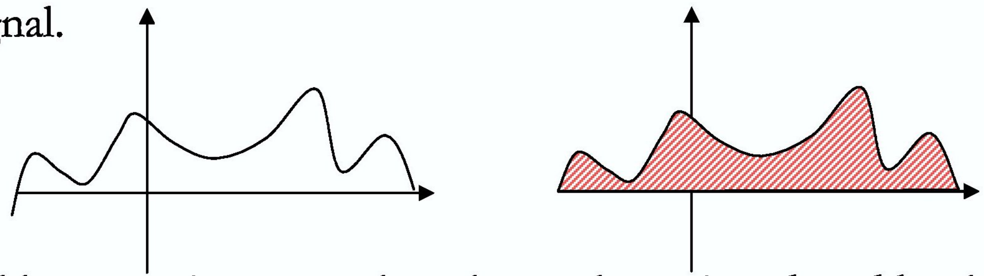
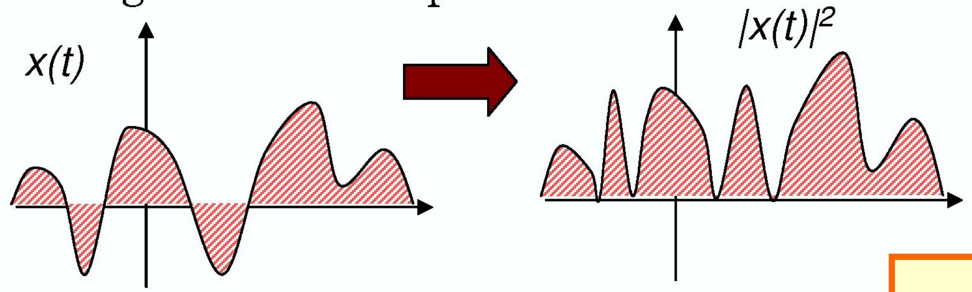
Energy & Power In Signals

- It is often useful to define the "size" or "strength" of a signal. That is, we would like to be able to use a single number that represents the average strength of the signal. How would we do that?
  - A reasonable answer would be to use the area under the curve. The larger the area, the stronger the signal.



  - But if the signal has negative areas, than the total area is reduced by the negative parts. Yet, a negative signal is not necessarily a weaker signal. In fact, -110 V will jolt you as much as +110 V will. So, we need another approach. Calculating the area under the "square of the absolute value of the signal" solves this problem



  - This area is defined as the energy of the (continuous time) signal.

$$E_x = \sum_{n=-\infty}^{\infty} \left(x[n] \right)^2$$

Energy & Power in Sequences

- Total ***energy*** of a (discrete) sequence $x[n]$ is similarly defined as

$$\boxed{\mathcal{E}_{\mathrm{x}} = \sum_{n=-\infty}^{\infty} x[n]\cdot x^{*}[n] = \sum_{n=-\infty}^{\infty} \left|x[n]\right|^{2}}$$

- An infinite length sequence with finite sample values may or may not have finite energy (see example 2.6, page 54, Mitra)
- A finite length sequence with finite sample values always has finite energy
- We define the energy of a sequence $x[n]$ over a finite interval $-K<n<K$ as

$$\boxed{\mathcal{E}_{x,K} = \sum_{n=-K}^{K} \left|x[n]\right|^{2}}$$

- However, if the signal is infinitely long, say, such as the 60Hz mains, the energy becomes infinite. Hence we define the ***average power*** of an ***aperiodic*** sequence as the "energy per unit time"

$$\boxed{P_{\mathrm{x}} = \lim_{K\to\infty} \frac{1}{2K+1} \sum_{n=-K}^{K} \left|x[n]\right|^{2}}$$

Energy & Power in Sequences

- Then $P_x = \lim_{K\to\infty} \frac{1}{2K+1} \sum_{n=-K}^{K} |x[n]|^2$ $\Rightarrow$ $P_x = \lim_{K\to\infty} \frac{1}{2K+1} \mathcal{E}_{x.K}$

- The average power of a ***periodic*** sequence $\tilde{x}[n]$ with a period N is given by

$$P_x = \frac{1}{N} \sum_{n=0}^{N-1} \left(\tilde{x}[n]\right)^2$$

  - The average power of an infinite-length sequence may be finite or infinite
  - A signal with finite energy is called an ***energy signal***. Energy signals has zero power!
  - A signal with finite (and nonzero) power is called a ***power signal***. A power signal, with nonzero power, has infinite energy.

Examples of Finite and Infinite Energy Signals

Infinite length sequence: $x_1[n] = \begin{cases} \dfrac{1}{n} & n \geq 1 \\ 0 & n \leq 0 \end{cases}$

Energy is: $\varepsilon_{x1} = \sum_{n=1}^{\infty} \left(\dfrac{1}{n} \right)^2 = \dfrac{\pi^2}{6}$ => ***finite***

Infinite length sequence: $x_2[n] = \begin{cases} \dfrac{1}{\sqrt{n}} & n \geq 1 \\ 0 & n \leq 0 \end{cases}$

Energy is: $\varepsilon_{x_2} = \sum_{n=1}^{\infty} \left(\dfrac{1}{n} \right)$ => ***infinite***

Example of a Power Signal

Infinite length sequence: $x[n] = \begin{cases} 3(-1)^n & n \geq 0 \\ 0 & n < 0 \end{cases}$

Energy : $\varepsilon_x = \sum_{n=0}^{\infty} 3^2(-1)^{2n} = \sum_{n=0}^{\infty} 9(1)^n$ => ***infinite***

Average Power : $P_x = \lim_{k\to\infty} \frac{1}{2K+1} 9 \sum_{k=0}^{K} 1^k = \lim_{k\to\infty} \frac{9(K+1)}{2K+1} = 4.5$

=> ***finite***

Energy & Power Sequences

Recap:

- A sequence with finite average power is called a *power signal*. Unless the power is zero, a power signal has infinite energy
  - A periodic sequence has a finite average power but infinite energy

- A sequence with finite energy is called an *energy signal*. An energy signal has zero average power.
  - A finite-length sequence has finite energy but zero average power

Discrete Time Systems

- Basic Operations on Discrete Sequences
- Discrete Systems
- Classification of Discrete Systems
  - Linearity
  - Shift-invariance
  - Causality
  - Memory
  - Stability
- Characterization of Discrete Systems
  - Impulse Response
  - Constant coefficient linear difference equations (CCLDE)
  - FIR systems
  - IIR systems

DISCRETE SYSTEMS

- A discrete-time system processes a given input sequence $x[n]$ to generates an output sequence $y[n]$ with more desirable properties
- In most applications, the discrete-time system is a single-input, single-output system:

$x[n]$ → DISCRETE TIME SYSTEM → $y[n]$

Input sequence

Output sequence

$$y[n] = \Re\{x[n]\}$$

The operator that acts on the input sequence

Classification of Systems

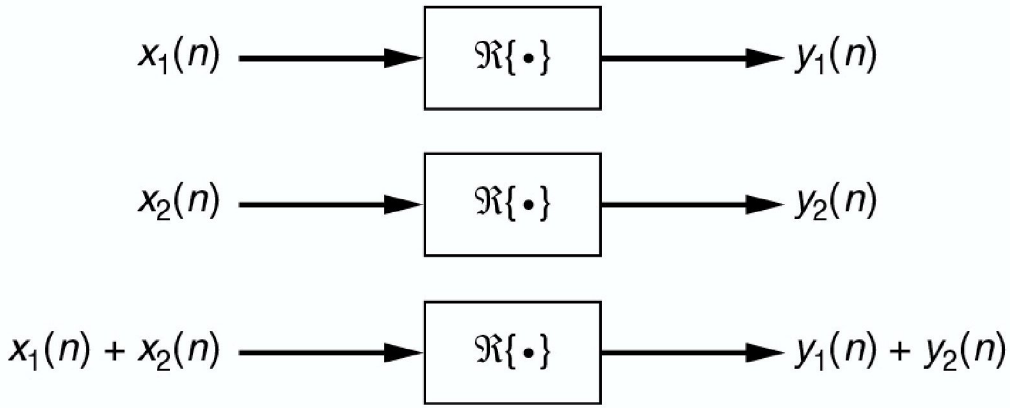
- Discrete time systems can be classified based on their properties
  - Discrete vs. continuous
  - Linear vs. non-linear
  - Shift-invariant or shift-variant
  - Causal vs. noncausal
  - Memoryless vs. with memory
  - Stable vs. unstable

LINEARITY

- Let $y_1[n]$ be the output due to an input $x_1[n]$ and $y_2[n]$ be the output due to an input $x_2[n]$. A system is said to be ***linear***, if the following superposition & homogeneity properties are satisfied

$$x[n] = \alpha\, x_1[n] + \beta\, x_2[n] \Longleftrightarrow y[n] = \alpha\, y_1[n] + \beta\, y_2[n]$$

$$\Re\{\alpha x_1[n] + \beta x_2[n]\} = \alpha . \Re\{x_1[n]\} + \beta . \{x_2[n]\}$$



- This property must hold for any arbitrary constants α and β , and for all possible inputs $x_1[n]$ and $x_2[n]$, and can also be generalized to any arbitrary number of inputs

An Example - Accumulator

➲ A discrete system whose input / output relationship is given as

$$y[n] = \sum_{\ell=-\infty}^{n} x[\ell]$$
$$= y[-1] + \sum_{\ell=0}^{n} x[\ell],$$

The second form is used, if the signal is causal, in which case y[-1] is the initial condition

is known as an ***accumulator***. The output at any given time, is simply the sum of all inputs up to that time.

➲ Is the accumulator linear? ___?___

➲ How about the accumulator for the causal systems? ___?___

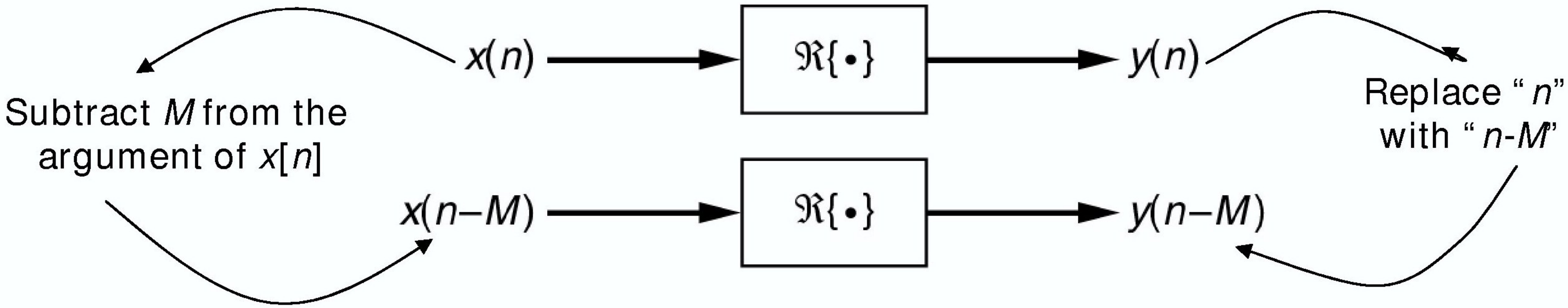
➲ How about the system $y[n]=ax_1[n]+b$ ___?___

➲ As an exercise, try $y[n] = x^2[n] - x[n-1]x[n+1]$

In class exercise

Shift - Invariance

- A system is said to be ***shift-invariant*** if $\Re(x[n\text{-}M])=y[n\text{-}M]$, for all n,m.



- For sequences and systems where the index n is related to discrete instants of time, this property is also called the ***time-invariance*** property

- Time-invariance property ensures that for a specified input, the output is independent of the time the input is being applied

Linear Time-Invariant Systems

- A system that satisfies both the linearity and the time (shift) invariance properties is called a ***linear time (shift) invariant*** system, LTI (LSI).

- We will see that these group of systems play a particularly important role in signal processing.
  - They are easy to characterize and analyze, and hence easy to design
  - Efficient algorithms have been developed over the years for such systems

CAUSALITY

- A system is said to be ***causal***, if the output at time n_0 does not depend on the inputs that come after n_0.

- In other words, in a causal system, the n_0th output sample $y[n_0]$ depends only on input samples $x[n]$ for $n \le n_0$ and does not depend on input samples for $n > n_0$.
- Here are some examples: Which systems are causal?

$$y[n] = \alpha_1 x[n] + \alpha_2 x[n-1] + \alpha_3 x[n-2] + \alpha_4 x[n-3]$$

$$y[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] + a_1 y[n-1] + a_2 y[n-2]$$

$$y[n] = y[n-1] + x[n]$$

$$y[n] = x_u[n] + \frac{1}{2}(x_u[n-1] + x_u[n+1])$$

$$y[n] = x_u[n] + \frac{1}{3}(x_u[n-1] + x_u[n+2]) + \frac{2}{3}(x_u[n-2] + x_u[n+1])$$

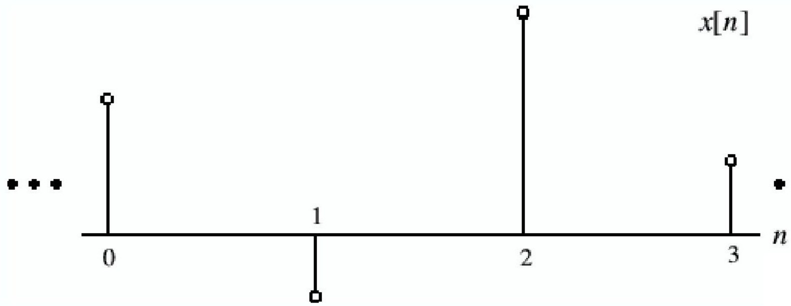
An Example - Upsampler

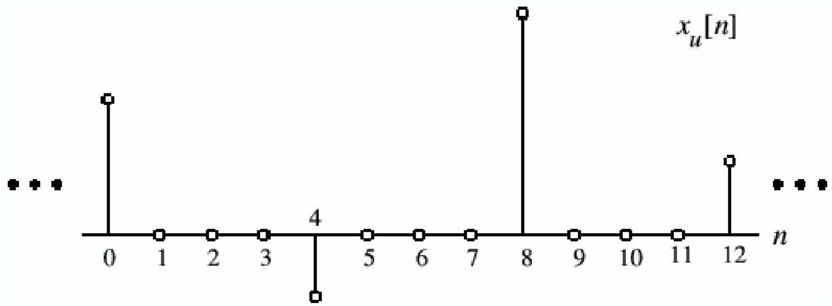
- A system whose input / output characteristics can be written as

$$x_u[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \ldots.. \\ 0, & \text{otherwise} \end{cases}$$

is known as an upsampler.

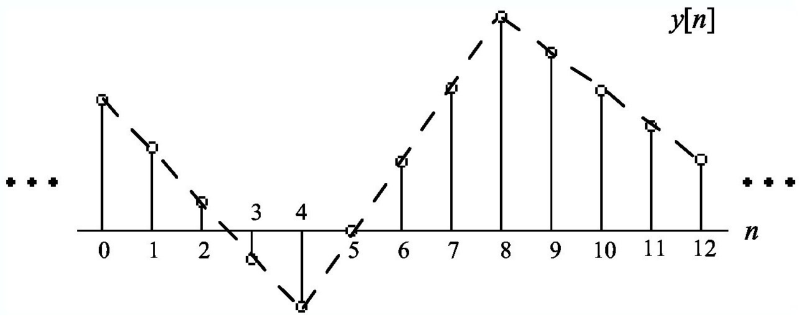
- This system inserts L zeros between every sample. If the samples are inserted based on their amplitudes, then the system is called an interpolator.





Is the upsampler a time-invariant system?

In class exercise



An Example - Downsampler

- A system whose input-output characteristic satisfies $y[n] = x[Mn]$ where M is a (+) integer, is called a ***downsampler*** or a ***decimator***.
  - Such a system reduces the number of samples by a factor of M by removing M samples from between every sample.

- Is this system
  - Linear?
  - Time (shift) invariant?
  - Causal?

Take home exercise

MEMORY

- A system is said to be ***memoryless*** if the output depends only on the current input, but not on any other past (or future) inputs. Otherwise, the system is said to have memory.

- Which of the following systems have memory?

$$y[n] = y[n-1] + x[n]$$

$$y[n] = x[Mn]$$

$$x_u[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, ... \\ 0, & \text{otherwise} \end{cases}$$

$$y[n] = \sum_{\ell=-\infty}^{n} x[\ell]$$

STABILITY

- There are several definitions of stability, which is of utmost importance in filter design. We will use the definition of stability in the BIBO sense

- A systems is said to be stable in the bounded input bounded output sense, if the system produces a finite (bounded) output for any finite (bounded) input, that is,

  - If y[n] is the response to an input x[n] that satisfies $|x[n]| \leq B_x < \infty$, and y[n] satisfies $|y[n]|\ ]| \leq B_y < \infty$, then the system is said to be stable in the BIBO sense.

- A system (filter) that is not stable is rarely of any practical use (except for very specialized applications), and therefore, most filters are designed to be BIBO stable.

An Example – Moving Average Filter

- An M – point ***moving average system*** (filter) is defined as

$$\boxed{y[n]=\frac{1}{M}\sum_{k=0}^{M-1}x[n-k]}$$

- What would you use such a system for?
- Is this system stable?

TRY THIS AT HOME

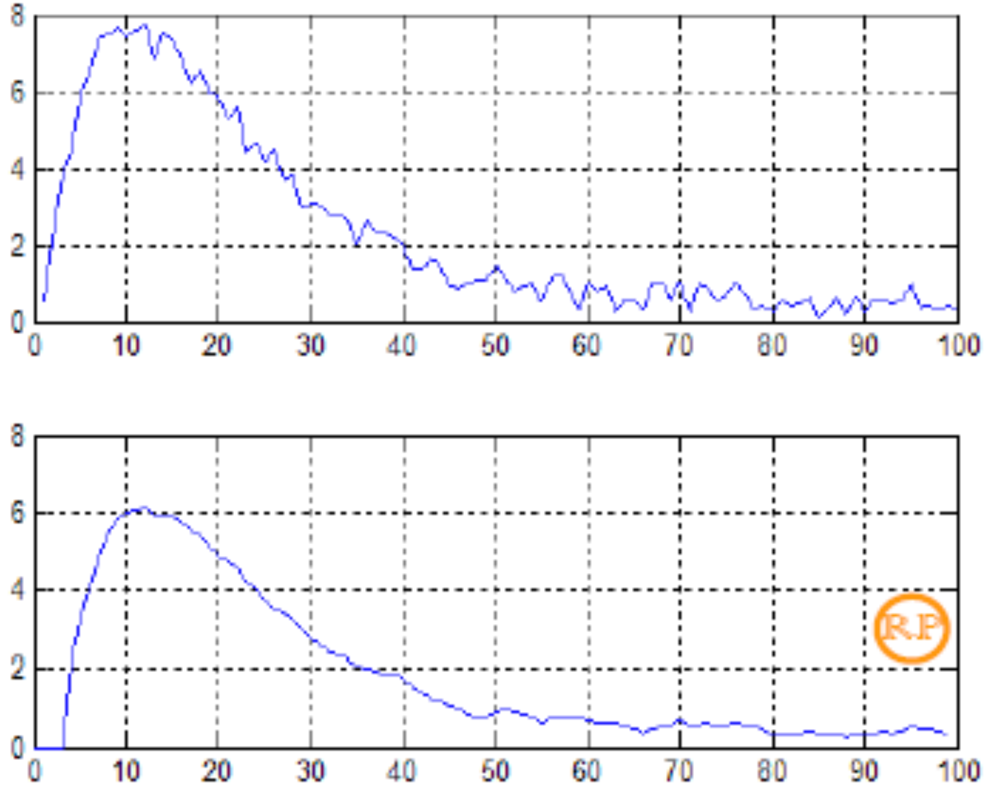
Matlab exercise

7 point moving average

```
n=0:99;
s=2*(n.*(0.9).^n);
d=rand(1, 100);
x=s+d;
subplot(211)
plot(x); grid

for i=7:100;
y(i)=(1/7)*sum(x(i-1)+x(i-2)+x(i-3)+x(i-4)+x(i-
5)+x(i-6));
end

subplot(212)
plot(n,y); grid
```



Basic Operations in schematic form

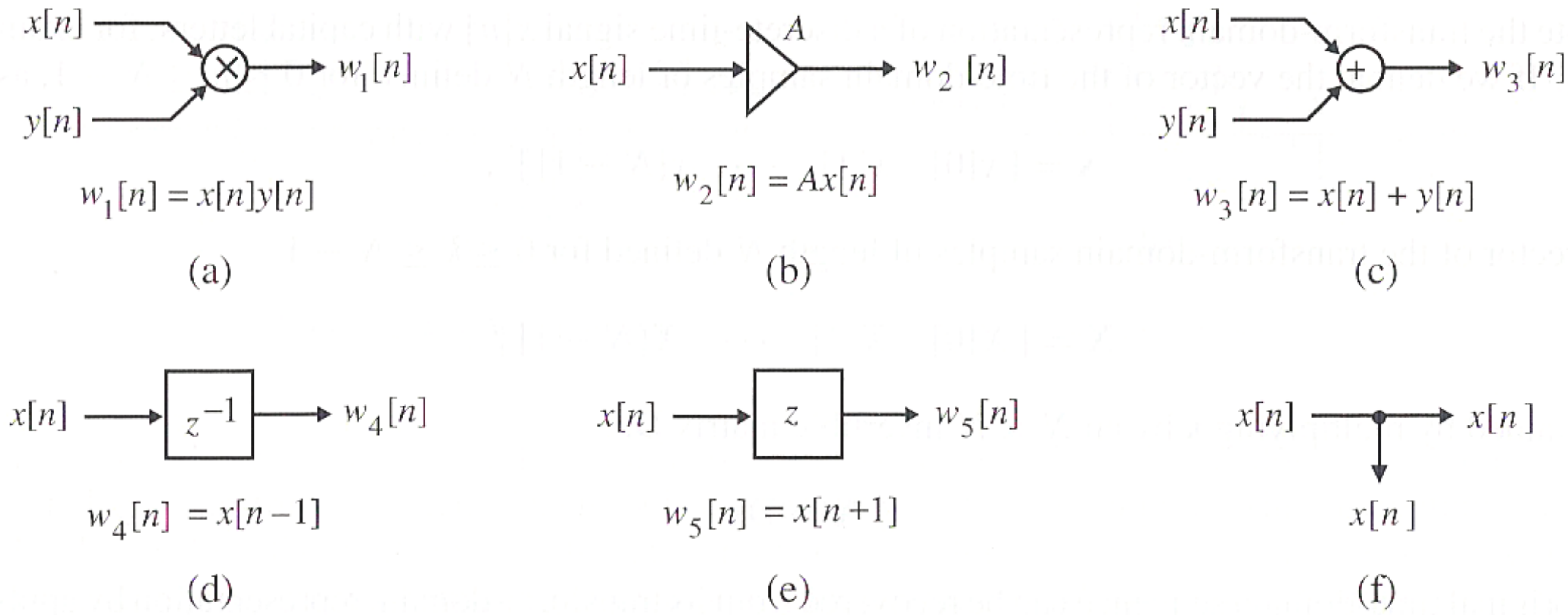
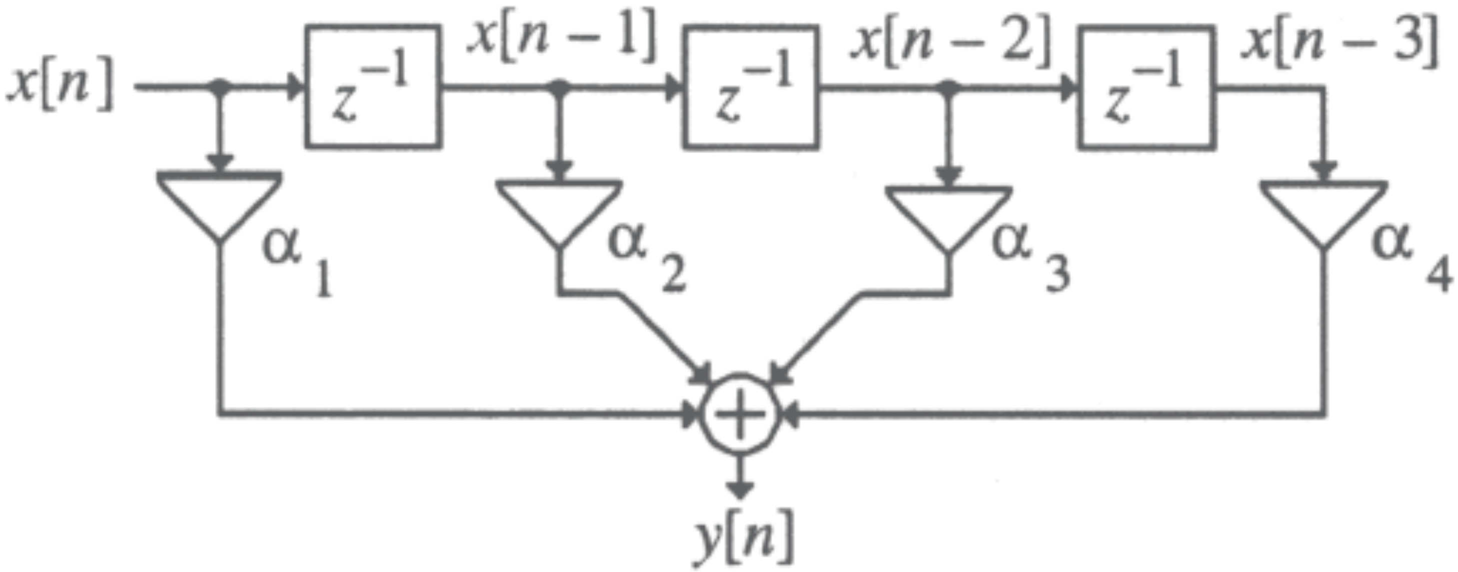


Figure 2.5: Schematic representations of basic operations on sequences: (a) modulator, (b) multiplier, (c) adder, (d) unit delay, (e) unit advance, and (f) pick-off node.

Block Diagram Implementation

- What does this system do…?



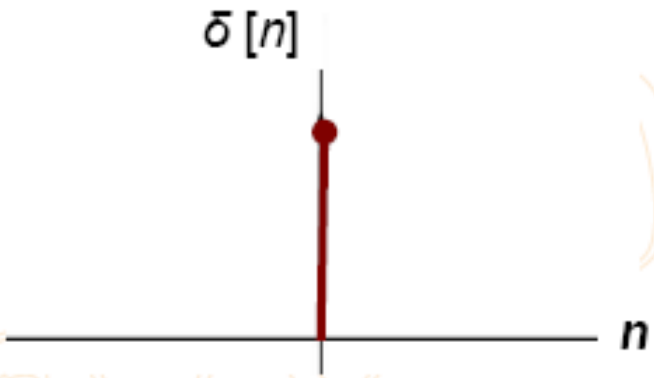
y[n]=…

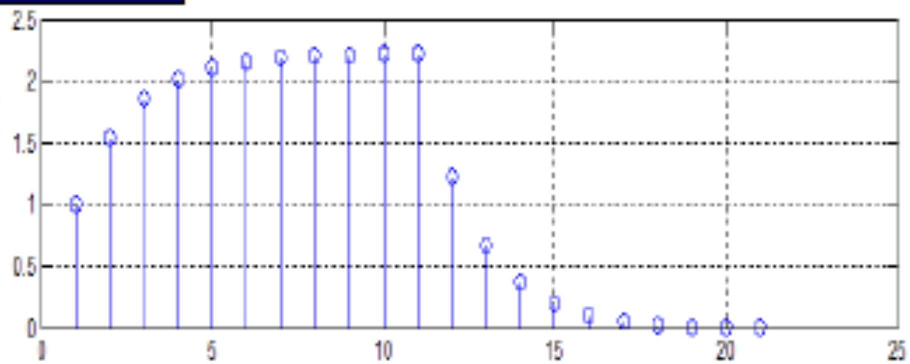
$$\alpha_1 x[n] + \alpha_2 x[n-1] + \alpha_3 x[n-2] + \alpha_4 x[n-3]$$

IMPULSE RESPONSE

- The response of a discrete system to a unit impulse sequence $\delta[n]$ is called the ***impulse response*** of the system, and it is typically denoted by $h[n]$

$x[n]=\delta[n]$ → **Discrete System** → $y[n]=h[n]$





$$h[n] = h[n] * \delta[n] = \sum_{m=-\infty}^{\infty} h[m]\,.\,\delta[n-m]$$

IMPULSE RESPONSE

- Ex: Consider the following system

$$y[n] = \alpha_1 x[n] + \alpha_2 x[n-1] + \alpha_3 x[n-2] + \alpha_4 x[n-3]$$

  - The impulse response of this system can be obtained by setting the input to the impulse sequence ➔

$$h[n] = \alpha_1 \delta[n] + \alpha_2 \delta[n-1] + \alpha_3 \delta[n-2] + \alpha_4 \delta[n-3]$$

$$\Updownarrow$$

$$\boxed{\begin{aligned} h[n] &= \sum_{m=-\infty}^{\infty} h[m] \cdot \delta[n-m] \\ &= \cdots h[-1]\delta[n+1] + h[0]\delta[n] + h[1]\delta[n-1] + \cdots \end{aligned}}$$

$$\Rightarrow \{h[n]\} = \{\underset{\uparrow}{\alpha_1}, \quad \alpha_2, \quad \alpha_3, \quad \alpha_4\}$$

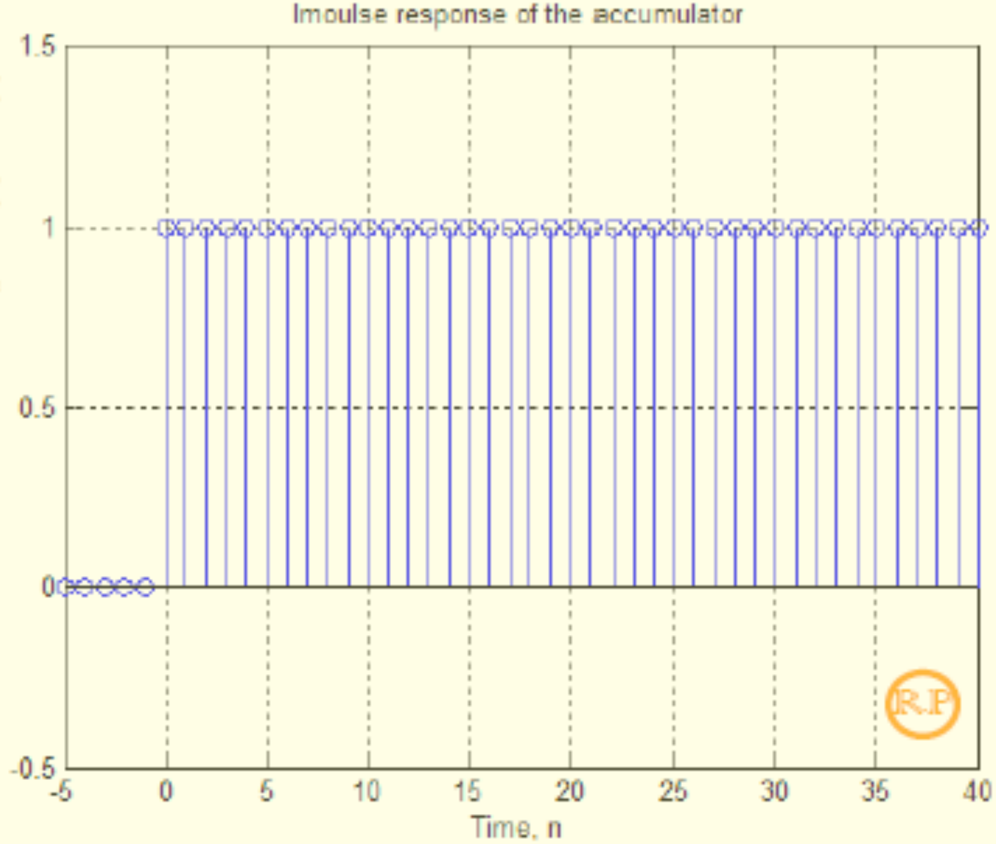
IMPULSE RESPONSE

- Recall the discrete time accumulator:

$$y[n] = \sum_{\ell=-\infty}^{n} x[\ell]$$

- What is the impulse response of this system?

h[n]=...



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IMPULSE RESPONSE

- So, what is the big deal?

- The impulse response plays a ***monumental*** role in characterization of LTI systems.
  - In fact, if you know the impulse response of a discrete LTI system, then you know the response of the system to any arbitrary input!
  - You tell me h[n], I will tell you the response to *any* x[n]

$x[n]$ → **Discrete System h[n]** → $y[n]$=x[n]*h[n]

Input sequence — Output sequence

$$\boxed{y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} x[m] \cdot h[n-m]}$$

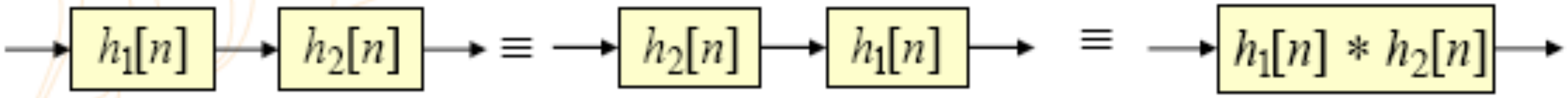
PROPERTIES

- There are several useful properties of the convolution and impulse response characterization:
  - An LTI system is BIBO sable, if its impulse response is absolutely summable:

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

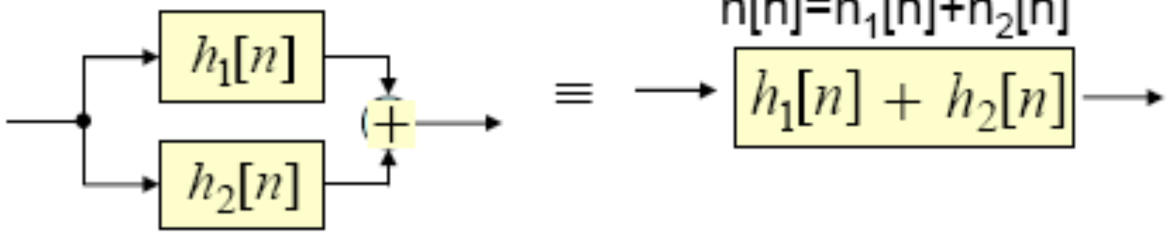
  - An LTI system is causal, if its impulse response is a causal sequence, i.e., $h[n]=0, n<0$
  - If more then one system are connected by series or parallel, the effective system impulse response can be obtained as follows:
    - Cascade

$h[n]=h_1[n]*h_2[n]$



    - Parallel

$h[n]=h_1[n]+h_2[n]$



EXAMPLE

- The system, known as factor-of-2 interpolator is given as

$$\boxed{h[n]=\delta[n]+\frac{1}{2}\left(\delta[n-1]+\delta[n+1]\right)}$$

that is, its impulse response is $h[n]=[0.5\ 1\ \ 0.5]$, $-1<n<1$.

- Is this system causal and stable?
  - Not causal. Because the impulse response is not a causal sequence
  - Stable, since summation of the absolute values of all h[n] values is finite
- A causal version of this system can be obtained, however, simply by delaying the output by one sample.

$$\boxed{h[n]=\delta[n-1]+\frac{1}{2}(\delta[n-2]+\delta[n])}$$

Properties

- A cascade connection of two stable is systems is always stable
- If a cascade system satisfies the following condition

$$h_1[n] * h_2[n] = \delta[n]$$

then h_1 and $h_2$are called ***inverse systems***

- An application of the inverse system concept is in the recovery of a signal $x[n]$ from its distorted version $x'[n]$ appearing at the output of a transmission channel
- If the impulse response of the channel is known, then $x[n]$ can be recovered by designing an inverse system of the channel

channel — inverse system

$$x[n] \longrightarrow \boxed{h_1[n]} \xrightarrow{x'[n]} \boxed{h_2[n]} \longrightarrow x[n]$$

$$h_1[n] \quad * \quad h_2[n] \quad = \delta[n]$$

EXERCISE

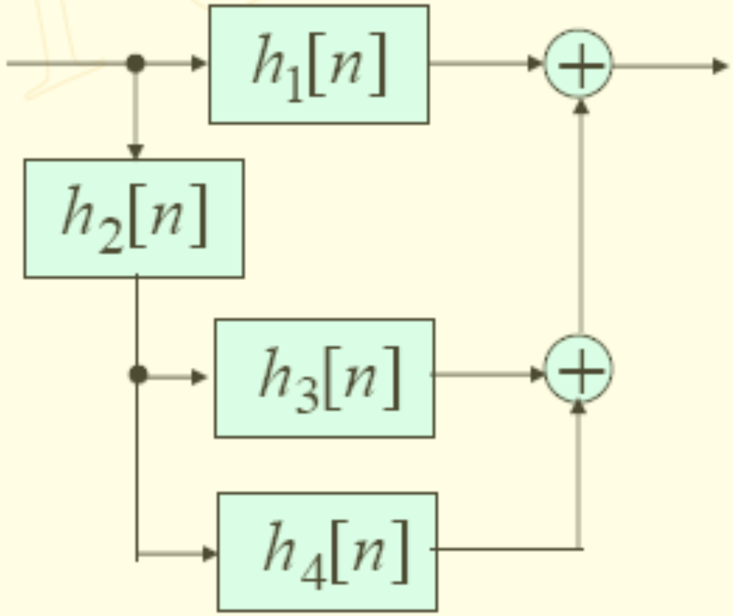
- Consider the discrete-time system where

$$h_1[n] = \delta[n] + 0.5\delta[n-1],$$

$$h_2[n] = 0.5\delta[n] - 0.25\delta[n-1],$$

$$h_3[n] = 2\delta[n],$$

$$h_4[n] = -2(0.5)^n u[n]$$



What is the overall system response?

FINITE IMPULSE RESPONSE SYSTEMS

➲ If the impulse response h[n] of a system is of finite length, that system is referred to as a ***finite impulse response (FIR)*** system

$$h[n]=0 \text{ for } n<N_1 \text{ and } n>N_2, \quad N_1<N_2$$

- The output of such a system can then be computed as a finite convolution sum

$$y[n]=\sum_{k=N_1}^{N_2} h[k]x[n-k]$$

- E.g., h[n]=[1 2 0 -1] is a FIR system (filter)

- FIR systems are also called ***nonrecursive systems*** (for reasons that will later become obvious), where the output can be computed from the current and past input values only – without requiring the values of ***previous outputs***.

INFINITE IMPULSE RESPONSE SYSTEMS

- If the impulse response is of infinite length, then the system is referred to as an ***infinite impulse response (IIR)*** system. These systems cannot be characterized by the convolution sum due to infinite sum.
  - Instead, they are typically characterized by constant coefficient linear difference equations (CCLDEs), as we will see later.
  - Recall accumulator and note that it can have an alternate – and more compact representation that makes the current output a function of previous inputs and outputs

$$y[n] = \sum_{\ell=-\infty}^{n} x[\ell] \quad \Longrightarrow \quad y[n] = y[n-1] + x[n]$$

  - The impulse response of this system (which is of infinite length), cannot be represented with a finite convolution sum. Note that, since the current output depends on the previous outputs, this is also called a ***recursive system***

Constant Coefficient Linear Difference Equations

- All discrete systems can also be represented using ***constant coefficient, linear difference equations,*** of the form

$$\boxed{y[n]+a_1 y[n-1]+a_2 y[n-2]+\cdots a_N y[n-N]=b_0 x[n]+b_1 x[n-1]+\cdots+b_M x[n-M]}$$

Outputs y[n] **Inputs x[n]**

$$\boxed{\sum_{i=0}^{N} a_i y[n-k]=\sum_{i=0}^{M} b_i x[n-k]}$$

Constant coefficients

  - Constant coefficients a_i and b_i are called *filter coefficients*
  - Integers M and N represent the maximum *delay* in the input and output, respectively. The larger of the two numbers is known as the *order of the filter*.
  - <u>Any</u> LTI system can be represented as two finite sum of products!

FIR SYSTEMS

- Note that the expression indicates the most general form of an LTI system:

$$\boxed{\sum_{i=0}^{N} a_i y[n-i] = \sum_{j=0}^{M} b_j x[n-j], \quad a_0 = 1}$$

  - If the current output y[n] does not depend on previous outputs y[n-i], that is if all a_i=0 (except a_0=1), then we have ***no recursion*** – such systems are FIR (***non-recursive***) systems

$$\boxed{y[n] = \sum_{j=0}^{M} b_j x[n-j]}$$

  - Note that the impulse response of an FIR system can easily be obtained from its CCLDE representation:

$$y[n] = \sum_{j=0}^{M} b_j x[n-j] \Rightarrow h[n] = \sum_{j=0}^{M} b_j \delta[n-j]$$

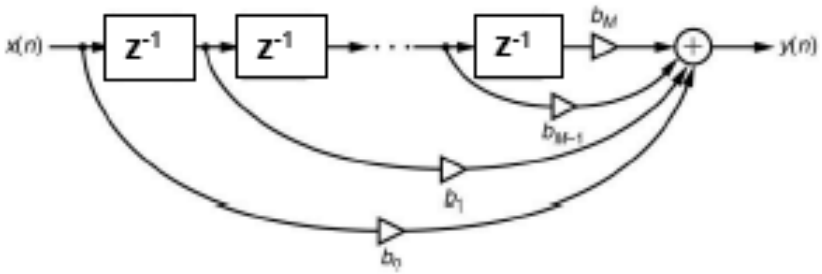
$$= b_0\delta[n] + b_1\delta[n-1] + \cdots + b_M\delta[n-M]$$

  - The sum of finite numbers will always be finite, therefore, the impulse response of this system will be finite, hence, finite impulse response.
  - Finite Impulse Response ←→ Nonrecursive

FIR SYSTEMS

$$y[n]=\sum_{j=0}^{M} b_j x[n-j]=b_0x[n]+b_1x[n-1]+\cdots+b_M x[n-M]$$

- Note that this representation looks similar to the definition of convolution. In fact, $\boldsymbol{y[n]=b_n * x[n]}$, that is the system output of an FIR filter is simply the convolution of input x[n] with the filter coefficients $\boldsymbol{b_n}$
- Since we already know that the output of a system is the convolution of its input with the system impulse response, it follows that ***filter coefficients b_n is the impulse response of an FIR filter!***
- The CCLDE representation of an FIR system can schematically be represented using the following diagram, known as the "***filter structure***"



The hardware implementation follows this structure exactly, using delay elements, adders and multipliers.