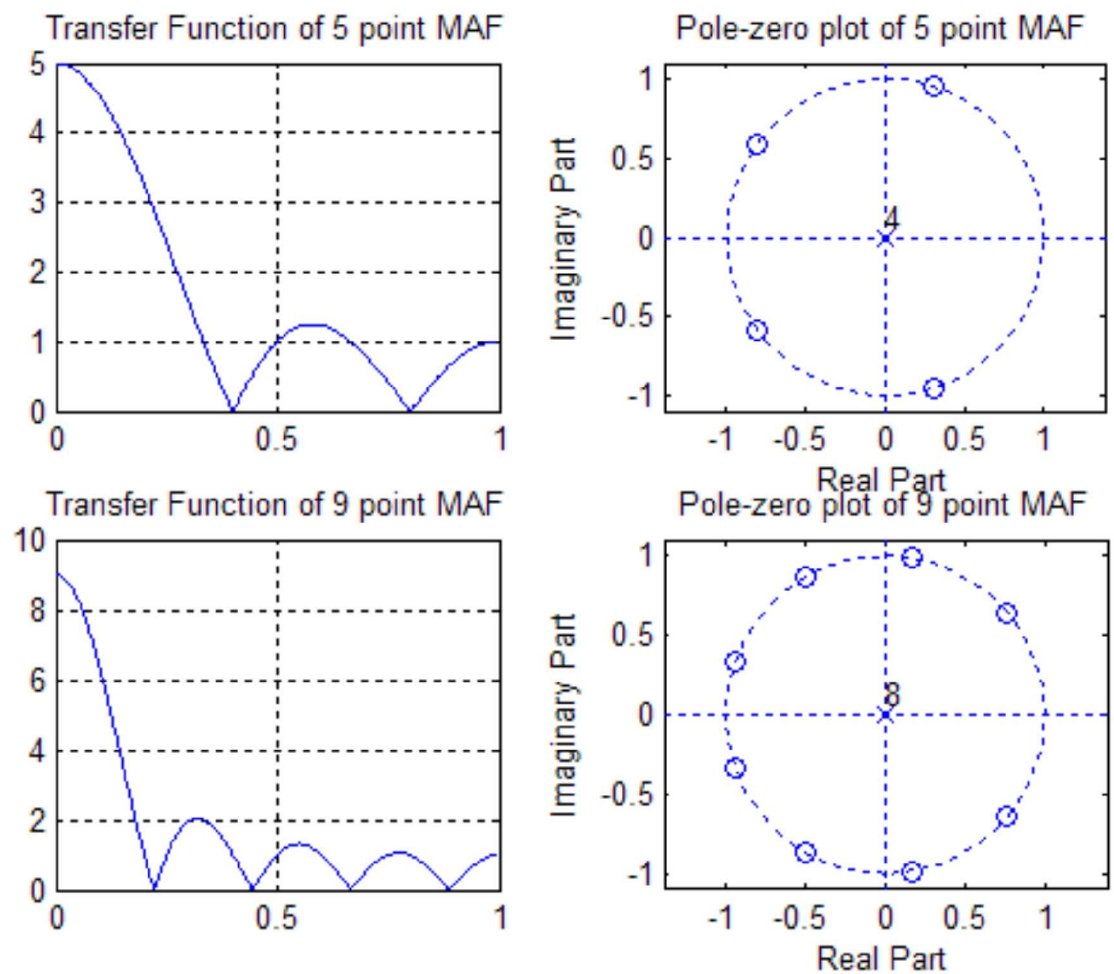
- Observe the following
  - The transfer function has M zeros on the unit circle at* $z = e^{j2\pi k/M}$,$0 \le k \le M-1$
  - There are M-1 poles at $z = 0$ and a single pole at $z = 1$
  - The pole at $z = 1$ exactly cancels the zero at $z = 1$
  - The ROC is the entire z-plane except $z = 0$



***To see this, try**
zplane(roots([1,0,0…,0, -1]))

Moving Average Filter

```
%h1=1/5* [1 1 1 1 1];
%h2=1/9 * [1 1 1 1 1 1 1 1 1];

b1=1/5*[1 1 1 1 1]; a1=1;
b2=1/9*[1 1 1 1 1 1 1 1 1]; a2=1;

[H1 w]=freqz(b1, 1, 512);
[H2 w]=freqz(b2, 1, 512);

[z1,p1,k1] = tf2zpk(b1,a1);
[z2,p2,k2] = tf2zpk(b2,a2);

subplot(221)
plot(w/pi, abs(H1)); grid
title('Transfer Function of 5 point MAF')
subplot(222)
zplane(b1,a1);
title('Pole-zero plot of 5 point MAF')
subplot(223)
plot(w/pi, abs(H2)); grid
title('Transfer Function of 9 point MAF')
subplot(224)
zplane(b2,a2);
title('Pole-zero plot of 9 point MAF')
```



Observe the effects of zeros and the poles !!!

Types of Transfer Functions

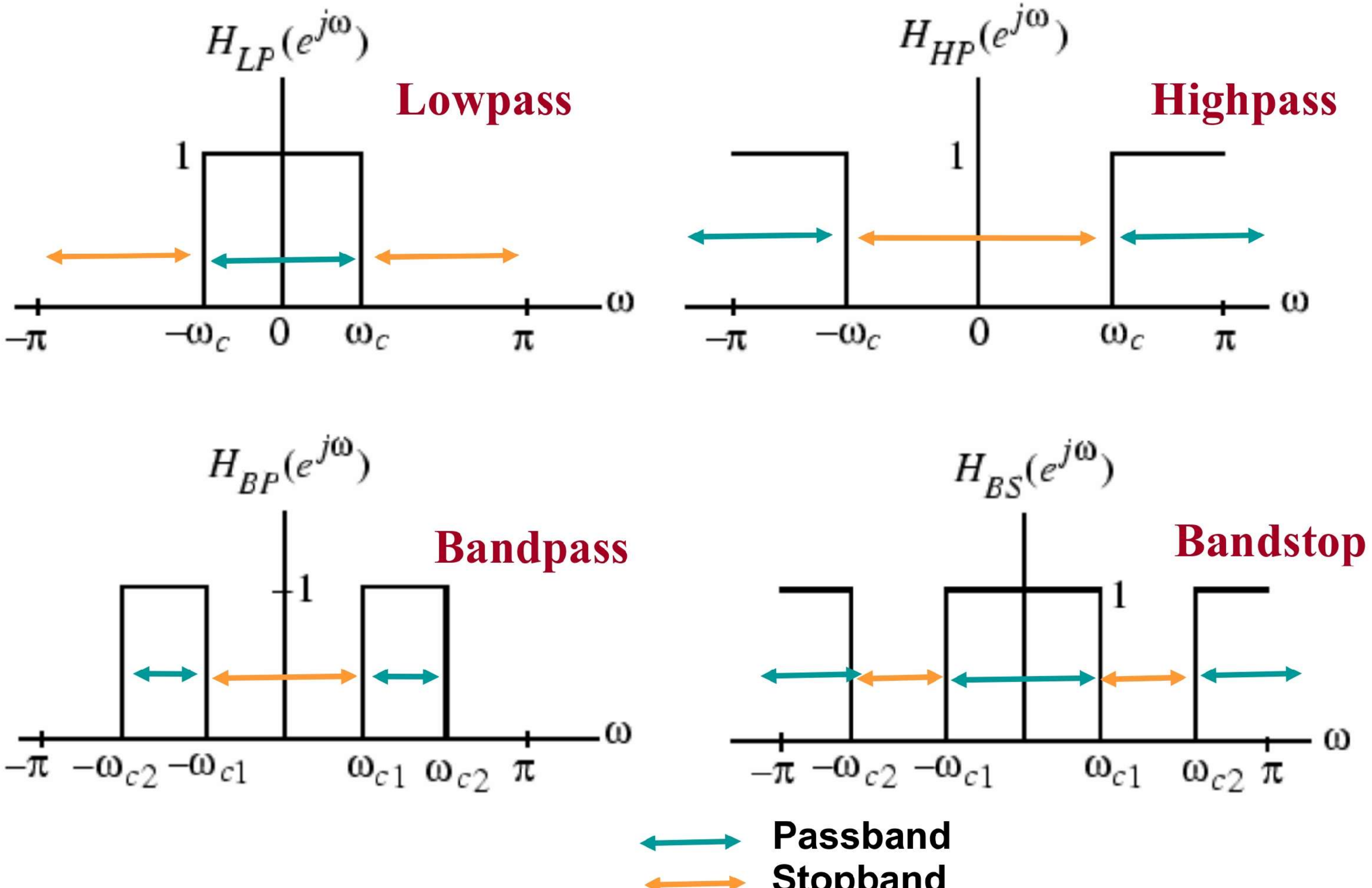
- The time-domain classification of an LTI digital transfer function sequence is based on the length of its impulse response:
  - Finite impulse response (FIR) transfer function
  - Infinite impulse response (IIR) transfer function
- Many other classifications are also used
  - For digital transfer functions with frequency-selective frequency responses, one classification is based on the shape of the magnitude function $|H(\omega)|$ or the form of the phase function $\theta(\omega)$
- Based on the magnitude spectrum, one of four types of ideal filters are usually defined
  - Low pass
  - High pass
  - Band pass
  - Band stop

IDEAL FILTER

- An ***ideal filter*** is a digital filter designed to pass signal components of certain frequencies ***without distortion***, which therefore has a frequency response equal to 1 at these frequencies, and has a frequency response equal to 0 at all other frequencies
- The range of frequencies where the frequency response takes the value of one is called the ***passband***
- The range of frequencies where the frequency response takes the value of zero is called the ***stopband***
- The transition frequency from a passband to stopband region is called the ***cutoff frequency***
- Note that an <u>ideal filter cannot be realized</u>. Why?

Ideal Filters

➲ The frequency responses of four common ideal filters in the [-π π] range are



Ideal Filters

- Recall that the DTFT of a rectangular pulse is a sinc function

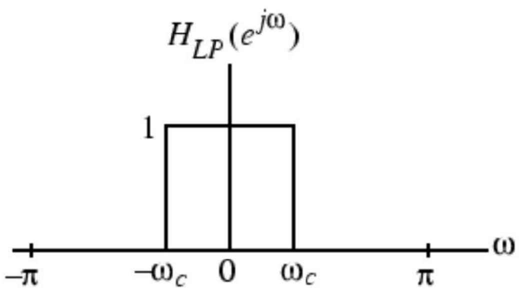
$$\boxed{x[n] = \mathrm{rect}_M[n] = \begin{cases} 1, & -M \le n \le M \\ 0, & \text{otherwise} \end{cases} \overset{\Im}{\Leftrightarrow} \sum_{n=-M}^{M} e^{-j\omega n} = \frac{\sin\left(M + 1/2\right)\omega}{\sin\left(\omega/2\right)}, \;\; \omega \neq 0}$$

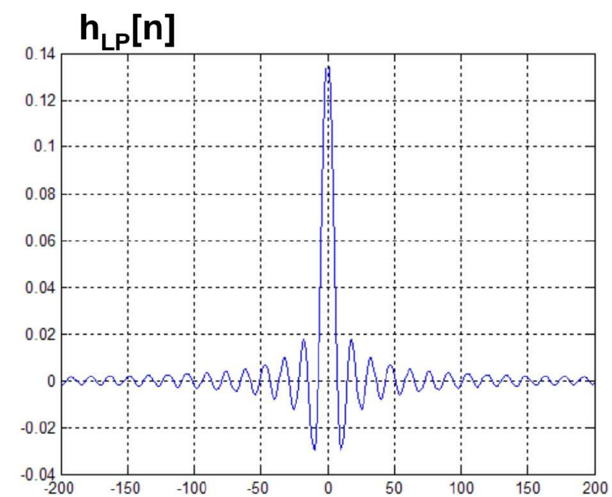
- From the duality theorem, the inverse DTFT of a rectangular pulse is also a sinc function. Since the ideal (lowpass) filter is of rectangular shape, its impulse response must be of sinc.

$$H_{LP}(\omega) = \begin{cases} 1, & 0 \le |\omega| \le \omega_c \\ 0, & \omega_c \le |\omega| \le \pi \end{cases}$$

$$h_{LP[n]} = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} 1 \cdot e^{j\omega n} d\omega$$

$$= \frac{\sin(\omega_c n)}{\pi n}, \; -\infty < n < \infty$$



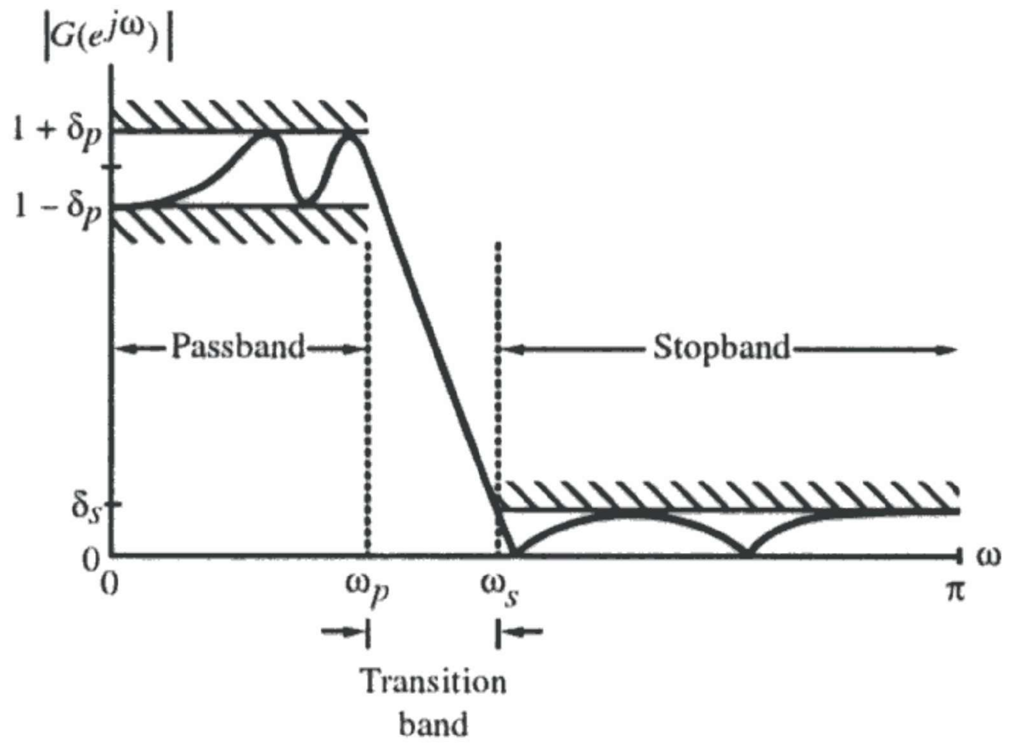


Ideal Filters

- We note the following about the impulse response of an ideal filter
  - $h_{LP}[n]$ is not absolutely summable
  - The corresponding transfer function is therefore not BIBO stable
  - $h_{LP}[n]$ is not causal, and is of doubly infinite length
  - The remaining three ideal filters are also characterized by doubly infinite, noncausal impulse responses and also are not absolutely summable
- Thus, the ideal filters with the ideal ***brick wall*** frequency responses cannot be realized with finite dimensional LTI filter

Realizable Filters

- In order to develop a stable and realizable filter transfer function
  - The ideal frequency response specifications are relaxed by including a transition band between the passband and the stopband
  - This permits the magnitude response to decay slowly from its maximum value in the passband to the zero value in the stopband
  - Moreover, the magnitude response is allowed to vary by a small amount both in the passband and the stopband
  - Typical magnitude response specifications of a lowpass filter therefore looks like ➔



Types of transfer Functions

- So far we have seen transfer functions characterized primarily according to their
  - Impulse response length (FIR / IIR)
  - Magnitude spectrum characteristics (LPF, HPF, BPF, BSF)
- A third classification of a transfer function is with respect to its phase characteristics
  - Zero phase
  - Linear phase
  - Generalized linear phase
  - Non-linear phase
- Recall that the phase spectrum tells us ___________________________
- In many applications, it is necessary that the digital filter designed does not distort the phase of the input signal components with frequencies in the passband

Phase and Group Delay

- A frequency selective system (filter) with frequency response $H(\omega)=|H(\omega)|\measuredangle H(\omega) = |H(\omega)|e^{j\theta(\omega)}$ changes the amplitude of all frequencies in the signal by a factor of $|H(\omega)|$, and adds a phase of $\theta(\omega)$ to all frequencies.
  - Note that both the amplitude change and the phase delay are functions of ω
  - The phase $\theta(\omega)$ is in terms of radians, but can be expressed in terms of time, which is called the ***phase delay***. The phase delay at a particular frequency ω_0 is given as

$$\tau_p(\omega_0) = -\frac{\theta(\omega_0)}{\omega_0}$$

Compare this to the phase delay in cont. time

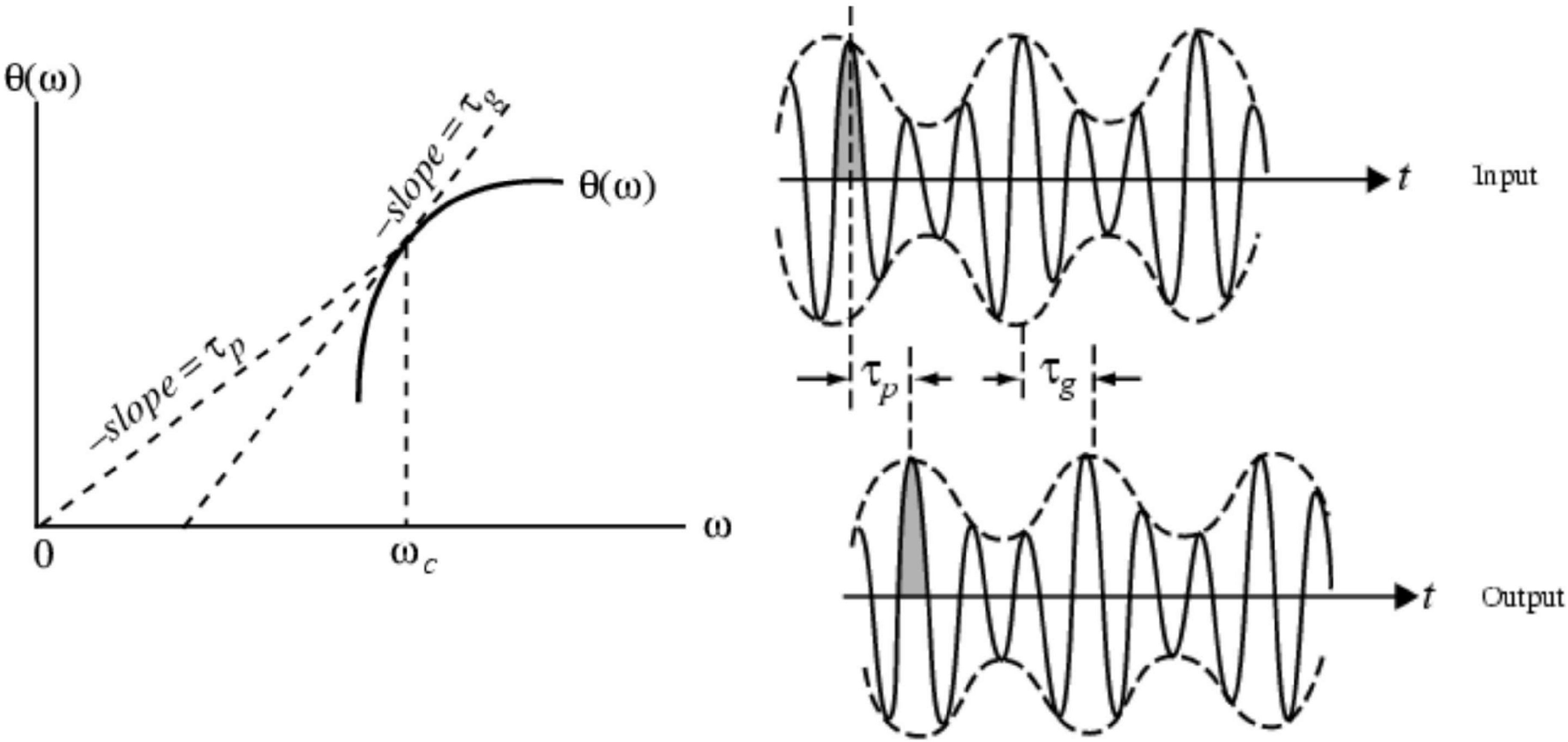
$$t_\theta = \frac{\theta}{2\pi f} = \frac{\theta}{\omega}$$

  - If an input system consists of many frequency components (which most practical signals do), then we can also define ***group delay***, the phase shift by which the envelope of the signal shifts. This can also be considered as the average phase delay – in seconds – of the filter as a function of frequency, given by

$$\tau_g(\omega_c) = -\left.\frac{d\theta(\omega)}{d\omega}\right|_{\omega=\omega_c}$$

Phase and Group Delay

- Note that both are slopes of the phase function, just defined slightly differently



ZERO PHASE FILTERS

- One way to avoid any phase distortion is to make sure the frequency response of the filter does not delay any of the spectral components. Such a transfer function is said to have a ***zero – phase*** characteristic.
- A zero – phase transfer function has no phase component, that is, the spectrum is purely real (no imaginary component) and non-negative
- However, it is NOT possible to design a causal digital filter with a zero phase.

Zero-Phase Filters

- Now, for *non-real-time* processing of real-valued input signals of finite length, zero-phase filtering can be implemented by relaxing the causality requirement
- A zero-phase filtering scheme can be obtained by the following procedure:
  - Process the input data (finite length) with a causal real-coefficient filter H(z).
  - Time reverse the output of this filter and process by the same filter.
  - Time reverse once again the output of the second filter

$$x[n] \longrightarrow \boxed{H(z)} \longrightarrow v[n] \qquad u[n] \longrightarrow \boxed{H(z)} \longrightarrow w[n]$$

$$u[n] = v[-n], \qquad y[n] = w[-n]$$

$$V(\omega) = H(\omega)X(\omega), \qquad W(\omega) = H(\omega)U(\omega)$$

$$U(\omega) = V^*(\omega), \qquad Y(\omega) = W^*(\omega) = H^*(\omega)U^*(\omega)$$

$$Y(\omega) = H^*(\omega)V(\omega) = H^*(\omega)H(\omega)X(\omega) = \left|H(\omega)\right|^2 X(\omega)$$

In Matlab

➲ The function **`filtfilt()`** implements the zero-phase filtering scheme

`filtfilt()`
Zero-phase digital filtering

y = filtfilt(b,a,x) performs zero-phase digital filtering by processing the input data in both the forward and reverse directions. After filtering in the forward direction, it reverses the filtered sequence and runs it back through the filter.

The resulting sequence has precisely zero-phase distortion and double the filter order. filtfilt minimizes start-up and ending transients by matching initial conditions, and works for both real and complex inputs.

Linear Phase

- Note that a zero-phase filter cannot be implemented for real-time applications. Why?

- For a causal transfer function with a nonzero phase response, the phase distortion can be avoided by ensuring that the transfer function has (preferably) a unity magnitude and a ***linear-phase*** characteristic in the frequency band of interest

$$H(\omega) = e^{-j\alpha\omega} \Rightarrow \left|H(\omega)\right| = 1 \qquad \angle H(\omega) = \theta(\omega) = -\alpha\omega$$

  - Note that this phase characteristic is linear for all ω in $[0\ 2\pi]$.
  - Recall that the phase delay at any given frequency ω_0 was
  - If we have linear phase, that is, $\theta(\omega)=-\alpha\omega$, then the total delay at any frequency ω_0 is $\tau_0 = -\theta(\omega_0)/\ \omega_0 = -\alpha\omega_0/\omega_0 = \alpha$
  - Note that this is identical to the group delay $d\theta(\omega)/d\omega$ evaluated at ω_0

$$\boxed{\tau_p(\omega_o) = -\frac{\theta(\omega_o)}{\omega_o}}$$

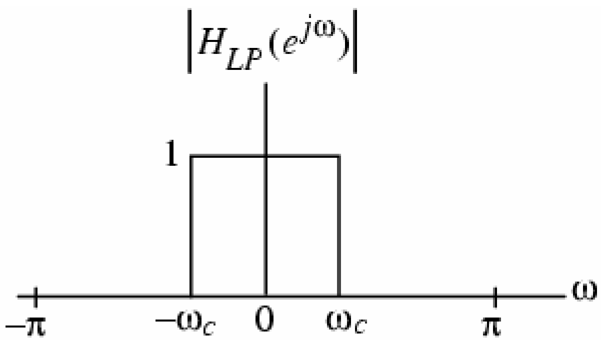
$$\boxed{\tau_g(\omega_0) = -\left.\frac{d\theta(\omega)}{d\omega}\right|_{\omega=\omega_0}}$$

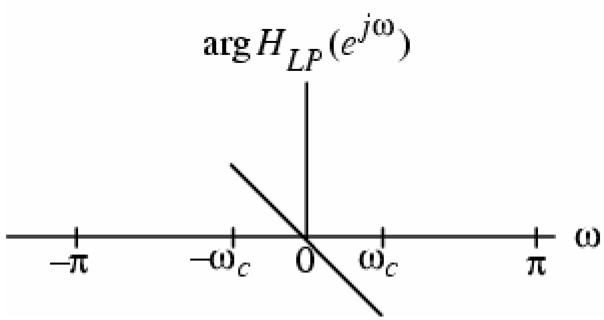
What is the Big Deal…?

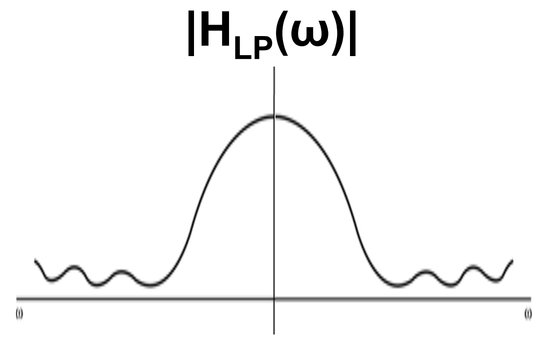
- The deal is huge!
- If the phase spectrum is linear, then the phase delay is independent of the frequency, and it is the same constant α for all frequencies.
- In other words, all frequencies are delayed by α seconds, or equivalently, the entire signal is delayed by α seconds.
  - Since the entire signal is delayed by a constant amount, there is no distortion!
- If the filter does not have linear phase, then different frequency components are delayed by different amounts, causing significant distortion.

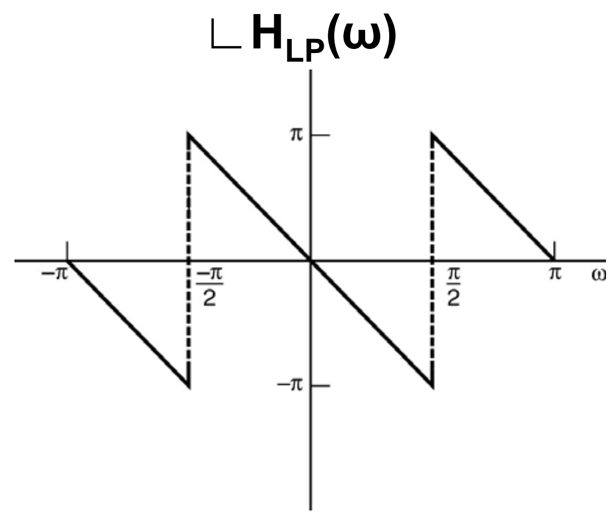
Take Home Message

- If it is desired to pass input signal components in a certain frequency range undistorted in both magnitude and phase, then the transfer function should exhibit a unity magnitude response and a linear-phase response in the band of interest









Generalized Linear Phase

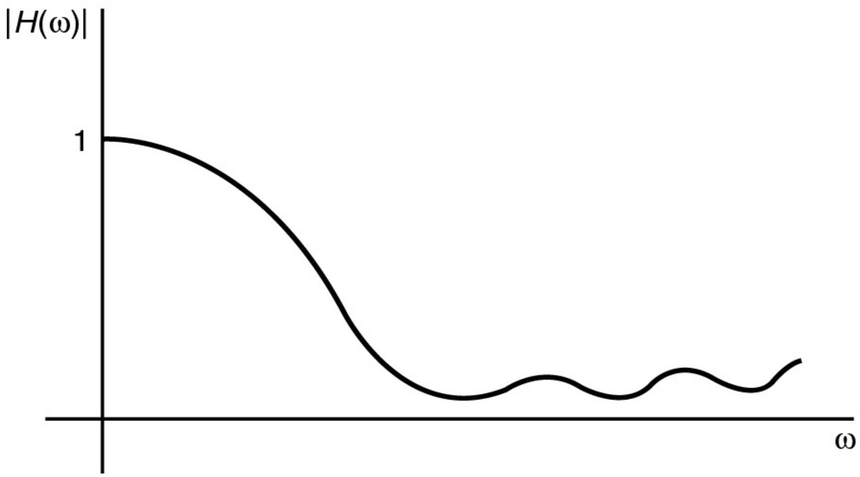
- Now consider the following system, where $G(\omega)$ is real (i.e., no phase)

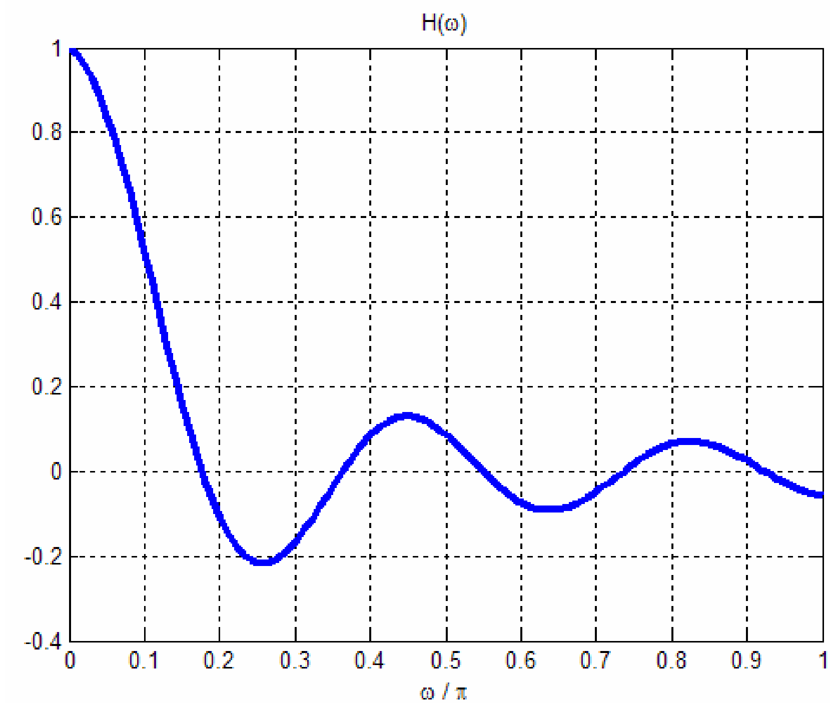
$$H(\omega) = e^{-j\alpha\omega} G(\omega)$$

- From our previous discussion, the term $e^{-j\alpha\omega}$ simply introduces a phase delay, that is, normally independent of frequency. Now,
  - If $G(\omega)$ is positive, the phase term is $\theta(\omega)=-\alpha\omega$, hence the system has linear phase.
  - If $G(\omega)<0$, then a 180º (π rad) phase term is added to the phase spectrum. Therefore, the phase response is $\theta(\omega)=-\alpha\omega+\pi$, the phase delay is no longer independent of frequency
  - We can, however, write $H(\omega)=-[e^{-j\alpha\omega} G(\omega)]$, and the function inside brackets has now linear phase → no distortion. The negative signs simply flips the signal.
  - As long as the system response does not change signs for different values of ω, it can be written as a linear phase transfer function. Therefore, such systems are said to have ***generalized linear phase***
  - In general, a system whose group delay is constant α, for all passband frequencies is considered as a generalized linear phase filter.

Approximately Linear Phase

- Consider the following transfer functions





- Note that above a certain frequency, say ω_c, the magnitude is very close to zero, that is most of the signal above this frequency is suppressed. So, if the phase response deviates from linearity above these frequencies, then signal is not distorted much, since those frequencies are blocked anyway.

Linear Phase Filters

- It is typically impossible to design a linear phase IIR filter, however, designing FIR filters with precise linear phase is very easy:
- Consider a causal FIR filter of length M+1 (order M)

$$H(z)=\sum_{n=0}^{N} h[n]z^{-n}=h[0]+h[1]z^{-1}+h[2]z^{-2}+\cdots+h[M]z^{-M}$$

  - This transfer function has linear phase, if its impulse response h[n] is either ***symmetric***

$$\boxed{h[n]=h[M-n],\quad 0\le n\le M}$$

or ***anti-symmetric***

$$\boxed{h[n]=-h[M-n],\quad 0\le n\le M}$$

LINEAR PHASE FILTERS

A linear phase solution for the causal FIR filter function $\boxed{H(z)=\sum_{n=0}^{N} h[n]z^{-n}}$

requires: $H(e^{j\omega})=e^{j(c\omega+\beta)}\breve{H}(\omega)$ where c and β are constants and $\breve{H}(\omega)$ is the amplitude response (zero-phase response), a real function of ω

for a real FIR the filter function has to be even: $\boxed{H(e^{j\omega})=H^*(e^{-j\omega})}$

which implies: $\boxed{e^{j(c\omega+\beta)}\breve{H}(\omega)=e^{-j(c(-\omega)+\beta)}\breve{H}(-\omega)}$ or: $\boxed{e^{j\beta}=e^{-j\beta}}$

so β must be 0 or π !

then we can write: $\boxed{\breve{H}(\omega)=\pm e^{-jc\omega}\breve{H}(e^{j\omega})=\pm\sum_{n=0}^{N} h[n]\cdot e^{-j\omega(c+n)}}$

LINEAR PHASE FILTERS

the anti-symmetric case then is: $\boxed{\breve{H}(-\omega) = \pm\sum_{l=0}^{N} h[l]\cdot e^{-j\omega(c+l)}}$

which, with a change of variable, $l = N - n$, can be written as :

$$\boxed{\breve{H}(-\omega) = \pm\sum_{n=0}^{N} h[N-n]\cdot e^{-j\omega(c+N-n)}}$$

Since $\breve{H}(\omega)$ is even, i.e. $\breve{H}(\omega) = \breve{H}(-\omega)$, the condition for the

corresponding impulse response is: $\boxed{h[n] = h[N-n], \quad 0 \le n \le N}$

with $\boxed{c = -\frac{N}{2}}$

For $\breve{H}(\omega)$ odd this becomes: $\boxed{h[n] = -h[N-n], \quad 0 \le n \le N, \quad c = -\frac{N}{2}}$

LINEAR PHASE FILTERS

This linear phase filter description can be generalised into a formalism for four type of FIR filters:

Type 1: symmetric sequence of odd length

Type 2: symmetric sequence of even length

Type 3: anti-symmetric sequence of odd length

Type 4: anti-symmetric sequence of even length

Linear Phase Filters

It is possible to generate other type of FIR transfer function from a given type transfer function. Take a given type I transfer function *H(z)*.

From this function we can generate at least three types of other functions:

$$E(z) = z^{-N/2} - H(z)$$

$$F(z) = (-1)^{N/2} H(-z)$$

$$G(z) = z^{-N/2} - (-1)^{N/2} H(-z)$$

with amplitude responses:

$$\breve{E}(\omega) = \breve{H}(\pi - \omega)$$

$$\breve{F}(\omega) = 1 - \breve{H}(\omega)$$

$$\breve{G}(\omega) = 1 - \breve{H}(\pi - \omega)$$

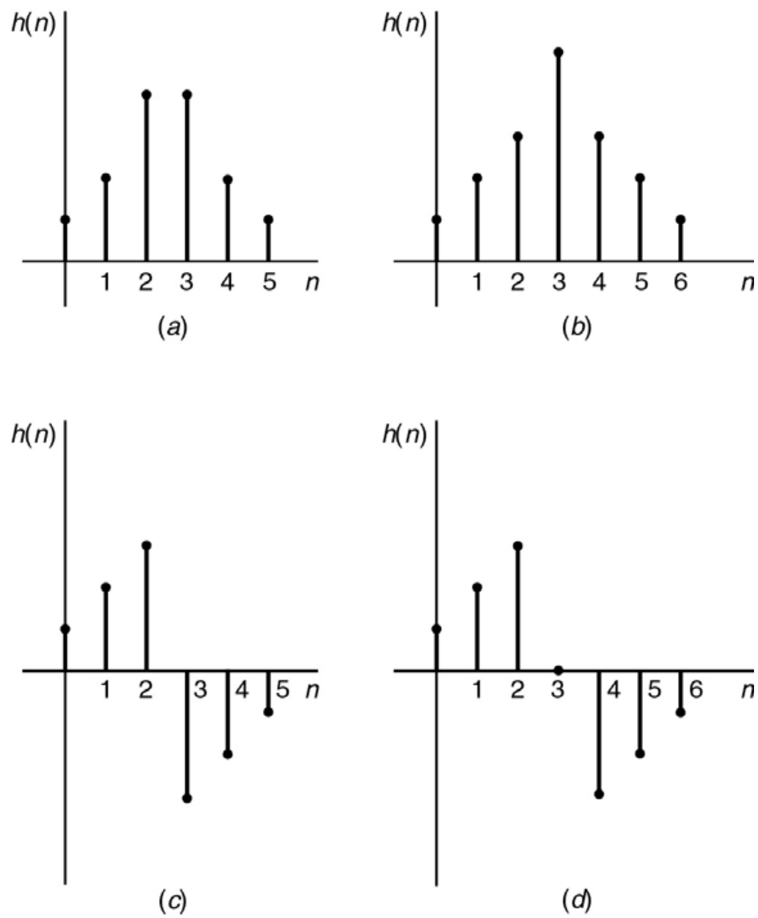
- if *H(z)* is a lowpass (bandpass) transfer function,
  then *E(z)* is a highpass (bandstop) transfer function; and vice versa.
- *F(z)* is of the same type as *H(z),* but with different width.
- *G(z)* is of the same type as *E(z),* but with different width.

Linear Phase Filters

- There are four possible scenarios: filter length even or odd, and impulse response is either symmetric or antisymmetric

FIR II: even length, symmetric

FIR I: odd length, symmetric



FIR IV: even length, antisymmetric

FIR III: odd length, antisymmetric

Note for this case that h[M/2]=0

FIR I and FIR II Types

- For symmetric coefficients, we can show that the frequency responses are of the following form:
- FIR II (M is odd, the sequence is symmetric and of even length)

$$\boxed{H(\omega) = 2e^{-j\underbrace{\frac{M}{2}}_{\alpha}\omega}\overbrace{\left(\sum_{i=0}^{(M-1)/2} h[i]\cos\left(\left(\frac{M}{2}-i\right)\omega\right)\right)}^{G(\omega)}}$$

  - Note that this of the form $H(\omega)=e^{-j\alpha\omega}G(\omega)$, where $\alpha=M/2$, and $G(\omega)$ is the real quantity (the summation term)➔ Output is delayed by M/2 samples!
- FIR I (M is even, sequence is symmetric and of odd length)

$$\boxed{H(\omega) = e^{-j\frac{M}{2}\omega}\left(h[\frac{M}{2}] + 2\sum_{i=1}^{M/2} h[i]\cos\left(\left(\frac{M}{2}-i\right)\omega\right)\right)}$$

  - Again, this system has linear phase (the quantity inside the parenthesis is a real quantity) and the phase delay is M/2 samples.

FIR III and FIR IV Types

- For antisymmetric sequences, we have h[n]=-h[M-n], which gives us ***sin*** terms in the summation expression:
- FIR IV (M is odd, the sequence is antisymmetric and of even length)

$$H(\omega) = 2e^{j\left[-\frac{M}{2}\omega + \frac{\pi}{2}\right]}\left(\sum_{i=0}^{(M-1)/2} h[i]\sin\left(\frac{M}{2} - i\right)\omega\right)$$

- FIR III (M is even, the sequence is antisymmetric and of odd length)

$$H(\omega) = 2e^{j\left[-\frac{M}{2}\omega + \frac{\pi}{2}\right]}\left(\sum_{i=1}^{M/2-1} h[i]\sin\left(\frac{M}{2} - i\right)\omega\right)$$

  - In both cases, the phase response is of the form $\theta(\omega)=-(M/2)\,\omega + \pi/2$, hence generalized linear phase. Again, in all of theses cases, the filter output is delayed by M/2 samples. Also, for all cases, if $G(\omega)<0$, an additional π term is added to the phase, which causes the samples to be flipped.

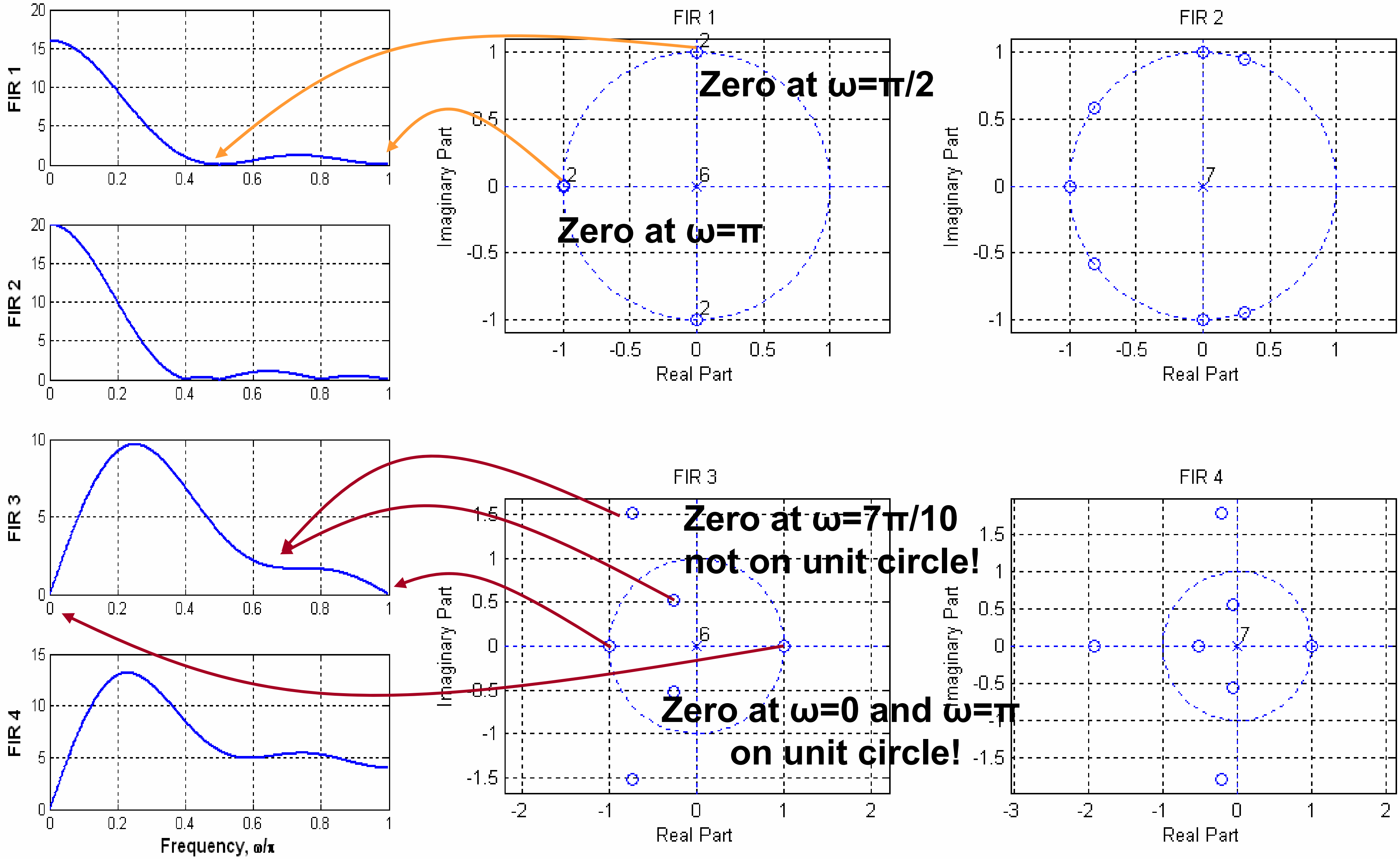
An Example – Matlab Demo

```
h1=[1 2 3 4 3 2 1]; % FIR 1
h2=[1 2 3 4 4 3 2 1]; % FIR 2
h3=[-1 -2 -3 0 3 2 1]; %FIR 3
h4=[-1 -2 -3 -4 4 3 2 1] ; % FIR 4


[H1 w]=freqz(h1, 1, 512); [H2 w]=freqz(h2, 1, 512);
[H3 w]=freqz(h3, 1, 512); [H4 w]=freqz(h4, 1, 512);

% Plot the magnitude and phase responses
% in angular frequency from 0 to pi
subplot(421); plot(w/pi, abs(H1));grid; ylabel('FIR 1')
subplot(422); plot(w/pi, unwrap(angle(H1)));grid;
subplot(423); plot(w/pi, abs(H2));grid; ylabel('FIR 2')
subplot(424); plot(w/pi, unwrap(angle(H2)));grid;
subplot(425); plot(w/pi, abs(H3));grid; ylabel('FIR 3')
subplot(426); plot(w/pi, unwrap(angle(H3)));grid;
subplot(427); plot(w/pi, abs(H4));grid
xlabel('Frequency, \omega/\pi'); ylabel('FIR 4')
subplot(428);  plot(w/pi, unwrap(angle(H4)));grid
xlabel('Frequency, \omega/\pi')
```

```
%Plot the zero - pole plots
figure
subplot(221)
zplane(h1,1); grid
title('FIR 1')
subplot(222)
zplane(h2,1);grid
title('FIR 2')
subplot(223)
zplane(h3,1);grid
title('FIR 3')
subplot(224)
zplane(h4,1);grid
title('FIR 4')
```

lin_phase_demo2.m

Zeros & Poles of an FIR Filter Example



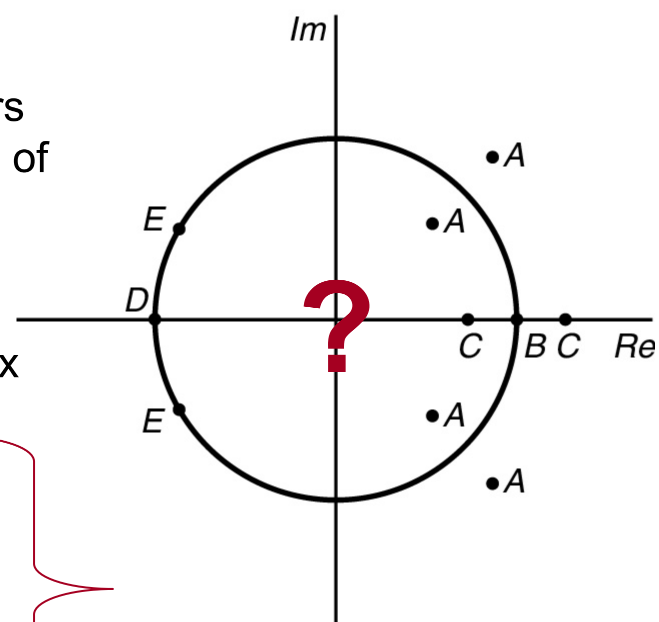
Zero Locations of Linear Phase Filters

- For linear phase filters, the impulse response satisfies

$$h[n] = \pm h[M-n] \Rightarrow H(z) = \pm \sum_{n=0}^{M} h(M-n) z^{-n} \qquad H(z) = \pm z^{-M} H(z^{-1})$$

  - We can make the following observations as facts

1. If z_0 is a zero of $H(z)$, so too is $1/z_0 = z_0^{-1}$
2. Real zeros that are not on the unit circle, always occur in pairs such as $(1-\alpha z^{-1})$ and $(1-\alpha^{-1} z^{-1})$, which is a direct consequence of Fact 1.
3. If the zero is complex, $z_0 = \alpha e^{j\theta}$, then its conjugate $\alpha e^{-j\theta}$ is also zero. But by the above statements, their symmetric counterparts $\alpha^{-1} e^{j\theta}$ and $\alpha^{-1} e^{-j\theta}$ must also be zero. So, complex zeros occur in quadruplets
4. If M is odd (filter length is even), and symmetric (that is, $h[n]=h[M-n]$), then $H(z)$ must have a zero at $z=-1$
5. If the filter is antisymmetric, that is, $h[n]=-h[M-n]$, then $H(z)$ must have a zero at $z=+1$ for both odd and even M
6. If the filter is symmetric and $h[n]=-h[M-n]$, and M is even, then $H(z)$ must have a zero at $z=-1$



Why…? (p. 376)

LINEAR PHASE FILTER ZERO LOCATIONS

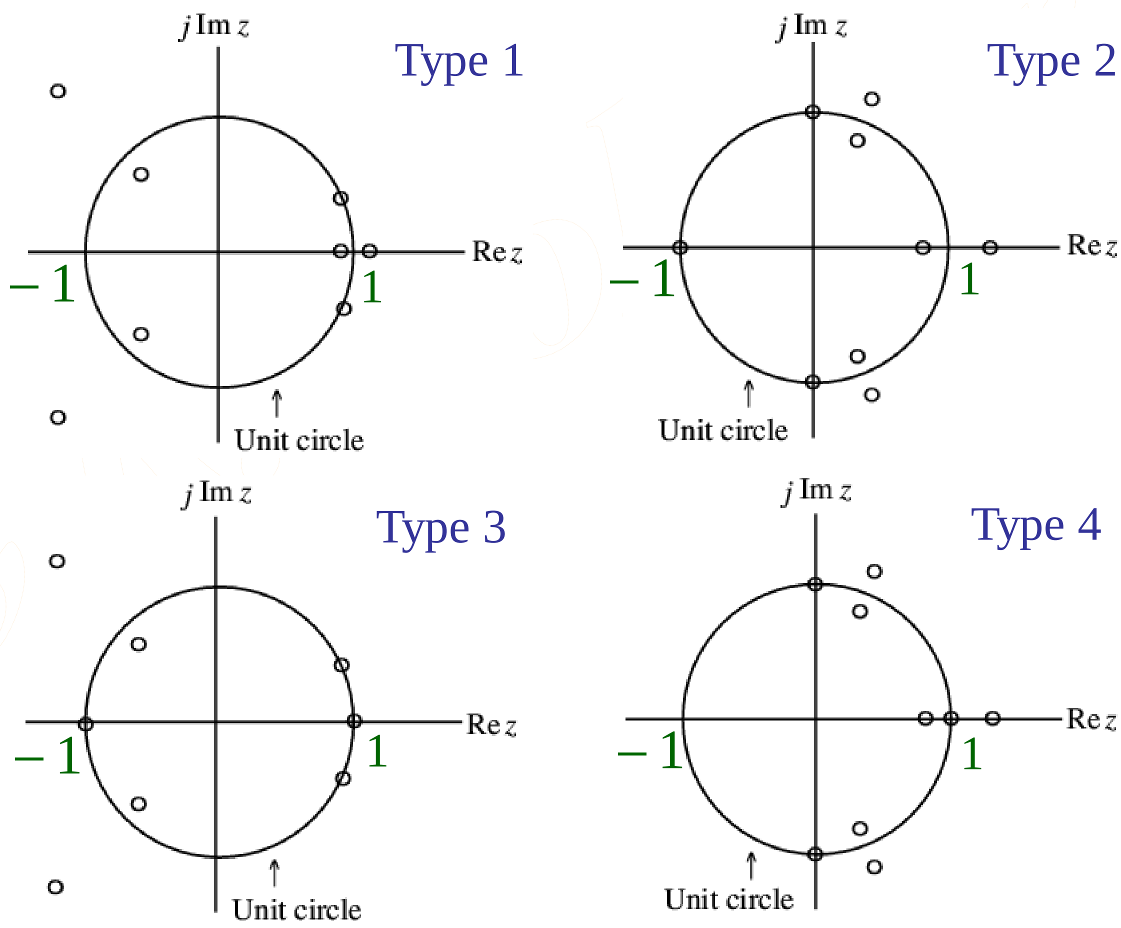
Type 1 FIR filter: Either an even number or no zeros at z = 1 and z=-1

Type 2 FIR filter: Either an even number or no zeros at z = 1, and an odd number of zeros at z=-1

Type 3 FIR filter: An odd number of zeros at z = 1 and and z=-1

Type 4 FIR filter: An odd number of zeros at z = 1, and either an even number or no zeros at z=-1

The presence of zeros at z=±1 leads to some limitations on the use of these linear-phase transfer functions for designing frequency-selective filters



A Type 2 FIR filter cannot be used to design a highpass filter since it always has a zero at z=-1

A Type 3 FIR filter has zeros at both z = 1 and z=-1, and hence cannot be used to design either a lowpass or a highpass or a bandstop filter

A Type 4 FIR filter is not appropriate to design a lowpass filter due to the presence of a zero at z = 1

Type 1 FIR filter has no such restrictions and can be used to design almost any type of filter