Today in DSP

- Introducing the Z- transform
  - Why do we need yet another transform?
  - Z-transform – relation to the DTFT
  - Some examples and observations
  - Region of convergence (ROC)
- Properties of z-transform
- Some common z-transforms
- Rational z-transforms
  - Z-transforms in the system theory
  - Poles and zeros of a system
  - Making physical sense of the z-transform – relation to filtering

Yet Another Transform?

- Think about all the transforms you have seen so far:
  - The Laplace transform
  - The Fourier series
  - The continuous Fourier transform
  - The discrete time Fourier transform
  - The discrete Fourier transform, and FFT
- Why need yet another…?
  - Convergence issues with the Fourier transforms
  - The DTFT of a sequence exists if and only if the sequence x[n] is absolutely summable, that is, if

$$\boxed{\sum_{n=-\infty}^{\infty} |x[n]| < \infty}$$

  - DTFT may not exist for many signals of practical or analytical signals, whose frequency analysis can therefore not be obtained through DTFT

The z-Transform
(The last one, really!)

- A generalization of the DTFT leads to the z-transform that may exist for many signals for which the DTFT does not.
  - DTFT is in fact a special case of the z-transform
    - …just like the Fourier transform is a special case of _______________(?)
- Furthermore, the use of the z-transform allows simple algebraic expressions to be used which greatly simplifies frequency domain analysis.
- Digital filters are designed, expressed, applied and represented in terms of the z-transform
- For a given sequence x[n], its z-transform X(z) is defined as

$$X(z)=\sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

where z lies in the complex space, that is, $z=a+jb=re^{j\omega}$

Z ←→ DTFT

- From the definition of the z-variable

$$X(z)=\sum_{n=-\infty}^{\infty} x[n]z^{-n}=\sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n}=\sum_{n=-\infty}^{\infty} x[n]r^{-n}e^{-j\omega n}$$

  - It follows that the DTFT is indeed a special case of the z-transform, specifically, z-transform reduces to DTFT for the special case of ***r*=1**, that is, **|*z*|=1**, provided that the latter exists.

$$X(\omega)=\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

  - The contour |z|=1 is a circle in the z-plane of unit radius → ***the unit circle***
  - Hence, the DTFT is really the z-transform evaluated on the unit circle
  - Just like the DTFT, z-transform too has its own, albeit less restrictive, convergence requirements, specifically, the infinite series $\sum_{n=-\infty}^{\infty} x[n]z^{-n}$ must converge.

- For a given sequence, the set R of values of z for which its z-transform converges is called the ***region of convergence (ROC).***

Convergence of the Z-Transform

- From our discussion with the DTFT, we know that the infinite series

$$X(z) = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] r^{-n} e^{-j\omega n}$$

converges if $x[n]r^{-n}$ is absolutely summable, that is, if

$$\boxed{\sum_{n=-\infty}^{\infty} \left| x[n] r^{-n} \right| < \infty}$$

- The area where this is satisfied defines the ROC, which in general is an annular region of the z-plane (since “z” is a complex number, constant z-values describe a circle in the z-plane)

$$R^{-} < |z| < R^{+} \quad \text{where } 0 \le R^{-} < R^{+} \le \infty$$

EXAMPLES

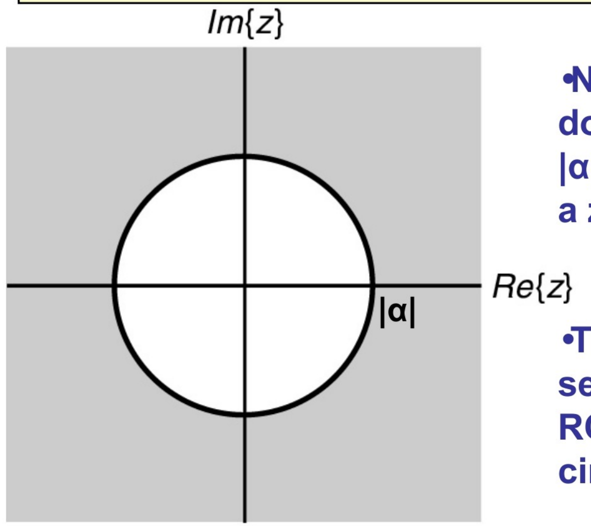
- Determine the z-transform and the corresponding ROC of the causal sequence $x[n]=\alpha^n u[n]$

$$X(z)=\frac{1}{1-\alpha z^{-1}}$$

for $\left|\alpha z^{-1}\right|<1$

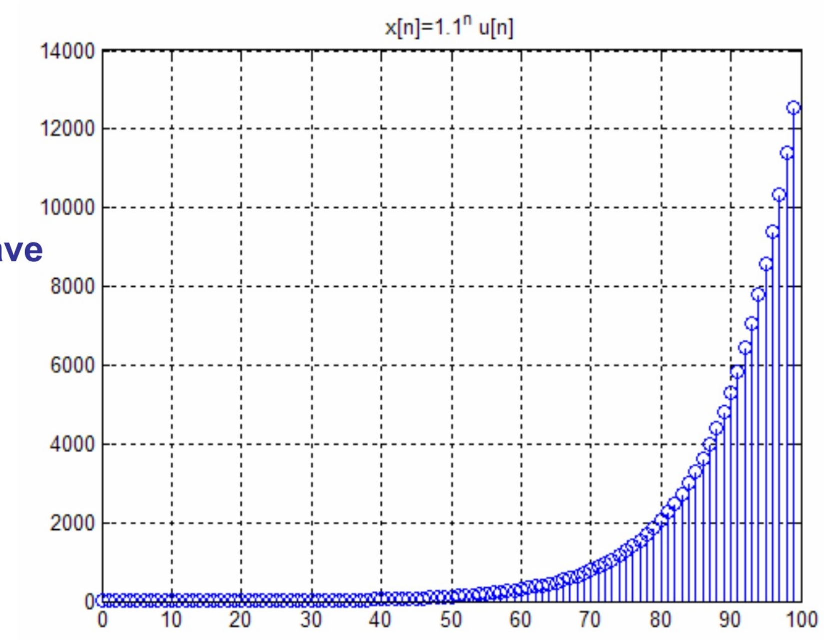
This power (geometric) series converges to

→ ROC is the annular region $|z| > |a|$



•Note that this sequence does not have a DTFT if |α|>1, however, it does have a z-transform!

•This is a right-sided sequence, which has an ROC that is outside of a circular area!



EXAMPLES

➲ The z-transform of the unit step sequence u[n] can be obtained from

$$X(z) = \frac{1}{1-\alpha z^{-1}}, \quad \text{for } \left|\alpha z^{-1}\right| < 1$$

by setting α=1 ➔ $U(z) = \dfrac{1}{1-z^{-1}} = \dfrac{z}{z-1}, \quad \text{for } \left|z^{-1}\right| < 1$

ROC is the annular regions |z|>1. Note that this sequence also does not have a DTFT!

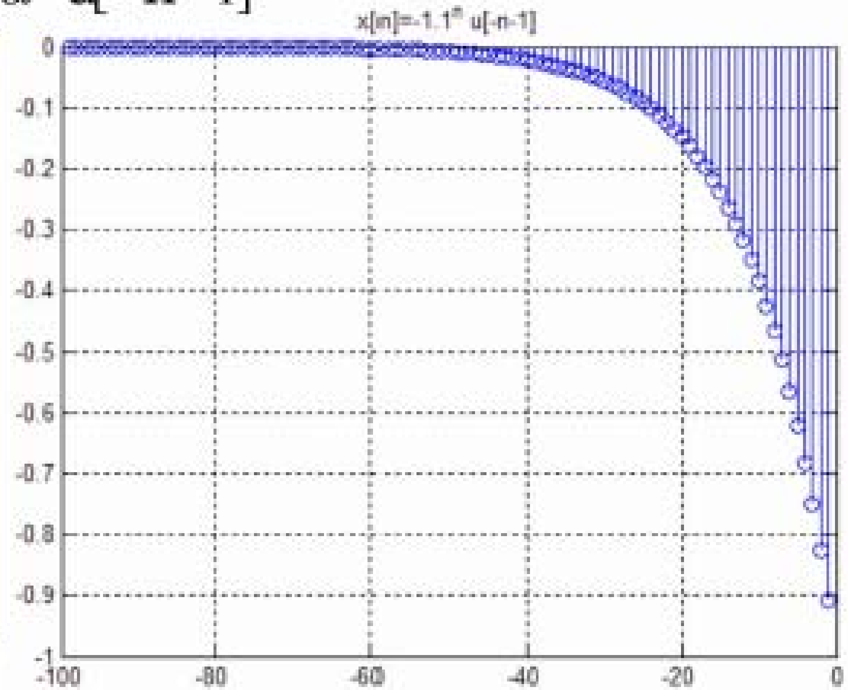
EXAMPLES

➲ Now consider the anti-causal sequence $y[n] = -\alpha^n u[-n-1]$

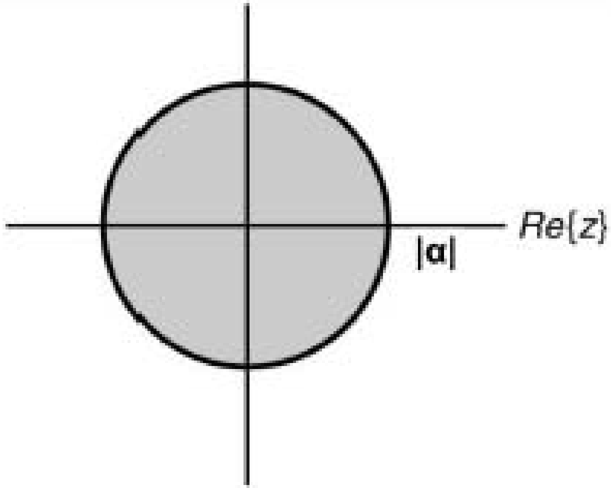
$$X(z) = -\sum_{n=-\infty}^{-1} \alpha^n z^{-n} = -\sum_{m=1}^{\infty} \alpha^{-m} z^m$$

$$= -\alpha^{-1} z \sum_{n=0}^{\infty} \alpha^{-n} z^n = \frac{1}{1-\alpha z^{-1}}$$

for $\quad |\alpha^{-1} z| < 1$



ROC is the annular region $|z| < |\alpha|$



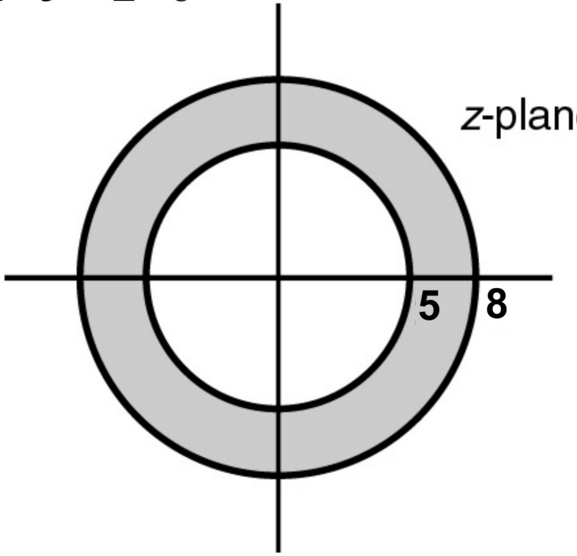
- The z-transforms of the two sequences $\alpha^n u[n]$ and $-\alpha^n u[-n-1]$ are identical even though the two parent sequences are different
- Only way a unique sequence can be associated with a z-transform is by specifying its ROC
- This is a left-sided sequence, which has an ROC that is inside of a circular area!

Some Observations

- If the two sequences $x[n] = \alpha^n u[n]$ and $y[n] = -\alpha^n u[-n-1]$ denote the impulse responses of a system (digital filter), then their z-transforms $z/(z-\alpha)$ represent the ***transfer functions*** of these systems.
- Both transfer functions have a pole at ***z=α***, which make the transfer function asymptotically approach to infinity at this value. Therefore, ***z=α*** is <u>not</u> included in either of the ROCs.
  - The circle with the radius of ***α*** is called the ***pole circle***. A system may have many poles, and hence many pole circles.
  - For right sided sequences, the ROC extend outside of the – ***outermost*** pole circle - , whereas for left sided sequences, the ROC is the inside of the – ***innermost*** pole circle.
  - For two-sided sequences, the ROC will be the intersection of the two ROC areas corresponding to the left and right sides of the sequence.
  - **Since DTFT is the z-transform evaluated on the unit circle, that is for $z=e^{j\omega}$, DTFT of a sequence exists <u>if and only if the ROC includes the unit circle</u>!**

EXAMPLE

- Consider $x[n]=5^n u[n]-8^n u[-n-1]$ ➔ $X(z)=\frac{z}{z-5}+\frac{z}{z-8}$
  - Corresponding ROCs are $|z|>5$ and $|z|<8$
  - Therefore the ROC for this signal is the annular region $5<|\mathbf{z}|<8$



  - Note that if the signal was $x[n]=8^n u[n]-5^n u[-n-1]$ the ROC would be empty! That is, the z-transform of this sequence does not exist!
  - **Now recall that DTFT was z-transform evaluated on the unit circle, that is for $z=e^{j\omega}$. Therefore, DTFT of a sequence exists (that is the series converges), <u>if and only if the ROC includes the unit circle</u>!**
  - The DTFT for the above example clearly does not exist, since the ROC does not include the unit circle!
  - Though, we must add that the existence of DTFT is not a guarantee for the existence of the z-transform.

Commonly Used z-Transform Pairs

Sequence	z-Transform	ROC
$\delta[n]$	1	All values of z
$u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$\alpha^n\, u[n]$	$\dfrac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
$(r^n \cos\omega_o n)\; u[n]$	$\dfrac{1-(r\cos\omega_o)z^{-1}}{1-(2r\cos\omega_o)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
$(r^n \sin\omega_o n)\; u[n]$	$\dfrac{(r\sin\omega_o)z^{-1}}{1-(2r\cos\omega_o)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

Z-TRANSFORM PROPERTIES

Let $x_1[n] \leftrightarrow X_1(z)$, $x_2[n] \leftrightarrow X_2(z)$, $h[n] \leftrightarrow H(z)$ be z-transform pairs, with individual ROC of R_{x1}, R_{x2}, and R_h, respectively. Also assume that any ROC is of the form $r_{in} < |z| < r_{out}$. Then the following hold:

Z-Transform Properties

Property	Time Domain	Z-domain	ROC
1. Linearity	$ax_1(n) + bx_2(n)$	$aX_1(z) + bX_2(z)$	$R_{x_1} \cap R_{x_2}$
2. Time shifting	$x(n - n_0)$	$z^{-n_0}X(z)$	R_x
3. Multiplication by discrete exponential	$a^n x(n)$	$X\left(\frac{z}{a}\right)$	$\|a\|r_i < \|z\| < \|a\|r_o$
4. Differentiation in the Z-domain	$nx(n)$	$-z\frac{dX(z)}{dz}$	R_x
5. Time reversal	$x(-n)$	$X(z^{-1})$	$1/r_o < \|z\| < 1/r_i$
6. Convolution in time	$x(n) * h(n)$	$X(z)H(z)$	$R_{x_1} \cap R_h$
7. Complex conjugation	$x^*(n)$	$(X(z^*))^*$	R_x
8. Multiplication in time	$x(n)h(n)$	$\frac{1}{2\pi j}\oint_{C_1} X(v)H\left(\frac{z}{v}\right)v^{-1}\,dv$	
9. Parseval's theorem	$\sum_{n=-\infty}^{\infty} x(n)h^*(n)$	$= \frac{1}{2\pi j}\oint_{C_1} X(v)H^*\left(\frac{1}{v^*}\right)v^{-1}d$	

Rational z-Transforms

- The z-transforms of LTI systems can be expressed as a ratio of two polynomials in z^{-1}, hence they are rational transforms.
  - Starting with the constant coefficient linear difference equation representation of an LTI system:

$$\sum_{i=0}^{N} a_i y[n-i] = \sum_{j=0}^{M} b_j x[n-j], \quad a_0 = 1$$

$$y[n] + a_1 y[n-1] + a_2 y[n-2] + \cdots a_N y[n-N] = b_0 x[n] + b_1 x[n-1] + \cdots + b_M x[n-M]$$

$$\Updownarrow \mathcal{Z} \qquad\qquad\qquad\qquad \Updownarrow \mathcal{Z}$$

$$Y(z) + a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) + \cdots a_N z^{-N} Y(z) = b_0 X(z) + b_1 z^{-1} X(z) + \cdots + b_M z^{-M} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}}$$

MATLAB uses this representation for all digital filters / systems / transfer functions!!!

Rational Z-Transforms

- A rational z-transform can be alternately written in factored form as

$$H(z)=\frac{b_0\prod_{\ell=1}^{M}(1-\zeta_\ell z^{-1})}{a_0\prod_{\ell=1}^{N}(1-p_\ell z^{-1})}=z^{(N-M)}\frac{p_0\prod_{\ell=1}^{M}(z-\zeta_\ell)}{d_0\prod_{\ell=1}^{N}(z-p_\ell)}$$

- At a root $z=\zeta_\ell$ of the numerator polynomial $H(\zeta_\ell)=0$, and as a result, these values of z are known as the ***zeros*** of $H(z)$
- At a root $z=p_\ell$ of the denominator polynomial $H(p_\ell)\rightarrow\infty$, and as a result, these values of z are known as the ***poles*** of $H(z)$
  - Note that H(z) has *M* finite zeros and *N* finite poles
  - If N > M there are additional *N-M* zeros at z = 0 (the origin in the z-plane)
  - If N < M there are additional *M-N* poles at z = 0
- Why is this important?
  - As we will see later, a digital filter is designed by placing appropriate number of zeros at the frequencies (z-values) to be suppressed, and poles at the frequencies to be amplified!

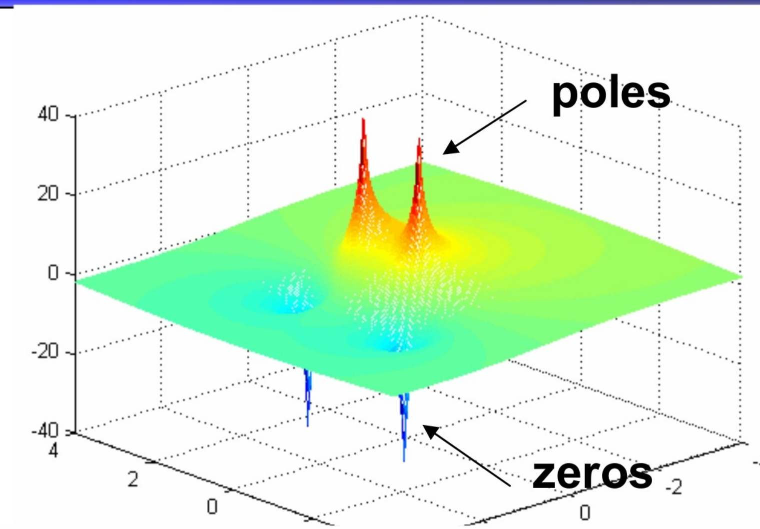
Some sense of Physical Interpretation of this Math Crap!

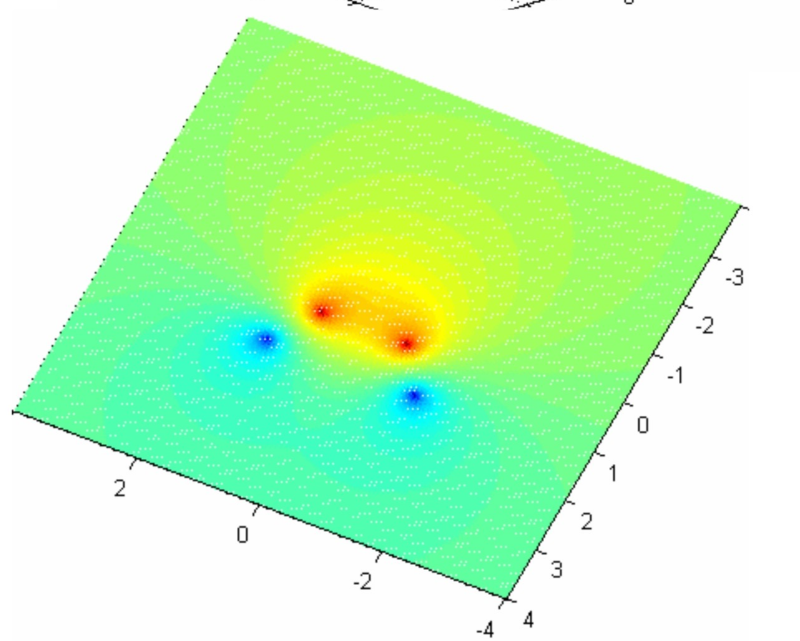
What does this look like???

$$G(z) = \frac{1 - 2.4\,z^{-1} + 2.88\,z^{-2}}{1 - 0.8\,z^{-1} + 0.64\,z^{-2}}$$

```
clear
close all
N=256;
rez=linspace(-4, 4, N);
imz=linspace(-4,4,N);
%create a uniform z-plane
for n=1:N
     z(n,:)=ones(1,N)*rez(n)+j*ones(1,N).*imz(1:N);
end
%Compute the H function on the z-plane
for n=1:N
   for m=1:N
      Hz(n,m)=H_fun(z(n,m));
   end
end
%Logarithmic mesh plot of the  H function
mesh(rez, imz, 20*log10(abs(Hz)))

 =======================================
 function Hz=H_fun(z);
 %Compute the transfer function
 Hz=(1-2.4*z^(-1)+2.88*z^(-2))/(1-0.8*z^(-1)+0.64*z^(-2));
```





In Matlab

- Matlab has simple functions to determine and plot the poles and zeros of a function in the z-plane:

tf2zpk() **Discrete-time transfer function to zero-pole conversion.**

[Z,P,K] = TF2ZPK(NUM,DEN) finds the zeros, poles, and gain:

$$H(z) = K \frac{(z-Z(1))(z-Z(2))\ldots(z-Z(n))}{(z-P(1))(z-P(2))\ldots(z-P(n))}$$

from a single-input, single-output transfer function in polynomial form:

$$H(z) = \frac{NUM(z)}{DEN(z)}$$

```
b=[1 -2.4 2.88];   a=[1 -0.8 0.64];
[z,p,k] = tf2zpk(b,a)
```

```
z =  1.2000 + 1.2000i
     1.2000 - 1.2000i
p =  0.4000 + 0.6928i
     0.4000 - 0.6928i
k=1
```

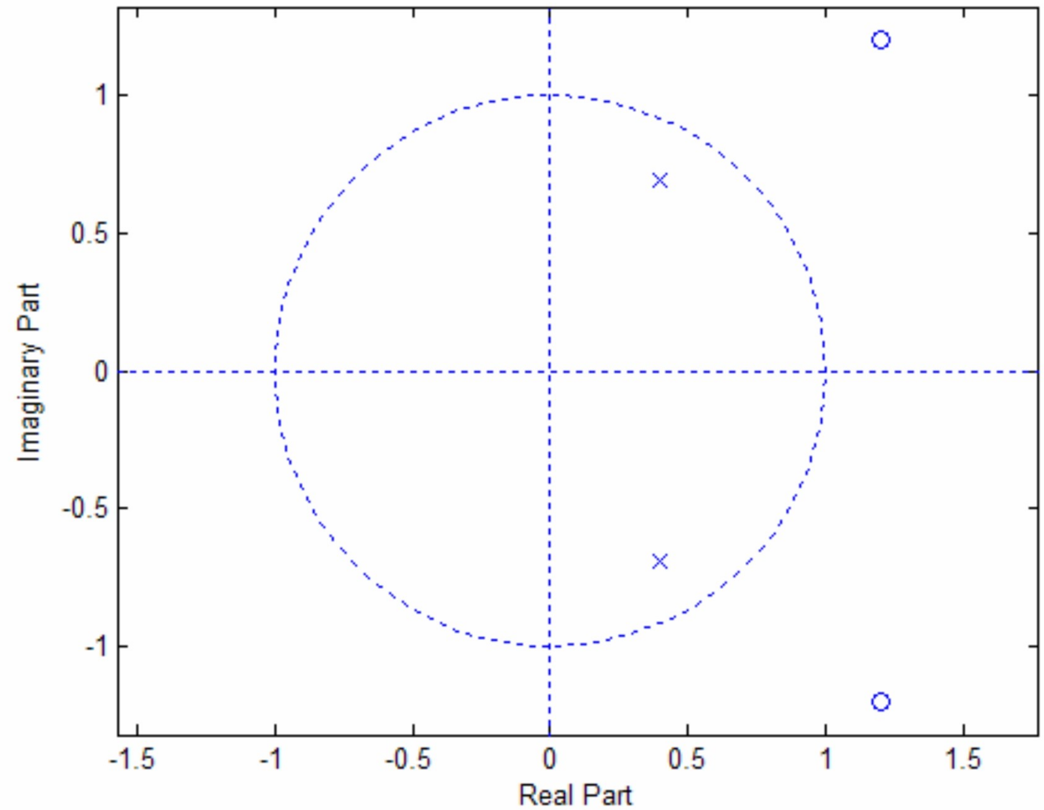
In Matlab

zplane Z-plane zero-pole plot.

zplane(Z,P) plots the zeros Z and poles P (in column vectors) with the unit circle for reference. Each zero is represented with a 'o' and each pole with a 'x' on the plot. Multiple zeros and poles are indicated by the multiplicity number shown to the upper right of the zero or pole.

ZPLANE(B,A) where B and A are row vectors containing transfer function polynomial coefficients plots the poles and zeros of B(z)/A(z).

```
b=[1 -2.4 2.88];   a=[1 -0.8 0.64];
zplane(b,a)
```

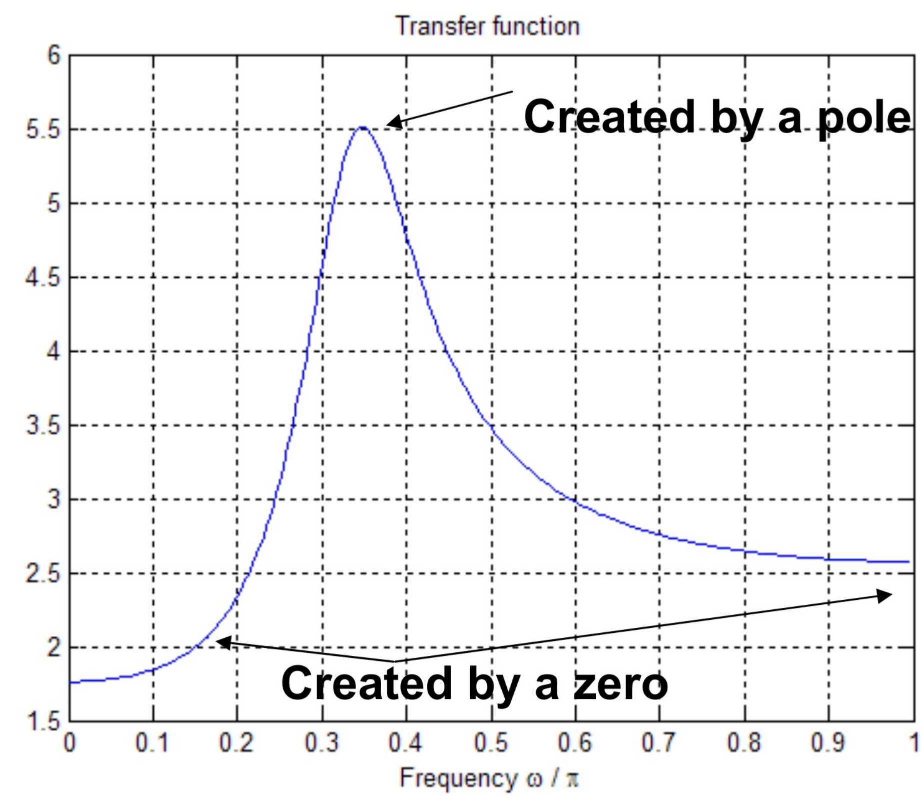


Some sense of Physical Interpretation of this Math Crap!

$$G(z) = \frac{1 - 2.4\,z^{-1} + 2.88\,z^{-2}}{1 - 0.8\,z^{-1} + 0.64\,z^{-2}}$$

```
[H w]=freqz([1 -2.4 2.88],[1 -0.8 0.64],256);
figure
plot(w/pi, abs(H))
grid
title('Transfer function')
xlabel('Frequency \omega / \pi')
```

This system has two zeros at z=1.2 ±j 1.2 and two poles at z=0.4 ± j0.6928



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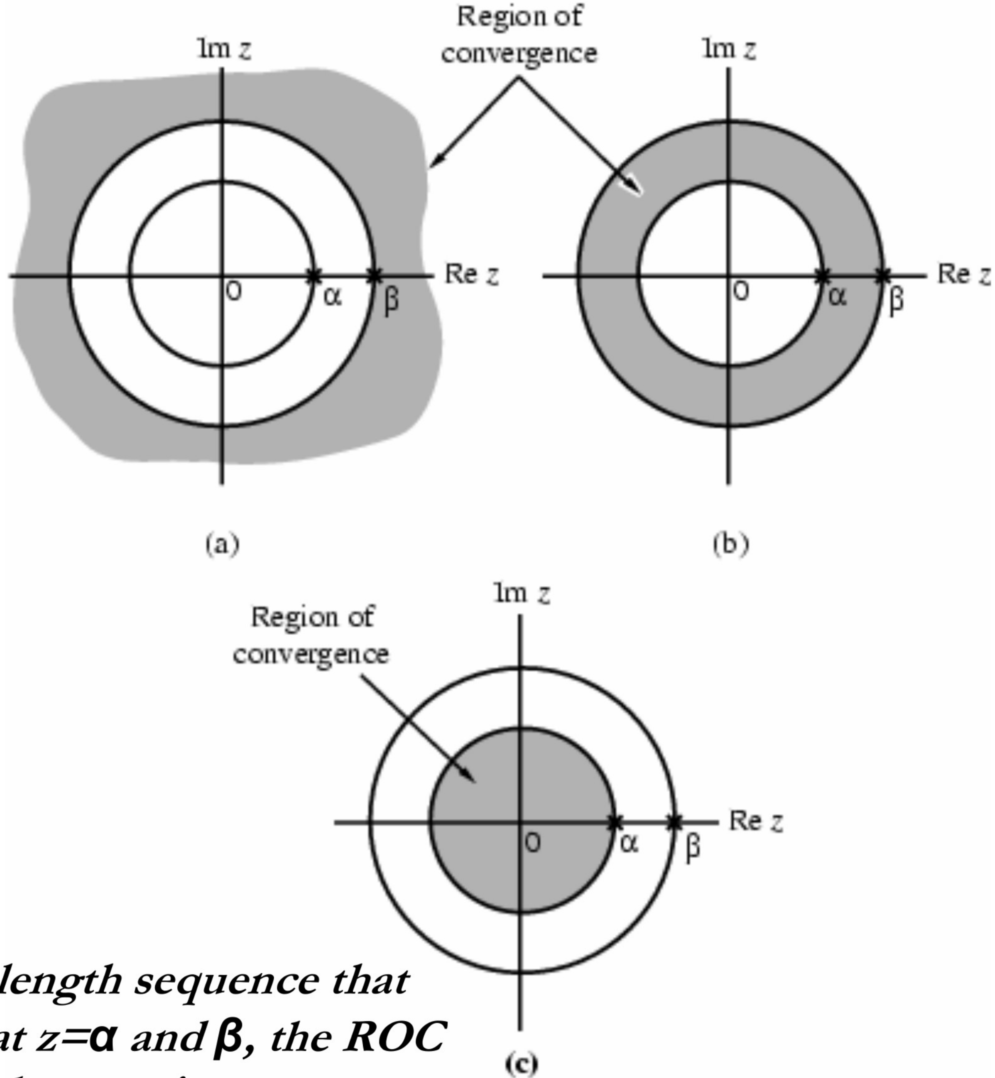
Poles & ROC

- The ROC of a rational z-transform cannot contain any poles and is bounded by the poles
  - For a right sided sequence, the ROC is outside of the largest pole
  - For a left sided sequence, the ROC is inside of the smallest pole
  - For a two sided sequence, some of the poles contribute to terms in the parent sequence for $n<0$ and other to terms for $n>0$. Therefore, the ROC is between the circular regions: outside of the largest pole coming from the $n>0$ sequence ***and*** inside of the smallest pole coming from the $n<0$ sequence

Poles and ROC

- The ROC of a rational z-transform cannot contain any poles and is bounded by the poles:
  - For a right sided sequence, the ROC is outside of the largest pole
  - For a left sided sequence, the ROC is inside of the smallest pole
  - For a two sided sequence, some of the poles contribute to terms in the parent sequence for n<0 and other to terms for n>0. Therefore, the ROC is between the circular regions: outside of the largest pole coming from the n>0 sequence ***and*** inside of the smallest pole coming from the n<0 sequence.
  - If the sequence is of finite length, then the ROC includes the entire z-plane, except possibly z=0 and/or z=∞.



For an infinite-length sequence that has two poles, at z=α and β, the ROC is one of these three options

Stability and DTFT

- Recall the main requirement for BIBO stability: $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$
  - This infinite sum may be difficult to calculate. Here is a much simpler way to determine if a system is stable.
    - But first, we show that if the above stated stability condition is satisfied, then the DTFT of the sequence exists:

$$|H(z)| = \left|\sum_{n=-\infty}^{\infty} h[n] z^{-n}\right| \le \sum_{n=-\infty}^{\infty} \left|h[n] z^{-n}\right| = \sum_{n=-\infty}^{\infty} |h[n]| \left|z^{-n}\right|$$

- If we evaluate this expression on the unit circle to determine if the DTFT exists:

$$|H(z)|_{z=e^{j\omega}} \le \left.\sum_{n=-\infty}^{\infty} |h[n]| \left|z^{-n}\right|\right|_{z=e^{j\omega}}$$

$$\Rightarrow \left|H\left(e^{j\omega}\right)\right| \le \sum_{n=-\infty}^{\infty} |h[n]|$$

h[n] BIBO stable $\longleftrightarrow$ The ROC includes the Unit Circle

- But if the system satisfies the stability condition, $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$ then it follows that

$$\Rightarrow \left|H\left(e^{j\omega}\right)\right| \le \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

Hence, if the stability condition is satisfied, DTFT must exist, or if DTFT exists, then the system is stable. Remember what is required – in terms of the ROC, for the DTFT to exist…?

Stability & ROC In terms of Zeros & Poles

- Recall that for a ***system*** to be causal, its impulse response must satisfy h[n]=0, n<0, that is for a causal system, the impulse response is ***right sided.*** Based on this, and our previous observations, we can make the following important conclusions:
  - **The ROC of a causal system extends outside of the outermost pole circle**
  - **The ROC of an anticausal system (whose h[n] is purely left-sided) lies inside of the innermost pole circle**
  - **The ROC of a noncausal system (whose h[n] two-sided) is bounded by two different pole circles**
  - **Now, for an LTI system to be stable it must be absolutely summable, or in other words, it must have a DTFT. But for a system to have a DTFT, its ROC must include the unit circle. ➔ <u>An LTI system is stable, if and only if the ROC of its transfer function H(z) includes the unit circle!</u>**
  - **Furthermore, a causal system's ROC lies outside of a pole circle. If that system is also stable, its ROC must include unit circle➔ Then <u>a causal system is stable, if and only if, all poles are inside the unit circle!</u> Similarly, an anticausal system is stable, if and only if its poles lie outside the unit circle.**
    - **An FIR filter is always stable, why?**

Inverse z-Transform

- The inverse z-transform can be obtained as a generalization of the inverse DTFT:

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) \cdot z^{n-1} dz$$

  - Since the variable ***z*** is defined on the complex (polar) plane, the integral is not a Cartesian integral, but rather a contour integral, where the contour *C* is any circular region that falls into the ROC of X(z).
  - Complicated procedure, yet several different procedures to compute
    1. Perform a long division for X(z), and invert each (simple) term individually – tedious, often does not result in a closed form (finds x[n] one by one for each ***n***)
    2. Direct evaluation of the contour integral using the Cauchy Residue theorem – tedious
    3. Partial fraction expansion – most commonly used procedure.

Cauchy Residue Theorem

- Cauchy's residue theorem states that the contour integral can be computed as a sum of the residues that lie inside the contour

$$x[n] = \frac{1}{2\pi j}\oint_C X(z)\cdot z^{n-1}dz = \sum\left[\begin{array}{l}\text{Residues of } X(z)\cdot z^{n-1} \text{ at the} \\ \text{poles inside the contour C}\end{array}\right]$$

  - whose special case include the following theorem

$$\oint_C z^{-l}dz = \begin{cases} 2\pi j, & l = 1 \\ 0, & \text{otherwise} \end{cases}$$

  - The details of this integration is reserved for a course in complex variables

Partial Fractional Expansion

The rational z-transform can be written as a ratio of polynomials

$$H(z) = \frac{\sum_{i=0}^{M} b_i z^{-i}}{\sum_{i=0}^{N} a_i z^{-i}}$$

which is equivalent to:

$$H(z) = \frac{z^{-M} \sum_{i=0}^{M} b_i z^{M-i}}{z^{-N} \sum_{i=0}^{N} a_i z^{N-i}}$$

to solve this we need to rewrite this expression a bit more in the form of functions which are Z-transforms of known basis functions

Partial Fractional Expansion

write this expansion in factorial form:

$$H(z) = \frac{b_0}{a_0} \frac{\prod_{i=1}^{M} (1 - c_i z^{-1})}{\prod_{i=1}^{N} (1 - d_i z^{-1})}$$

with zeros at c_i and poles at d_i

If $M < N$ and the poles are all 1st order then $H(z)$ can be written as:

$$H(z) = \sum_{i=0}^{N} \frac{A_i}{1 - d_i z^{-1}}$$

and the coefficients can be found via:

$$A_i = (1 - d_i z^{-1}) H(z) \Big|_{z = d_i}$$

Partial Fractional Expansion

Example 1: $$H(z)=\frac{1}{(1-\frac{3}{4}z^{-1}+\frac{3}{8}z^{-2})}=\frac{1}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})}$$ $|z|>1/2$

this can be written as: $$H(z)=\frac{A_1}{(1-\frac{1}{4}z^{-1})}+\frac{A_2}{(1-\frac{1}{2}z^{-1})}$$

with: $$A_1=\left.\frac{1-\frac{1}{4}z^{-1}}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})}\right|_{z=\frac{1}{4}}=-1;\qquad A_2=\left.\frac{1-\frac{1}{2}z^{-1}}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})}\right|_{z=\frac{1}{2}}=2$$

so: $$H(z)=\frac{-1}{(1-\frac{1}{4}z^{-1})}+\frac{2}{(1-\frac{1}{2}z^{-1})}$$

$h[n]$ is causal

$$h[n]=2\left(\tfrac{1}{2}\right)^n\mu[n]-\left(\tfrac{1}{4}\right)^n\mu[n]$$

Partial Fractional Expansion

if $M \geq N$ a polynomial must be added of order $M - N$:

$$H(z) = \sum_{k=0}^{M-N} B_k z^{-k} + \sum_{i=0}^{N} \frac{A_i}{1 - d_i z^{-1}}$$

values of B_k are obtained by long division of the numerator of $H(z)$ by the denominator.

in the case of multiple poles of order s at z = d_i while all other poles are simple another term is required:

$$H(z) = \sum_{k=0}^{M-N} B_k z^{-k} + \sum_{k=0, k \neq i}^{N} \frac{A_k}{1 - d_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{\left(1 - d_i z^{-1}\right)^m}$$

Partial Fractional Expansion

the coefficients C_m can be determined by taking derivatives and evaluating the function at the poles:

$$C_m = \frac{1}{(s-m)!(-d_i)^{s-m}} \left\{ \frac{d^{s-m}}{dw^{s-m}} [(1-d_i w)^s H(w^{-1})] \right\}_{w=d_i^{-1}}$$

if there are more multiple-order poles then there will be an additional term for each multiple-order pole:

$$H(z) = \sum_{k=0}^{M-N} B_k z^{-k} + \sum_{k=0,k\neq i,k\neq j}^{N} \frac{A_k}{1-d_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{\left(1-d_i z^{-1}\right)^m} + \sum_{n=1}^{t} \frac{D_n}{\left(1-d_j z^{-1}\right)^n} + \ldots$$

Partial Fraction Expansion

- Re-express the rational z-transform as a partial fraction expansion of simpler terms, whose inverse z-transforms are known.
  - Slightly different procedure depending on whether the system has simple poles or multiple poles
  - A rational H(z) can be expressed as

$$H(z) = \frac{Y(z)}{X(z)} = \frac{P(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}} = \frac{\sum_{i=0}^{M} b_i z^{-i}}{\sum_{i=0}^{N} a_i z^{-i}}$$

  - If $M \geq N$ then H(z) can be re-expressed through long division

$$\boxed{H(z) = \sum_{\ell=0}^{M-N} \eta_\ell z^{-\ell} + \frac{P_1(z)}{D(z)}}$$

$$H(z) = \frac{2 + 0.8\,z^{-1} + 0.5\,z^{-2} + 0.3\,z^{-3}}{1 + 0.8\,z^{-1} + 0.2\,z^{-2}} \Rightarrow H(z) = -3.5 + 1.5\,z^{-1} + \frac{5.5 + 2.1\,z^{-1}}{1 + 0.8\,z^{-1} + 0.2\,z^{-2}}$$

where the degree of $P_1(z)$ is less than N. The rational fraction $P_1(z)/D(z)$ is then called a ***proper fraction*** or ***proper polynomial*** .

Partial Fractional Expansion

Example 2:

$$H(z) = \frac{2z^2 + 4z + 2}{2z^2 - 3z + 1} = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = \frac{(1 + z^{-1})^2}{(1 - \frac{1}{2}z^{-1})(1 - z^{-1})}$$

As the numerator is the same order as the denominator we need to write *H(z)* as:

$$H(z) = B_0 + \frac{A_1}{(1 - \frac{1}{2}z^{-1})} + \frac{A_2}{(1 - z^{-1})} \qquad |z| > 1$$

we first need to determine B_o and a residual expression with a lower order numerator by dividing the denominator into the numerator

$$B_0 = + \frac{1 - 2z^{-1} + z^{-2}}{(1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2})} = 2 + \left[\frac{5z^{-1} - 1}{(1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2})} \right]$$

Partial Fractional Expansion

the next step is to further solve the second term: $\left[\dfrac{5z^{-1}-1}{(1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2})}\right]$

and solve for A_1 and A_2:

$$A_1 = 2+\frac{5z^{-1}-1}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}(1-\tfrac{1}{2}z^{-1})\Bigg|_{z=\frac{1}{2}} = -9$$

$$A_2 = 2+\frac{5z^{-1}-1}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}(1-z^{-1})\Bigg|_{z=1} = +8$$

so: $$H(z) = 2-\frac{9}{(1-\frac{1}{2}z^{-1})}+\frac{8}{(1-z^{-1})}$$

$h[n]$ is causal $\longrightarrow$ $$h[n] = 2\delta[n]-9\left(\tfrac{1}{2}\right)^n\mu[n]+8\mu[n]$$

Exercises

- Solve the following as an exercise – the solutions are given (next slide) so that you can check your answer:

1. $X(z)=\dfrac{z}{2z^2-3z+1}$ (i) $\text{ROC}:|z|<1/2$ as well as (ii) $\text{ROC}:|z|>1$

2. $X(z)=\dfrac{z}{z(z-1)(z-2)^2}$ $\text{ROC}:|z|>2$

3. $X(z)=\dfrac{2z^3-5z^2+z+3}{(z-1)(z-2)}$ $\text{ROC}:|z|<1$

4. $X(z)=\dfrac{2+z^{-2}+3z^{-4}}{z^2+4z+3}$ $\text{ROC}:|z|>0$

Answers

1. (i) $x[n] = -u[-n-1]+(0.5)^n u[-n-1]$; (ii) $x[n]=u[n]-(0.5)^n u[n]$
2. $x[n] = (1-2^n+n*2^{n-1})u[n]$
3. $x[n] = 2\delta[n+1]+1.5\delta[n]+u[-n-1]-(0.5)(2)^n u[-n-1]$
4. $x[n] = 1[(-1)^{n-1}-(-3)^{n-1}]u[n-1]$
$+0.5[(-1)^{n-3}-(-3)^{n-3}]u[n-3]$
$+1.5[(-1)^{n-5}-(-3)^{n-5}]u[n-5]$

In Matlab

`[r,p,k]= residuez(num,den)` develops the partial-fraction expansion of a rational z-transform with numerator and denominator coefficients given by vectors ***num*** and ***den.*** Vector ***r*** contains the residues, vector ***p*** contains the poles, vector ***k*** contains the direct term constants (the coefficients of z-terms if the ratio is made proper)

`[num,den]=residuez(r,p,k)` converts a z-transform expressed in a partial-fraction expansion form to its rational form.

See the matlab help files on **`residuez`** and **`residue`** for more details!

➲ In **`residuez`** function, you do not need to divide by "z"! The coefficient normally used for 1/z will be in "k" variable. You can also use the continuous equivalent of this function, **`residue`**, for which you **do** need to divide the original function by "z" before you obtain the coefficients to get the correct results!

EXAMPLE IN MATLAB

- Consider the first question in the exercise, and write it in terms of z^{-1}:

$$X(z) = \frac{z}{2z^2 - 3z + 1} = \frac{z}{z^2\left(2 - 3z^{-1} + z^{-2}\right)} = \frac{z^{-1}}{2 - 3z^{-1} + z^{-2}}$$

  - Then, b=[0 1], a=[2 -3 1];
  - **[r p k]=residuez (b,a);** returns
    - ➔ r=[1, -1], p=[1 ½], k=[]

$$X(z) = \frac{1}{1-(1)z^{-1}} + \frac{-1}{1-(0.5)z^{-1}}$$

$$\Rightarrow x[n] = -u[-n-1] - (-1)(0.5)^n u[-n-1], \quad |z| < \frac{1}{2}$$