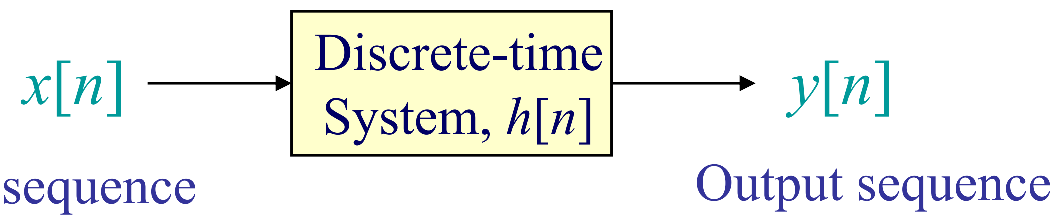
Discrete Convolution

- **Discrete Convolution:** The operation by far the most commonly used in DSP, but also most commonly misused, abused and confused by uninitiated (=students).

- At the heart of any DSP system:



$$
\begin{aligned}
y[n] &= x[n] * h[n] \\
&= \sum_{m=-\infty}^{\infty} x[m] \cdot h[n-m] \\
&= \sum_{m=-\infty}^{\infty} h[m] \cdot x[n-m]
\end{aligned}
$$

$h[n]$: Impulse response of the system

Discrete Convolution

- The "*n*" dependency of *y*[*n*] deserves some care: for each value of "*n*" the convolution sum must be computed ***separately*** over all values of a dummy variable "*m*". So, for each "*n*"

1. Rename the independent variable as ***m***. You now have ***x***[***m***] and ***h***[***m***]. Flip ***h***[***m***] over the origin. This is ***h***[***-m***]
2. Shift ***h***[***-m***] as far left as possible to a point "***n***", where the two signals barely touch. This is ***h***[***n-m***]
3. Multiply the two signals and sum over all values of ***m***. This is the convolution sum for the specific "***n***" picked above.
4. Shift / move ***h***[***-m***] to the right by one sample, and obtain a new ***h***[***n-m***]. Multiply and sum over all ***m***.
5. Repeat 2~4 until ***h***[***n-m***] no longer overlaps with ***x***[***m***], i.e., shifted out of the ***x***[***m***] zone.

$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} x[m] \cdot h[n-m] = \sum_{m=-\infty}^{\infty} h[m] \cdot x[n-m]$$

Useful Expressions

The following expressions are often useful in calculating convolutions of analytical discrete signals

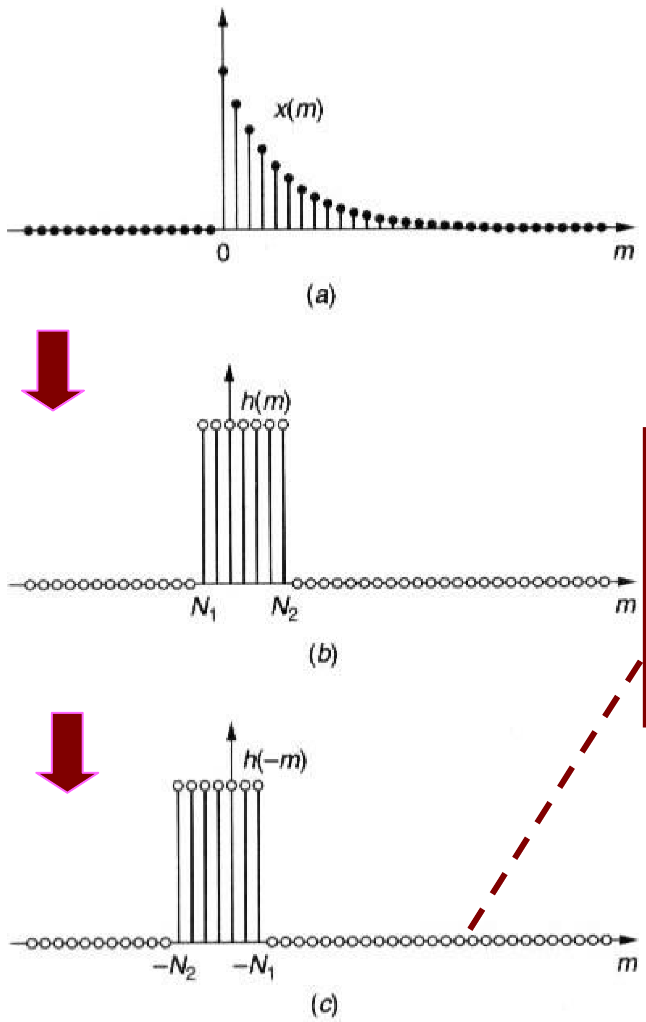
$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, \quad |a| < 1$$

$$\sum_{n=k}^{\infty} a^n = \frac{a^k}{1-a}, \quad |a| < 1$$

$$\sum_{n=m}^{N} a^n = \frac{a^m - a^{N+1}}{1-a}, \quad a \neq 1$$

$$\sum_{n=0}^{N-1} a^n = \begin{cases} \dfrac{1-a^N}{1-a}, & |a| \neq 1 \\ N, & a = 1 \end{cases}$$

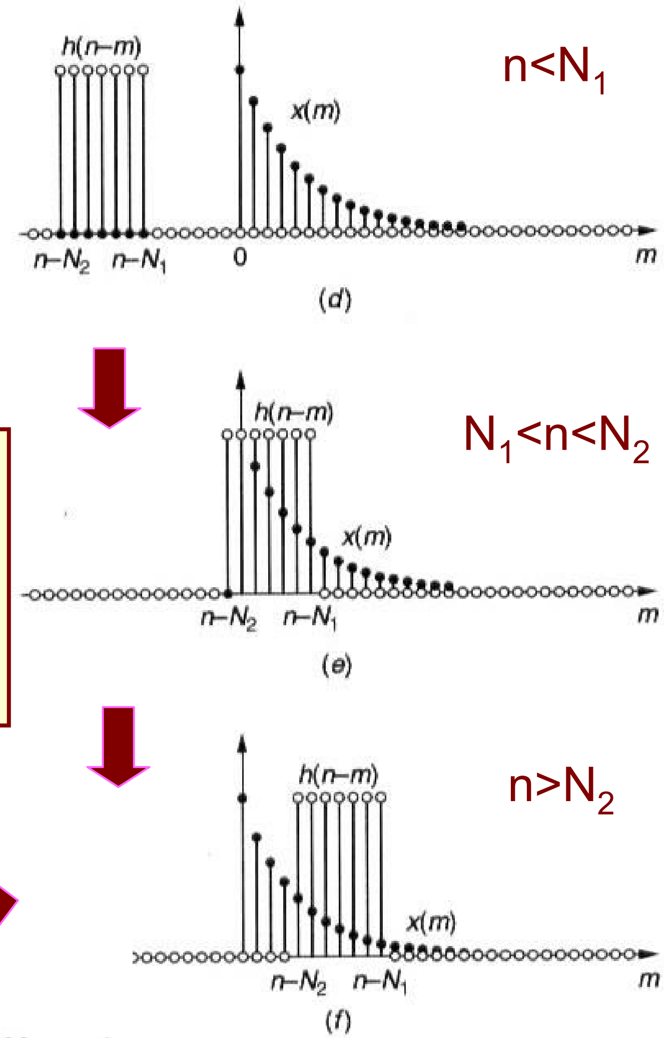
CONVOLUTION EXAMPLE



$$x[n] = a^n u[n]$$

$$h[n] = \begin{cases} 1 & N_1 \le n \le N_2 \\ 0 & otherwise \end{cases}$$

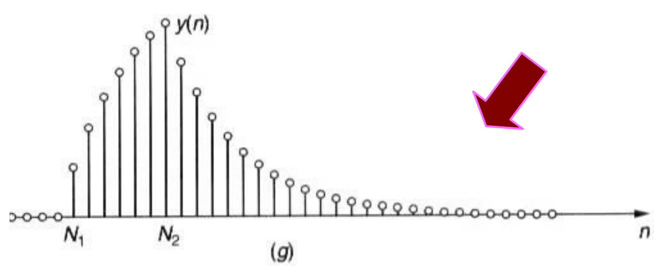
$$y[n] = \begin{cases} 0 & n < N_1 \\ \dfrac{1-a^{n-N_1+1}}{1-a} & N_1 \le n < N_2 \\ a^{n-N_2}\,\dfrac{1-a^{N_2-N_1+1}}{1-a} & n \ge N_2 \end{cases}$$



$n<N_1$

$N_1<n<N_2$

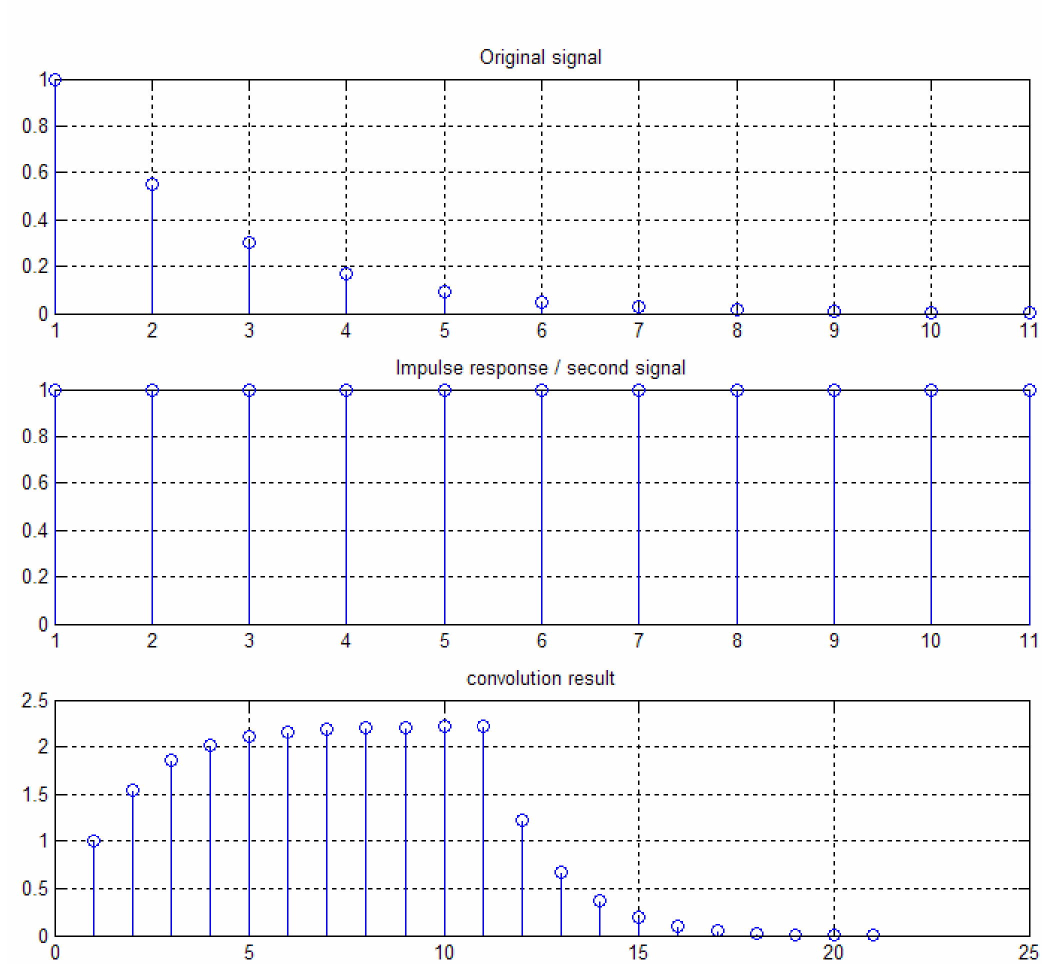
$n>N_2$



In Matlab

- Matlab has the built-in convolution function, **conv(.)**
- Be careful, however in setting the time axis

```
n=-3:7;
x=0.55.^(n+3);
h=[1 1 1 1 1 1 1 1 1 1 1];
y=conv(x,h);
subplot(311)
stem(x)
title('Original signal')
subplot(312)
stem(h)  % Use stem for discrete sequences
title('Impulse response / second signal')
subplot(313)
stem(y)
title(' convolution result')
```



Correlation

An efficient way to compare signals with each other and search for similarities

The mathematical formulation of correlation is the cross correlation sequence:

$$r_{xy}[l] = \sum_{n=-\infty}^{\infty} x[n]y[n-l]$$

(note the difference with convolution)

and for the time reversed cross correlation sequence it can easily be shown that it is related to the original sequence as:

$$r_{yx}[l] = \sum_{n=-\infty}^{\infty} y[n]x[n-l] = \sum_{m=-\infty}^{\infty} y[m+l]x[m] = r_{xy}[-l]$$

Correlation

autocorrelation is a cross correlation sequence of a series with itself:

$$r_{xx}[l] = \sum_{n=-\infty}^{\infty} x[n]x[n-l]$$

note that the zero lag term of the autocorrelation sequence is just the total enery of the signal:

$$r_{xx}[0] = \sum_{n=-\infty}^{\infty} x^2[n] = \varepsilon_x$$

and that an autocorrelation sequence is even for real *x*[*n*] since:

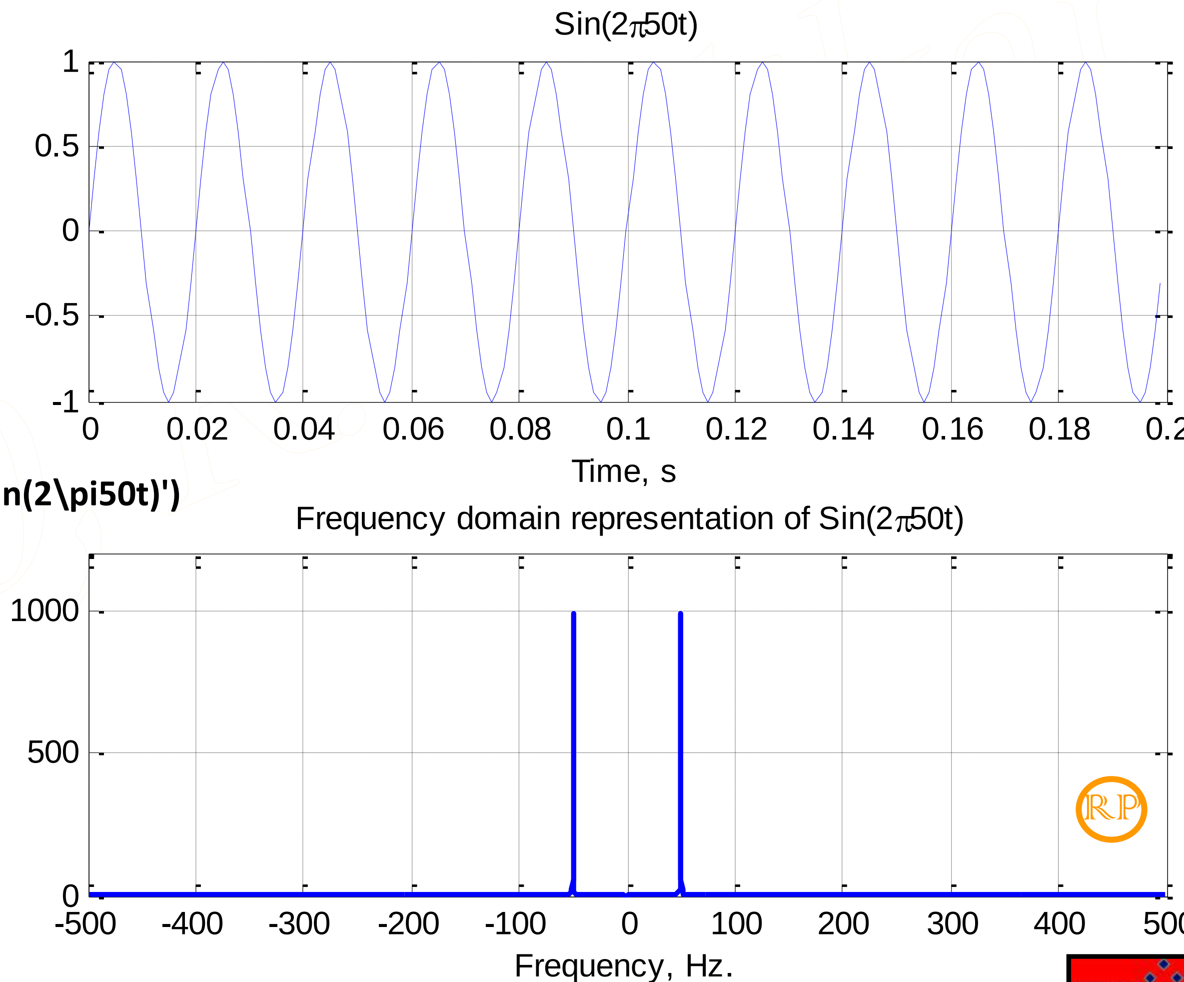
$$r_{xx}[l] = r_{xx}[-l]$$

The Frequency Domain

- Time domain operation are often not very informative and/or efficient in signal processing
- An alternative representation and characterization of signals and systems can be made in transform domain
  - Much more can be said, much more information can be extracted from a signal in the transform / frequency domain.
  - Many operations that are complicated in time domain become rather simple algebraic expressions in transform domain
  - Most signal processing algorithms and operations become more intuitive in frequency domain, once the basic concepts of the frequency domain are understood.

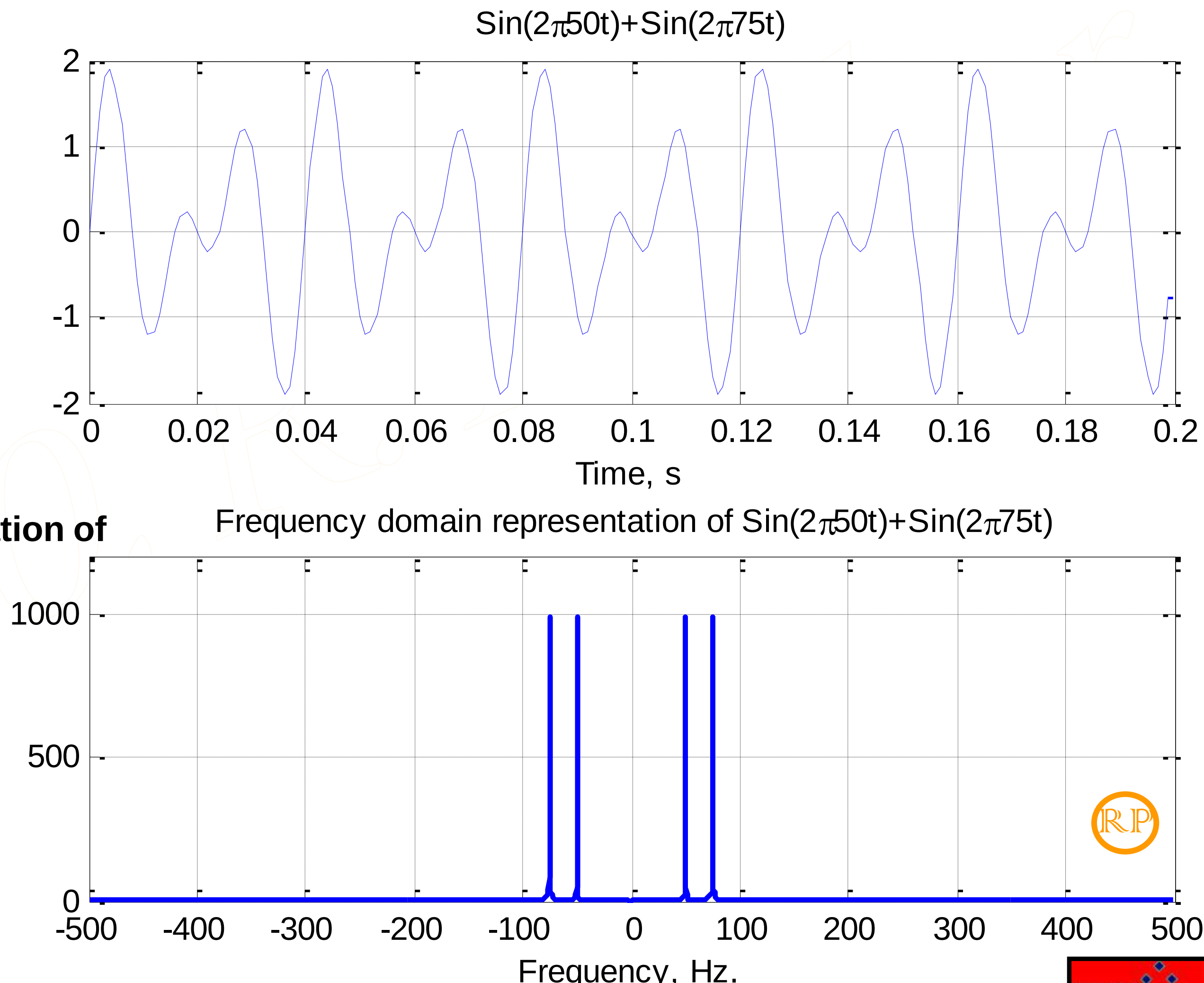
Frequency Domain An Example

```
t=-1:0.001:1;
x=sin(2*pi*50*t);
subplot(211)
plot(t(1001:1200),x(1:200))
grid
title('Sin(2\pi50t)')
xlabel('Time, s')
subplot(212)
X=abs(fft(x));
X2=fftshift(X);
f=-499.9:1000/2001:500;
plot(f,X2);
grid
title(' Frequency domain representation of Sin(2\pi50t)')
xlabel('Frequency, Hz.')
```



Frequency Domain Another Example

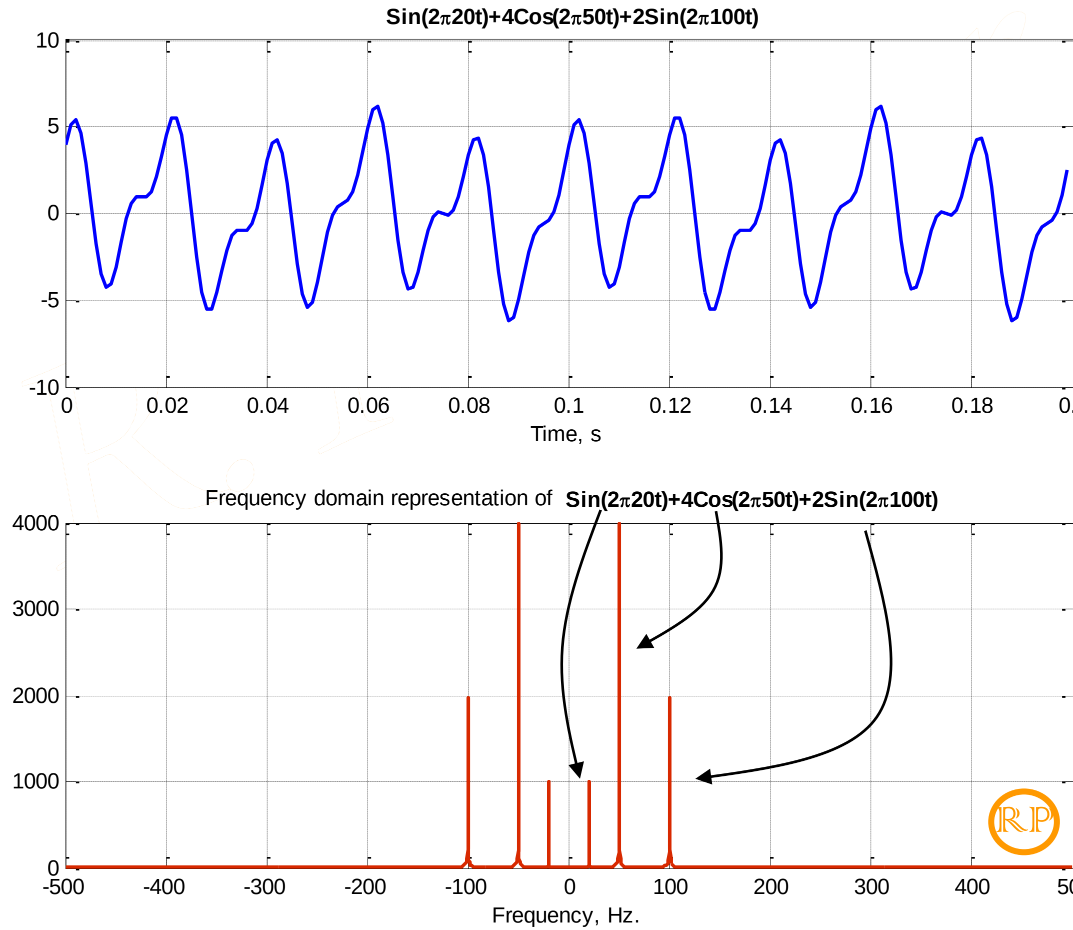
```
t=-1:0.001:1;
x=sin(2*pi*50*t)+sin(2*pi*75*t);
subplot(211)
plot(t(1001:1200),x(1:200))
grid
title('Sin(2\pi50t)+Sin(2\pi75t)')
xlabel('Time, s')
subplot(212)
X=abs(fft(x));
X2=fftshift(X);
f=-499.9:1000/2001:500;
plot(f,X2);
grid
title(' Frequency domain representation of Sin(2\pi50t)+Sin(2\pi75t)')
xlabel('Frequency, Hz.')
```



Frequency Domain And…Another Example

Freq_domain_example.m

```
t=-1:0.001:1;
x=sin(2*pi*20*t)+4*cos(2*pi*50*t)+2*sin(2*pi*100*t);
subplot(211)
plot(t(1001:1200),x(1:200))
grid
title('Sin(2\pi20t)+4Cos(2\pi50t)+2Sin(2\pi100t)')
xlabel('Time, s')
subplot(212)
X=abs(fft(x));
X2=fftshift(X);
f=-499.9:1000/2001:500;
plot(f,X2);
grid
title(' Frequency domain representation of Sin(2\pi20t)+4Cos(2\pi50t)+2Sin(2\pi100t)')
xlabel('Frequency, Hz.')
```



Magnitude spectrum

THE FOURIER TRANSFORM

The frequency domain representation of a time domain signal can be obtained through Fourier transform.

Spectrum: **A compact representation of the frequency content of a signal that is composed of sinusoids**

Fourier Who…?

Jean B. Joseph Fourier
(1768-1830)

*"An **arbitrary** function, continuous or with discontinuities, defined in a finite interval by an arbitrarily capricious graph can always be expressed as a sum of sinusoids"*

J.B.J. Fourier

December 21, 1807

$$F[k] = \int f(t) e^{-j2\pi kt/N} dt \qquad f(t) = \frac{1}{2\pi} \sum_{i=0}^{N-1} F[k] e^{j2\pi kt/N}$$

Jean B. J. Fourier

- He announced his discovery in a prize paper on the theory of heat (1807).
  - The judges: Laplace, Lagrange, Poisson and Legendre
- Three of the judges found it incredible that ***sum of sines and cosines could add up to anything but an infinitely differential function***, but...
  - Lagrange: Lack of mathematical rigor and generality ➔Denied publication….
    - Became famous with his other previous work on math, assigned as chair of Ecole Polytechnique
    - Napoleon took him along to conquer Egypt
    - Return back after several years
    - Barely escaped Giyotin!
  - After 15 years, following several attempts and disappointments and frustration, he published his results in **Theorie Analytique de la Chaleur** in 1822 (Analytical Theory of Heat).
  - In 1829, Dirichlet proved Fourier's claim with very few and non-restricting conditions.
  - Next 150 years: His ideas expanded and generalized. 1965: Cooley and Tukey--> Fast Fourier Transform ➔ Computational simplicity ➔ King of all transforms… Countless number of applications engineering, finance, applied mathematics, etc.

Fourier Transforms

- Fourier Series (FS)
  - Fourier's original work: A periodic function can be represented as a finite, weighted sum of sinusoids that are integer multiples of the fundamental frequency Ω_0 of the signal. ➔ These frequencies are said to be harmonically related, or simply ***harmonics***.
- (Continuous) Fourier Transform (FT)
  - Extension of Fourier series to non-periodic functions: Any continuous aperiodic function can be represented as an infinite sum (integral) of sinusoids. The sinusoids are no longer integer multiples of a specific frequency anymore.
- Discrete Time Fourier Transform (DTFT)
  - Extension of FT to discrete sequences. Any discrete function can also be represented as an infinite sum (integral) of sinusoids.
- Discrete Fourier Transform (DFT)
  - Because DTFT is defined as an infinite sum, the frequency representation is not discrete (but continuous). An extension to DTFT is DFT, where the frequency variable is also discretized.
- Fast Fourier Transform (FFT)
  - Mathematically identical to DFT, however a significantly more efficient implementation. FFT is what signal processing made possible today!

Dirichlet Conditions (1829)

- Before we dive into Fourier transforms, it is important to understand for which type of functions, Fourier transform can be calculated.
- Dirichlet put the final period to the discussion on the feasibility of Fourier transform by proving the necessary conditions for the existence of Fourier representations of signals
  - The signal must have finite number of discontinuities
  - The signal must have finite number of extremum points within its period
  - The signal must be absolutely integrable within its period

$$\int_{t_0}^{t_0+T} |x(t)|\, dt < \infty$$

- How restrictive are these conditions…?

Fourier Series

- Any periodic signal *x(t)* whose fundamental period is T_0 (hence, fundamental frequency $f_0=1/T_0$, $\Omega_0=2\pi f_0$), can be represented as a finite sum of complex exponentials (sines and cosines)
  - That is, a signal however arbitrary and complicated it may be, can be represented as a sum of simple building blocks

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\Omega_0 kt}$$

  - Note that each complex exponential that makes up the sum is an integer multiple of Ω_0, the fundamental frequency
  - Hence, the complex exponentials are ***harmonically related***
  - The coefficients c_k , aka Fourier (series) coefficients, are possibly complex
    - **Fourier series (and all other types of Fourier transforms) are complex valued !** **That is, there is a magnitude and phase (angle) term to the Fourier transform!**

Fourier Series

- This is the ***synthesis equation***: x(t) is synthesized from its building blocks, the complex exponentials at integer multiples of Ω_0

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\Omega_0 t}$$

$k\Omega_0$: "k^th" integer multiple – ***k^th harmonic*** of the ***fundamental frequency Ω_0***

c_k: Fourier coefficients – how much of k^{th} harmonic exists in the signal

$|c_k|$: Magnitude of the k^{th} harmonic ➔ magnitude spectrum of $x(t)$

$< c_k$: Phase of the k^{th} harmonic ➔ phase spectrum of $x(t)$

} **Spectrum of *x*(*t*)**

- How to compute the Fourier coefficients, c_k?

Fourier Series

- The coefficients c_k can be obtained through the ***analysis equation***.

$$c_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) e^{-jk\Omega_0 t} dt$$

The limits of the integral can be chosen to cover any interval of T_0, for example, $[-T_0/2 \; T_0/2]$ or $[0 \; T_0]$ or $[-T_0 \; 0]$.

- Note that, while $x(t)$ is a sum, c_k are obtained through an integral of complex values.
  - Why / how…?
- More importantly, if $x(t)$ is real, then the coefficients satisfy $c_{-k}=c^*_k$, that is $|c_{-k}|=|c_k|$ ➔ why?

Trigonometric Fourier Series

- Using the Euler's rule, we can represent the complex Fourier series in two trigonometric forms:

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\Omega_0 t) + b_k \sin(k\Omega_0 t) \right)$$

$$a_k = \frac{2}{T_0} \int_{T_0} x(t) \cos(k\Omega_0 t) dt$$

$$b_k = \frac{2}{T_0} \int_{T_0} x(t) \sin(k\Omega_0 t) dt$$

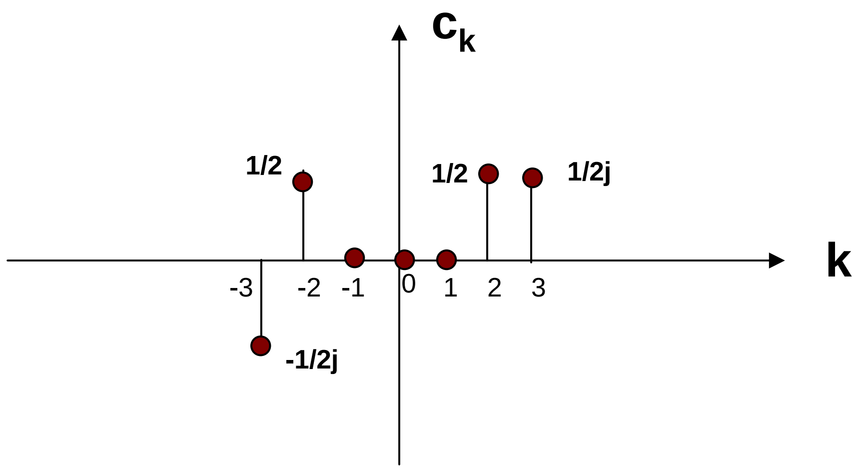
- As you might have already guessed the trigonometric Fourier coefficients, $\boldsymbol{a_k}$ and $\boldsymbol{b_k}$, are not independent of the complex Fourier coefficients

$$a_0 = 2c_0 \qquad a_k = c_k + c_{-k} \qquad b_k = j(c_k - c_{-k})$$

$$c_k = \frac{a_k - jb_k}{2}, \qquad c_{-k} = \frac{a_k + jb_k}{2}$$

Quick Facts and An Example

- Fourier series are computed for **periodic** signals (continuous or discrete). A periodic signal has finite number of ***discrete*** spectral components.
  - Each spectral component is represented by c_k and c_{-k} Fourier series coefficients, k=1,2,…,N.
  - Each ***k*** represents one of the spectral components at integer multiples of Ω_0, the fundamental frequency of the signal. These ***discrete spectral components*** at Ω_0, $2\Omega_0$,…, $N\Omega_0$ are called ***harmonics***.
    - For example, the signal x(t)=cos4t+sin6t has two (four, if you count the negative frequencies) spectral components. The fundamental frequency is $\Omega_0=2$, and $c_{-3}=-1/2j$, $c_3=1/2j$, $c_{-2}=c_2=1/2$

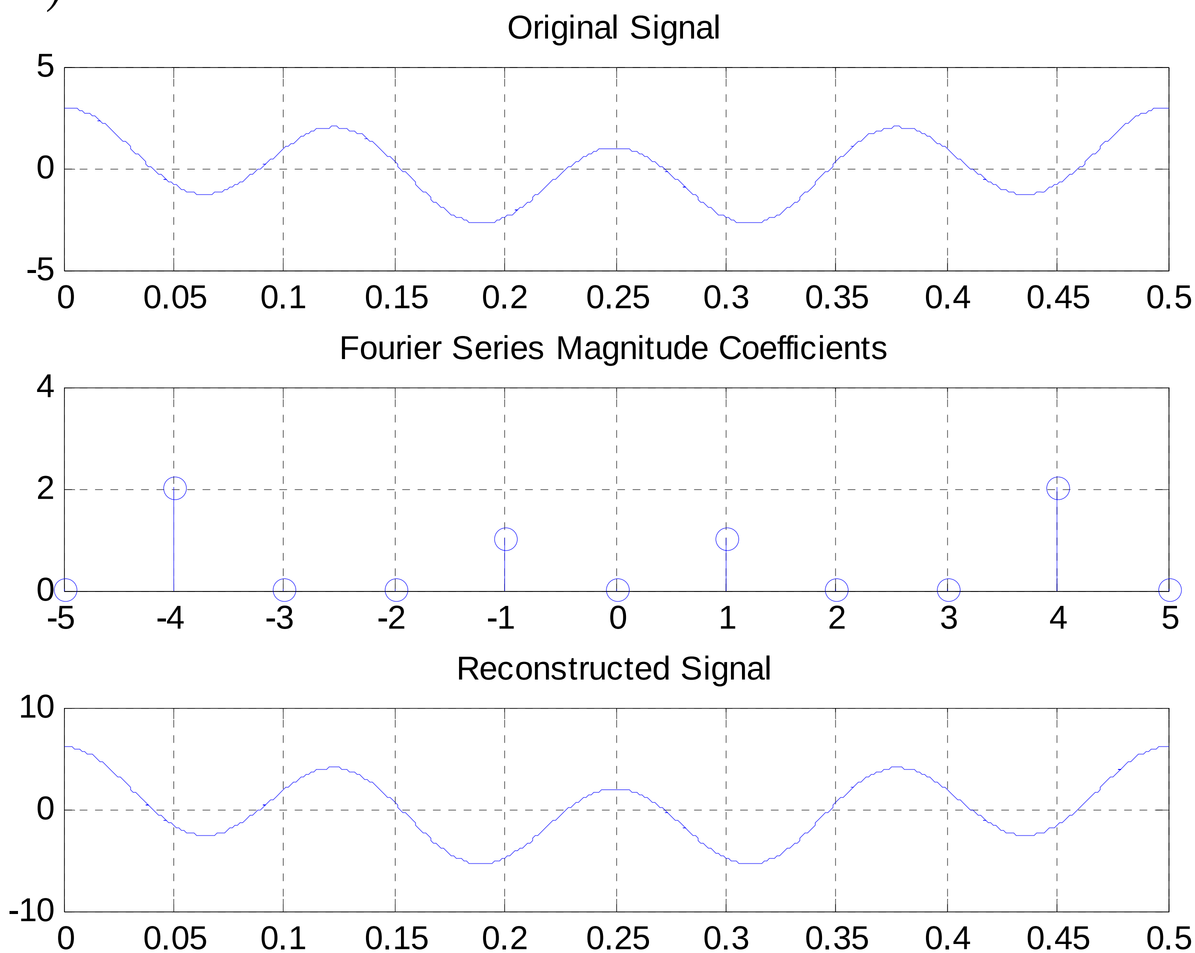


k=0→ Ω=0 rad/s
k=1→ Ω=2 rad/s
k=2→ Ω=4 rad/s
k=3→ Ω=6 rad/s

Another Example

$$x(t) = \cos(4\pi t) + 2\cos(16\pi t)$$

t=0:0.001:0.5;
x=cos(2*pi*2*t)+2*cos(2*pi*8*t);
w0=2*pi*2;
K=5;
[X x_recon]=fourier_series(x,K,t,w0);



p.s. fourier_series() is not a standard Matlab function. You will be writing this function as part of next lab.

Quick Facts About Fourier Series

- If a signal is even, then all b_k=0, and if a signal is odd, then all a_k=0
- If $x(t)$ is real, then the Fourier series are symmetric, that is, $c_k = c_{-k}$
  - This is inline with our interpretation of frequency as two phasors rotating at the same rate but opposite directions.
  - For real signals, the relationship between complex and trigonometric representation of Fourier series further simplifies to $a_k = 2\Re[c_k] \quad b_k = -2\,\mathrm{Im}[c_k]$
  - We also have a third representation for real signals:

$$x(t) = C_0 + \sum_{k=1}^{\infty} C_k \cos(k\Omega_0 t - \theta_k)$$

$$C_0 = \frac{a_0}{2}, \quad C_k = \sqrt{a_k^2 + b_k^2}, \quad \theta_k = \tan^{-1}\frac{b_k}{a_k}$$

k^{th} harmonic

Phase angles

DC component

Harmonic amplitudes

SUMMARY

- We can often tell much more about a signal by looking at its frequency content, that is, its spectrum.
- For continuous time signals, that are also periodic with a fundamental frequency of Ω_0, Fourier series gives us the spectrum of the signal
  - The Fourier series of such a signal is a series of impulses at integer multiples of Ω_0. These impulses in the frequency domain represent the harmonics of the signal
  - Remember: the term $e^{j\Omega_o}$ represents one spectral component at frequency Ω_0
  - $\text{Cos}(\Omega_0 t)$ has two such complex exponentials in it, at $\pm\Omega_0$. Therefore, each cosine at a particular frequency Ω_0 consists of two spectral components, one at each of $\pm\Omega_0$.

$$Cos\left(\Omega_0 t\right) = \frac{e^{j\Omega_o t} + e^{-j\Omega_o t}}{2}$$

$$Sin\left(\Omega_0 t\right) = \frac{e^{j\Omega_o t} - e^{-j\Omega_o t}}{2j}$$

The Discrete Fourier Transform

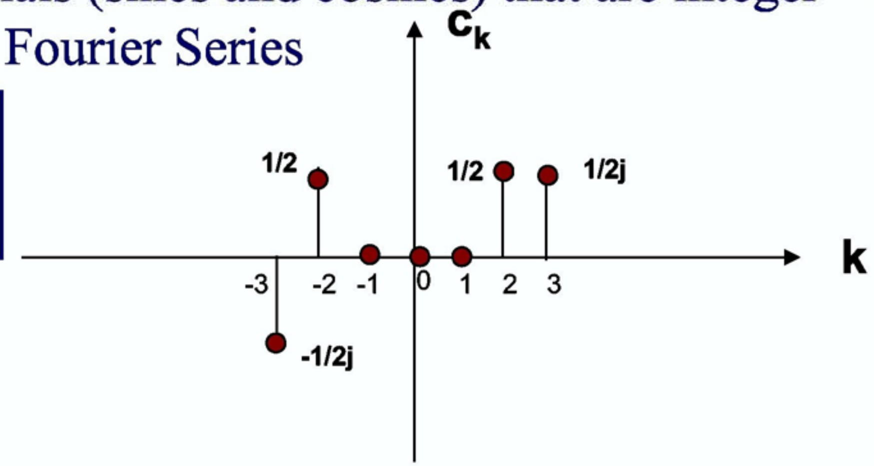
- The Discrete Time Fourier Transform
  - Definition
  - Key theorems
    - DTFT of the output of an LTI system
    - The frequency response
    - Periodicity of DTFT
    - Definition of discrete frequency
    - Existence of DTFT
  - DTFTs of some important sequences
  - DTFT properties

Fourier Series and Fourier Transform

- Any periodic signal *x(t)* whose fundamental period is T_0, can be represented as a finite and discrete sum of complex exponentials (sines and cosines) that are integer multiples of Ω_0, the fundamental frequency: Fourier Series

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\Omega_0 kt}$$

$$c_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) e^{-jk\Omega_0 t} dt$$



- A non-periodic continuous time signal can also be represented as an (infinite and continuous) sum of complex exponentials: Fourier Transform

$$X(\Omega) = \Im\left(x(t)\right) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$

$$\overset{\Im}{\Leftrightarrow}$$

$$x(t) = \Im^{-1}\left(X(\Omega)\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega) e^{j\Omega t} d\Omega$$

$$x(t) \overset{\Im}{\Leftrightarrow} x(\Omega)$$

$$X(\Omega) = \left|X(\Omega)\right| \angle X(\Omega) = \left|X(\Omega)\right| e^{j\phi(\Omega)}$$

KEY FACTS TO REMEMBER

- All FT pairs provide a transformation between time and frequency domains: The frequency domain representation provides how much of which frequencies exist in the signal ➔ More specifically, how much $e^{j\Omega t}$ exists in the signal for each Ω.
- In general, the frequency representation is complex (except when the signal is even).
  - $|X(\Omega)|$: The magnitude spectrum ➔ the power of each Ω component
  - Ang $X(\Omega)$: The phase spectrum ➔ the amount of phase delay for each Ω component
- The ***FS is discrete in frequency*** domain, since it is the discrete set of exponentials – integer multiples of Ω_0 – that make up the signal. This is because only a finite number of frequencies are required to construct a periodic signal.
- The ***FT is continuous in frequency*** domain, since exponentials of a continuum of frequencies are required to reconstruct a non-periodic signal.
- Both transforms are ***non-periodic*** in frequency domain.

Discrete –Time Fourier Transform (DTFT)

- Similar to continuous time signals, discrete time sequences can also be periodic or non-periodic, resulting in discrete-time Fourier series or discrete – time Fourier transform, respectively.
- Most signals in engineering applications are non-periodic, so we will concentrate on DTFT.
- We will represent the discrete frequency as ω, measured in radians/sample.

$$x[n] \overset{\mathfrak{I}}{\Leftrightarrow} X(\omega)$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

and

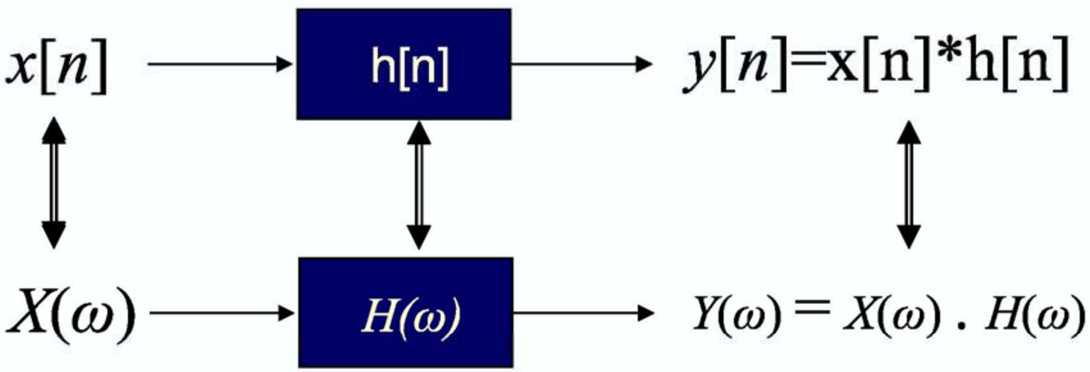
$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} \, d\omega$$

Quick facts:

- Since x[n] is discrete, we can only add them, hence summation
- The sum of x[n], weighted with continuous exponentials, is continuous ➔ the DTFT X(ω) is continuous (non-discrete)
- Since X(ω) is continuous, x[n] is obtained as a continuous integral of X(ω), weighed by the same complex exponentials.
- x[n] is obtained as an integral of X(ω), where the integral is over an interval of **2π.** ➔ This is our first clue that DTFT is periodic with **2π** in frequency domain.
- X(ω) is sometimes denoted as $X(e^{j\omega})$ in some books, including yours. While $X(e^{j\omega})$ is more accurate, we will use X(ω) for brevity.

Important Theorems

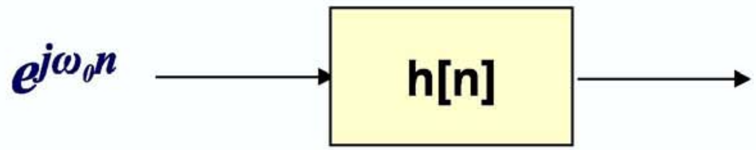
- There are several important theorems related to DTFT.
- Theorem 1:
  - If x[n] is input to an LTI system with an impulse response of h[n], then the DTFT of the output is the product of X(ω) and H(ω)



Frequency Response

➲ Theorem 2:

- If the input to an LTI system with an impulse response of h[n] is a complex exponential $e^{j\omega_0 n}$, then the output is the ***SAME*** complex exponential whose magnitude and phase are given by $|H(\omega)|$ and $<H(\omega)$, evaluated at $\omega = \omega_0$.



$$y[n] = \sum_{k=-\infty}^{\infty} h[k] e^{j\omega_0 (n-k)}$$

$$= \underbrace{\left(\sum_{k=-\infty}^{\infty} h[k] e^{j\omega_0 k} \right)}_{H(\omega_0)} e^{j\omega_0 n}$$

$$\boxed{y[n] = H(\omega_0) e^{j\omega_0 n}}$$

If the system input is a complex exponential at a specific frequency ω_0, then the system output is the same exponential, at the same frequency ω_0 but weighted by a complex amplitude that is a function of the input frequency. This complex amplitude, $H(\omega_0)$, is the DTFT of system impulse function h[n], evaluated at ω_0, and it is called the *frequency response* of the system.

Frequency Response

- This theorem constitutes the fundamental cornerstone for the concept of ***frequency response***. $H(\omega)$, the DTFT of h[n], is called the frequency response of the system

- Why is it important?
  - If a sinusoidal sequence with frequency ω_0 is applied to a system whose frequency response is $H(\omega)$, then the output can be obtained simply by evaluating $H(\omega)$ at $\omega = \omega_0$.
  - Since all signals can be written as a superposition of sinusoids at different frequencies, then the output to an arbitrary input can be obtained as the superposition of $H(\omega_0)$ for each component that makes up the input signal!

PERIODICITY OF DTFT

- Theorem 3:
  - The DTFT of a discrete sequence is periodic with the period 2π, that is

$$X(\omega) = X(\omega + 2\pi k) \quad \text{for any integer } k$$

- The periodicity of DTFT can be easily verified from the definition:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

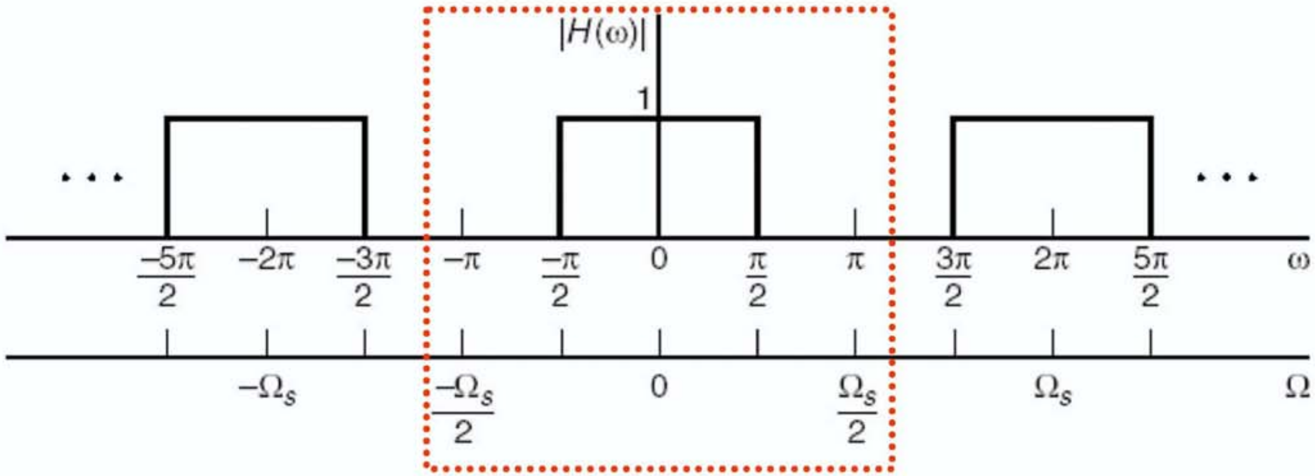
Why…

$$X(\omega + 2\pi k) = \sum_{n=-\infty}^{\infty} x[n] e^{-j(\omega + 2\pi k)n} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} e^{-j2\pi k n}$$

$$= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = X(\omega) \quad \forall k$$

Implications of the Periodicity Property

$$H(\omega) = H(\omega + 2\pi)$$



➪ ***Theorem 4 (You-will-flunk-if-you-do-not-understand-this-fact*** **theorem):**

The discrete frequency 2π rad. corresponds to the sampling frequency Ω_s used to sample the original continuous signal x(t) to obtain x[n].

➪Proof: $x(t) = A\sin(\Omega t - \theta) \Rightarrow x(nT_s) = A\sin(\Omega T_s n - \theta)$

➪ $\omega = \Omega T_s$ ➔ For $\Omega = \Omega_s$, we have $\omega = \Omega_s T_s = 2\pi f_s T_s = 2\pi$

EXISTENCE OF DTFT

➲ Theorem 5:

- The DTFT of a sequence exists if and only if, the sequence x[n] is absolutely summable, that is, if

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

because:

This quantity is always ≤ 1

$$|X(\omega)| = \left| \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \right| \le \sum_{n=-\infty}^{\infty} |x[n]| \cdot \left| e^{-j\omega n} \right| \le \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

- Hence, if x[n] is absolutely summable, then $|X(\omega)|$ is finite, which means that $X(\omega)$ exists.
- We should add that this is sufficient, but not required to have a DTFT. Certain sequences that do not satisfy this requirement also have DTFTs. These will be discussed later within z-transform.

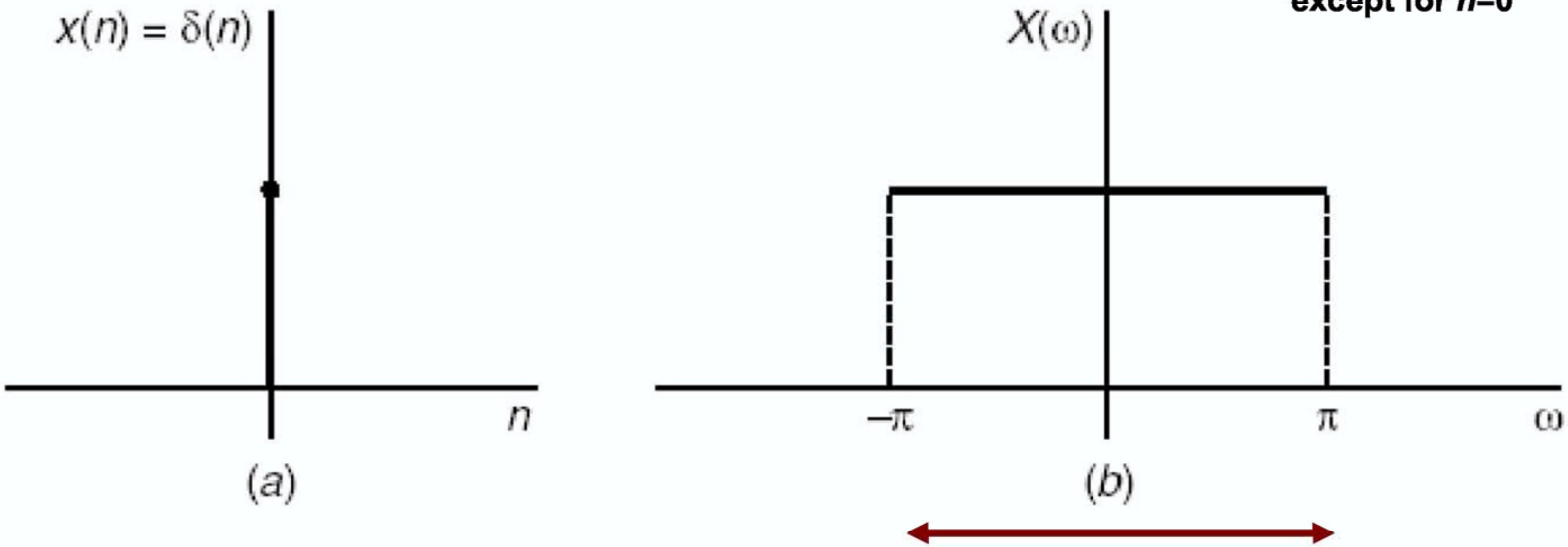
Important DTFT Pairs
Impulse Function

- The DTFT of the impulse function is "1" over the entire frequency band.

$$\Im\{\delta[n]\} = 1$$

$$\Delta(\omega) = \sum_{n=-\infty}^{n=\infty} \delta[n] e^{-j\omega n} = 1$$

Summations terms are all zero, except for *n*=0

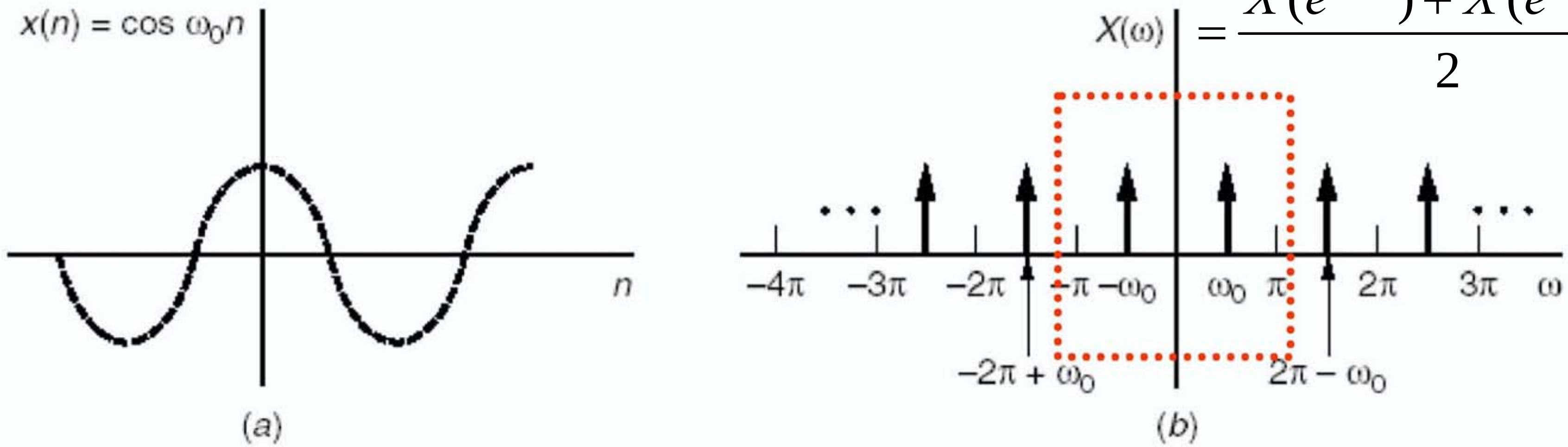


Extend of the frequency band
in discrete frequency domain

Important DTFT Pairs
The sinusoid at ω_0

➲ By far the most often used DTFT pair (it is less complicated then it looks):

$$x[n] = e^{j\omega_0 n} \quad ; \quad X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$$



$$= \frac{X(e^{j\omega_0 n}) + X(e^{-j\omega_0 n})}{2}$$

$$x[n] = e^{j\omega_0 n} \overset{\mathfrak{I}}{\Leftrightarrow} 2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_0 \pm 2\pi m)$$

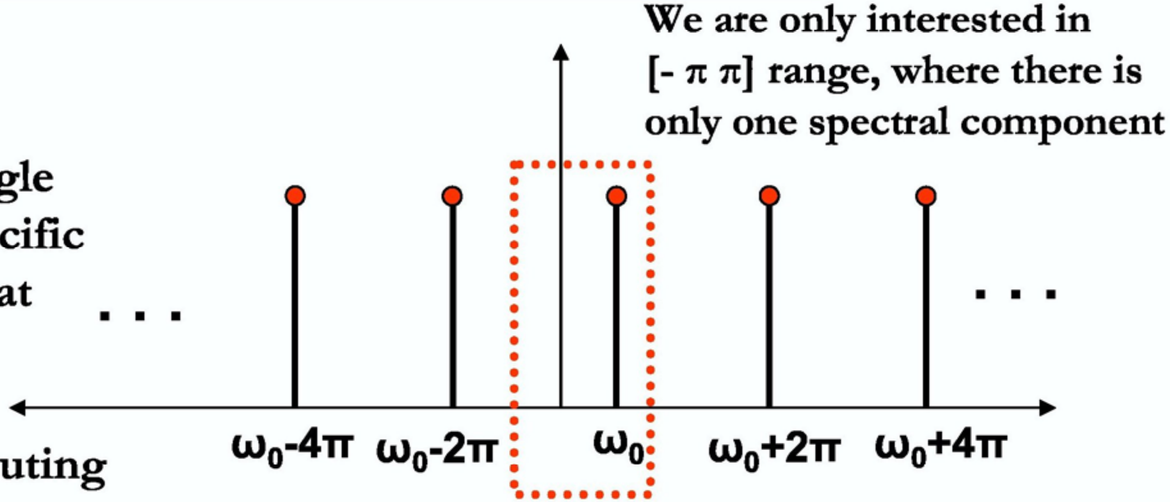
The above expression can also be obtained from the DTFT of the complex exponential through the Euler's formula.

Important DTFT Pairs
The Complex Exponential

- The DTFT of the complex exponential:

$$x[n] = \alpha^n \mu[n] \quad ((\alpha) < 1); X(\omega_0) = \frac{1}{1 - \alpha e^{-j\omega_0}}$$

Hence, the spectrum of a single complex exponential at a specific frequency is an impulse at that frequency.



This can be verified by computing the inverse DTFT of X(ω) given above, as in the previous example.

Important DTFT Pairs
Real Exponential

$$x[n] = \alpha^n u[n] \overset{\Im}{\Leftrightarrow} \frac{1}{1-\alpha e^{-j\omega}}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n] e^{-j\omega n} = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n}$$

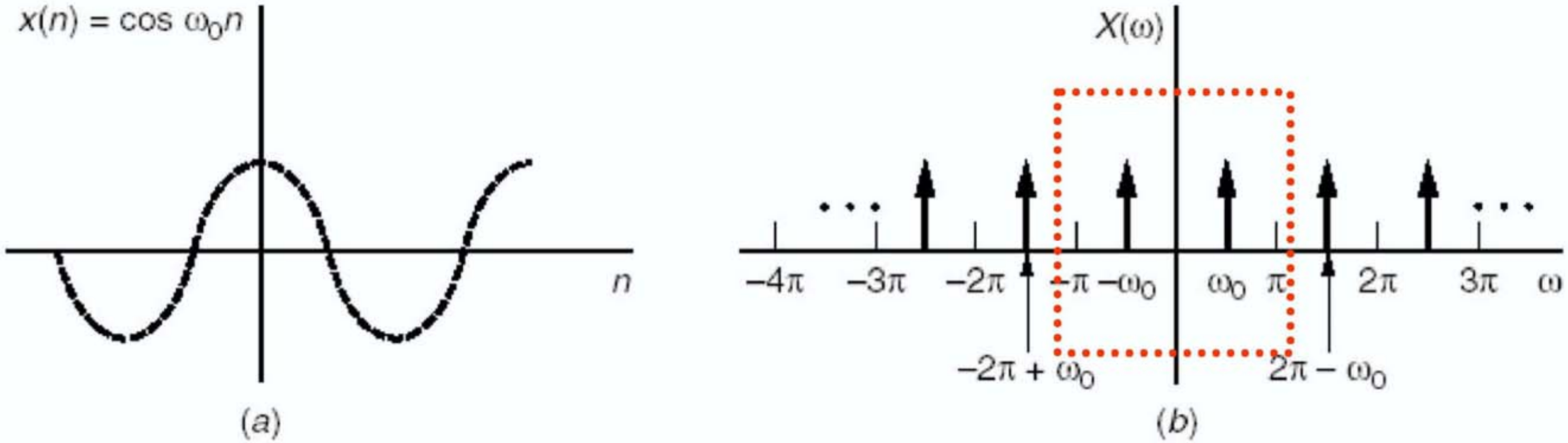
???

$$= \sum_{n=0}^{\infty} \left(\alpha e^{-j\omega}\right)^n = \frac{1}{1-\alpha e^{-j\omega}}$$

Important DTFT Pairs
The sinusoid at ω_0

➲ By far the most often used DTFT pair (it is less complicated then it looks):

$$x[n] = e^{j\omega_0 n} \quad ; \quad X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$$



$$x[n] = e^{j\omega_0 n} \overset{\mathfrak{I}}{\Leftrightarrow} 2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_0 \pm 2\pi m)$$

The above expression can also be obtained from the DTFT of the complex exponential through the Euler's formula.

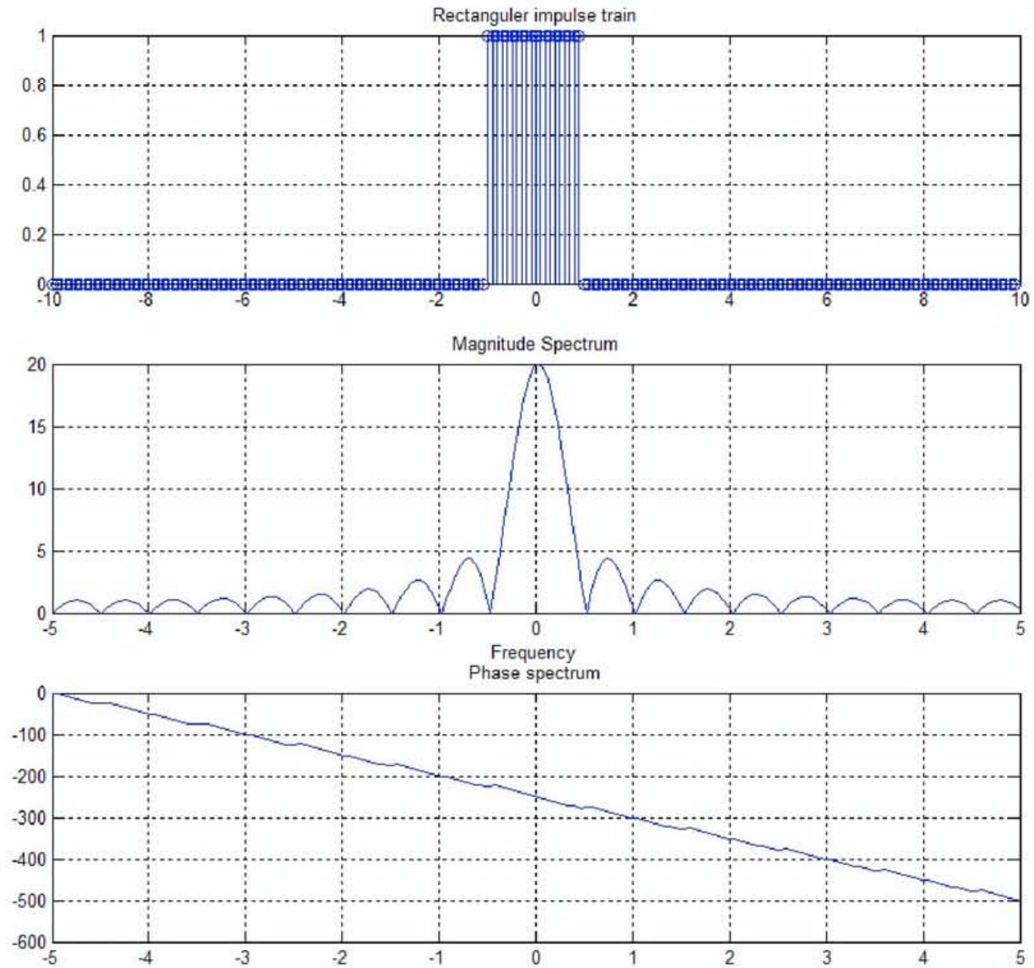
Important DTFT Pairs
Rectangular Pulse

- Also very commonly used in DSP, as it provides the FT of an ideal lowpass filter (we will see this later)

$$x[n] = \mathrm{rect}_M[n] = \begin{cases} 1, & -M \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

$$\overset{\mathfrak{I}}{\Leftrightarrow}$$

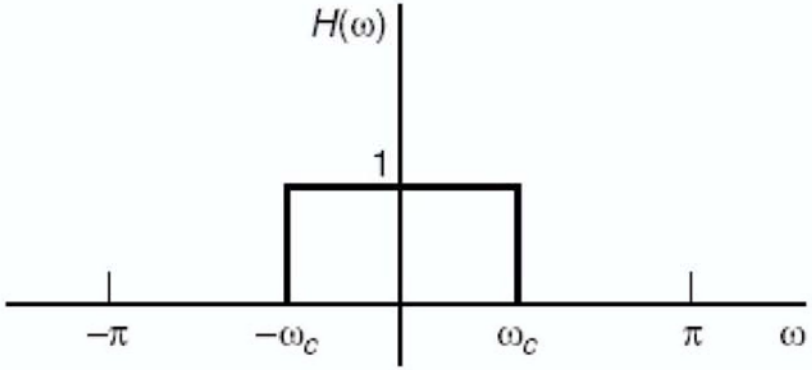
$$\sum_{n=-M}^{M} e^{-j\omega n} = \frac{\sin\left(M + 1/2\right)\omega}{\sin\left(\omega/2\right)}, \quad \omega \neq 0$$



Ideal Lowpass Filter

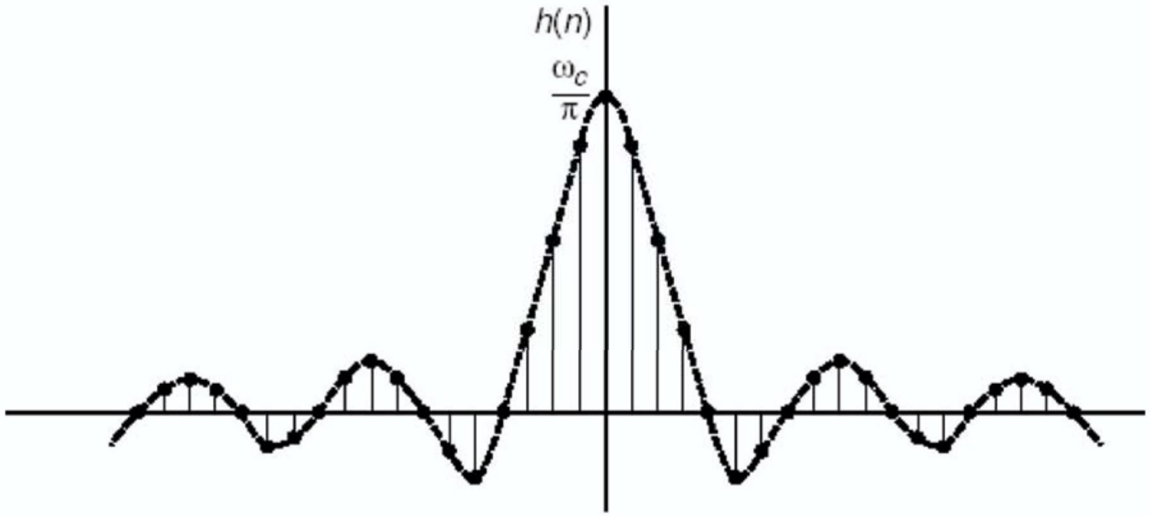
➲ The ideal lowpass filter is defined as

$$H(\omega)=\begin{cases}1, & |\omega|\le\omega_c\\ 0, & \omega_c\le\omega\le\pi\end{cases}$$



↳ Taking its inverse DTFT, we can obtain the corresponding impulse function h[n]:

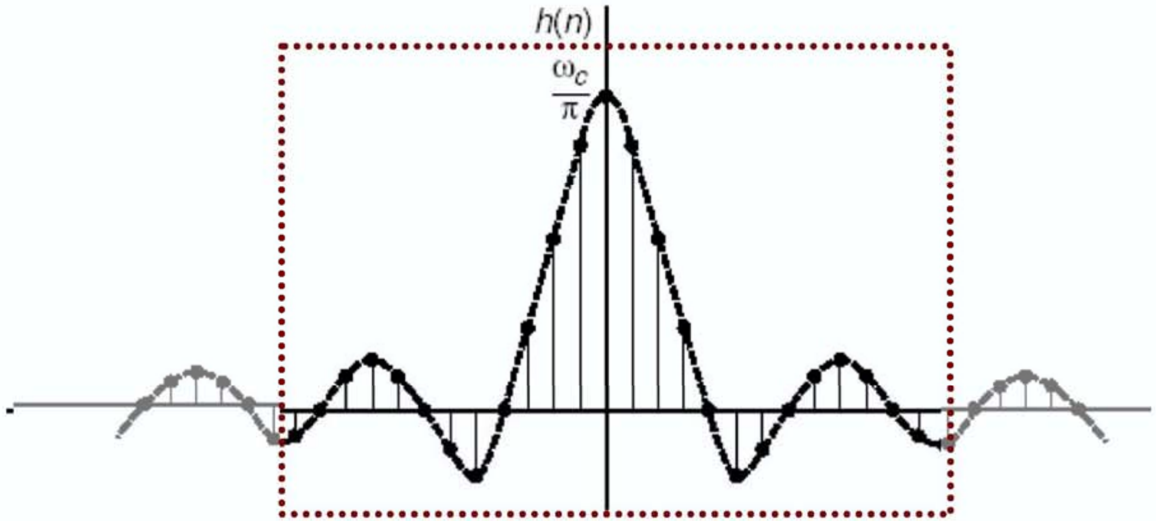
$$h[n]=\frac{\sin\omega_c n}{\pi n}$$



(b)

Ideal Lowpass Filter

- Note that:
  - The impulse response of an ideal LPF is infinitely long → This is an IIR filter. In fact h[n] is not absolutely summable → its DTFT cannot be computed → an ideal h[n] cannot be realized!
  - One possible solution is to truncate h[n], say with a window function, and then take its DTFT to obtain the frequency response of a realizable FIR filter.



Some Useful Matlab Functions

- Matlab cannot explicitly calculate the DTFT, since the frequency axis is continuous. However, it can calculate an approximation of the DTFT using a given number of points.
- y=**fft**(x, N) – Calculates the discrete Fourier transform of the signal x at N points. If N is not provided, length of y is the same as x. DFT is a sampled version of the DTFT, where the samples are taken at N equidistant points around the unit circle from 0 to π
- [h,w] = **freqz**(b,a,N,'whole') – Calculates the frequency response of a filter whose CCLDE coefficients are given as b and a, using N number of points around the unit circle. If 'whole' is included, it returns a frequency base of w from 0 to 2π, otherwise, from 0 to π.
- **y=abs**(x)- Calculates the absolute value of signal x. For complex values signals, the output is the magnitude (spectrum) of the complex argument.
- y=**angle**(x) – Calculates the phase (spectrum) of the signal x.
- q = **unwrap**(p) corrects the radian phase angles in a vector p by adding multiples of 2π when absolute jumps between consecutive elements of p are greater than the default jump tolerance of π radians.
- y = **fftshift**(x) rearranges the outputs of **fft** by moving the zero-frequency component to the center of the array. It is useful for visualizing a Fourier transform with the zero-frequency component in the middle of the spectrum.

Other Important Properties of DTFT

- We will study the following properties of the DTFT:
  - Linearity ➔ DTFT is a linear operator
  - Time reversal ➔ $x[-n] \leftrightarrow X(-\omega)$
  - Time shift ➔ $x[n-n_0] \leftrightarrow X(\omega)e^{-j\omega n_0}$
  - Frequency shift ➔ $x[n]\, e^{j\omega_0 n} \leftrightarrow X(\omega - \omega_0)$
  - Convolution in time ➔ $x[n]*y[n] \leftrightarrow X(\omega).Y(\omega)$
  - Convolution in frequency
  - Differentiation in frequency ➔ $nx[n] \leftrightarrow j\,(dX(\omega)/d\omega)$
  - Parseval's theorem ➔ Conservation of energy in time and frequency domains
  - Symmetry properties

$$x[n] \overset{\mathfrak{I}}{\Leftrightarrow} X(\omega) \qquad y[n] \overset{\mathfrak{I}}{\Leftrightarrow} Y(\omega)$$

Linearity & Differentiation In Frequency

$$x[n] \overset{\mathfrak{I}}{\Leftrightarrow} X(\omega)$$

- The DTFT is a linear operator

$$ax[n] + by[n] \overset{\mathfrak{I}}{\Leftrightarrow} aX(\omega) + bY(\omega)$$

- Multiplying the time domain signal with the independent time variable is equivalent to differentiation in frequency domain.

$$nx[n] \leftrightarrow j\frac{dX(\omega)}{d\omega}$$

Time Reversal , Time & Frequency Shift

- A reversal in of the time domain variable causes a reversal of the frequency variable

$$x[-n] \overset{\Im}{\Leftrightarrow} X(-\omega)$$

- A shift in time domain by *m* samples causes a phase shift of $e^{-j\omega m}$ in the frequency domain

$$x[n-n_0] \leftrightarrow e^{-j\omega n_0} X(\omega)$$

  - Note that the magnitude spectrum is unchanged by time shift. Why?

- A shift in frequency domain by ω_0 causes a time delay of $e^{j\omega_0 n}$

$$e^{j\omega_0 n} x[n] \leftrightarrow X(\omega-\omega_0)$$

Convolution

- Convolution in time domain is equivalent to multiplication in frequency domain

$$x[n] * h[n] \overset{\Im}{\Leftrightarrow} X(\omega) \cdot H(\omega)$$

  - This is one of the fundamental theorems in filtering. It allows us to compute the filter response in frequency domain using the frequency response of the filter.

- Multiplication in time domain is equivalent to convolution in frequency domain

$$x[n] \cdot h[n] \overset{\Im}{\Leftrightarrow} \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\gamma) H(\omega - \gamma) d\gamma$$

PARSEVAL'S THEOREM

- The energy of the signal , whether computed in time domain or the frequency domain, is the same!

General form:

$$\sum_{n=-\infty}^{\infty} g[n].h^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(\omega).H^*(\omega)\,d\omega$$

Energy of a continuous periodic function

$$\sum_{n=-\infty}^{\infty} g[n].g^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(\omega).G^*(\omega)\,d\omega = \varepsilon_g$$

Symmetry Properties of DTFT

- Your text lists several symmetry properties of DTFT (pg. 125-130)
  - While all of these properties are important for academic reasons, the following are important for practical reasons:
    - The Fourier transform of a real signal is conjugate symmetric: the magnitude spectrum is an even function of ω (symmetric), whereas the phase spectrum is an odd function of ω (antisymmetric). That is, for a real signal x[n]

$$X^*(\omega) = X^*(-\omega) \Rightarrow |X(\omega)| = |X(-\omega)|, \quad \angle X(\omega) = -\angle X(-\omega)$$

    - The Fourier transform of a symmetric signal is real! More generally, if

$$x[n] \overset{\mathfrak{I}}{\longleftrightarrow} X(\omega) = X_{real}(\omega) + jX_{imag}(\omega)$$

then, the following is true:

$$x_{even}[n] \overset{\mathfrak{I}}{\longleftrightarrow} X_{real}(\omega), \qquad x_{odd}[n] \overset{\mathfrak{I}}{\longleftrightarrow} X_{imag}(\omega)$$

since the even part of any signal is necessarily symmetric, it follows that the DTFT of any symmetric signal must also necessarily be real!

Properties of Fourier Transform

Property	Signal	Fourier transform
	$x(t)$	$X(\omega)$
	$x_1(t)$	$X_1(\omega)$
	$x_2(t)$	$X_2(\omega)$
Linearity	$a_1 x_1(t) + a_2 x_2(t)$	$a_1 X_1(\omega) + a_2 X_2(\omega)$
Time shifting	$x(t - t_0)$	$e^{-j\omega t_0} X(\omega)$
Frequency shifting	$e^{j\omega_0 t} x(t)$	$X(\omega - \omega_0)$
Time scaling	$x(at)$	$\frac{1}{\lvert a \rvert} X\left(\frac{\omega}{a}\right)$
Time reversal	$x(-t)$	$X(-\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Time differentiation	$\frac{dx(t)}{dt}$	$j\omega X(\omega)$
Frequency differentiation	$(-jt)x(t)$	$\frac{dX(\omega)}{d\omega}$

Property	Signal	Fourier Transform
Integration	$\int_{-\infty}^{t} x(\tau)\, d\tau$	$\pi X(0)\delta(\omega) + \frac{1}{j\omega} X(\omega)$
Convolution	$x_1(t) * x_2(t)$	$X_1(\omega) X_2(\omega)$
Multiplication	$x_1(t) x_2(t)$	$\frac{1}{2\pi} X_1(\omega) * X_2(\omega)$
Real signal	$x(t) = x_e(t) + x_o(t)$	$X(\omega) = A(\omega) + jB(\omega)$
		$X(-\omega) = X^*(\omega)$
Even component	$x_e(t)$	$\mathrm{Re}\{X(\omega)\} = A(\omega)$
Odd component	$x_o(t)$	$j\,\mathrm{Im}\{X(\omega)\} = jB(\omega)$
Parseval's relations		

$$\int_{-\infty}^{\infty} x_1(\lambda) X_2(\lambda)\, d\lambda = \int_{-\infty}^{\infty} X_1(\lambda) x_2(\lambda)\, d\lambda$$

$$\int_{-\infty}^{\infty} x_1(t) x_2(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\omega) X_2(-\omega)\, d\omega$$

$$\int_{-\infty}^{\infty} \lvert x(t) \rvert^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lvert X(\omega) \rvert^2\, d\omega$$

The table is from Signals and Systems, H.P. Hsu (Schaum's series), which uses ω for continuous frequencies. Replace all ω with Ω to be consistent with our, and the textbook notation.

Common Fourier Transform Pairs

$x(t)$	$X(\omega)$
$\delta(t)$	1
$\delta(t-t_0)$	$e^{-j\omega t_0}$
1	$2\pi\delta(\omega)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
$\sin\omega_0 t$	$-j\pi[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
$u(t)$	$\pi\delta(\omega)+\dfrac{1}{j\omega}$
$u(-t)$	$\pi\delta(\omega)-\dfrac{1}{j\omega}$
$e^{-at}u(t),\ a>0$	$\dfrac{1}{j\omega+a}$
$te^{-at}u(t),\ a>0$	$\dfrac{1}{(j\omega+a)^2}$

$x(t)$	$X(\omega)$
$e^{-a\|t\|},\ a>0$	$\dfrac{2a}{a^2+\omega^2}$
$\dfrac{1}{a^2+t^2}$	$e^{-a\|\omega\|}$
$e^{-at^2},\ a>0$	$\sqrt{\dfrac{\pi}{a}}\,e^{-\omega^2/4}$
$p_a(t)=\begin{cases}1 & \|t\|<a\\ 0 & \|t\|>a\end{cases}$	$2a\dfrac{\sin\omega a}{\omega a}$
$\dfrac{\sin at}{\pi t}$	$p_a(\omega)=\begin{cases}1 & \|\omega\|<a\\ 0 & \|\omega\|>a\end{cases}$
$\operatorname{sgn} t$	$\dfrac{2}{j\omega}$
$\sum_{k=-\infty}^{\infty}\delta(t-kT)$	$\omega_0\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_0),\ \omega_0=\dfrac{2\pi}{T}$