Statistical Signal Processing
An Example of a Final Exam

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The exam duration is 3 hours ($9^{00} - 12^{00}$).
The number of points given to each question is indicated next to it. The grade will be based on your answers to all questions.
Indicate clearly the steps in your solution and provide sufficient text.

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1. The variable x is a discrete random variable which can attain the values $\vec{x} = [0, 1, 2, 3]^*$ with the following probability density function:

$$p(x) = \begin{cases} 2\theta/3 & \text{if } x = 0; \\ \theta/3 & \text{if } x = 1; \\ 2(1-\theta)/3 & \text{if } x = 2; \\ (1-\theta)/3 & \text{if } x = 3. \end{cases}$$

Suppose we obtained 10 independent observations of x with the values: $(2, 1, 3, 2, 3, 0, 1, 0, 2, 1)$.

- **Write the general PDF of this data vector, $\vec{\mathbf{x}}$, $p(x_1, x_2, \ldots, x_{10})$.**
- **What is the maximum likelihood estimator of θ?**
- **Find the mean and variance of x for the maximum likelihood θ?**

(33 points)

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2. A probability density function that describes the arrival rate of photons to a certain detector,

$$p(T) = \begin{cases} \alpha\, e^{-\alpha T} & \text{for } T \geq 0; \\ 0 & \text{Otherwise.} \end{cases}$$

where T represents time interval between arrivals of photons and α is the arrival rate. A total of $N+1$ photons have arrived with time intervals $T_1, T_2, \ldots, T_N$. We would like to estimate the value of the parameter α.

- **Write the expression for the Likelihood function, $p(T_1, T_2, \ldots, T_N|\alpha)$, for this problem.**
- **Calculate the maximum likelihood estimator for α.**

Now assume that the parameter is a random variable with the *prior*:

$$p(\alpha) = \begin{cases} \alpha\beta^2\, e^{-\alpha\beta} & \text{for } \alpha \geq 0; \\ 0 & \text{Otherwise.} \end{cases}$$

where β is a given constant.

- **What is the *posterior* pdf $p(\alpha|T_1, T_2, \ldots, T_N)$?**
- **Find the MAP estimator of α**

Note that $\int_0^\infty y^n e^{-y} dy = n!$.

(33 points)

3. A number of short questions are given here, each is 11 point.

   1. **Assume you are given the measured value of a quantity $d = s + \epsilon$, where $\epsilon \sim \mathcal{N}(0, \sigma^2)$. You also know the correlation function of the signal s between two points separated by time interval t as $\xi(t) = s_0^2 e^{-(t/\tau)^2}$. Write an expression for the value of this quantity at every point is space. Express you answer in terms of s_0, ξ and σ**
   2. **Explain intuitively the central limit theorem**
   3. **Two variables $\mathbf{x} = \{x_1, x_2\}$ with zero mean and the following covariance matrix,**

      $$\mathbf{C} = \begin{pmatrix} \sigma^2 & \sigma^2/2 \\ \sigma^2/2 & \sigma^2 \end{pmatrix}$$

      The variables follow a bivariate Gaussian distribution

      $$p(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^2 \det(\mathbf{C})}} e^{-\frac{1}{2}\mathbf{x}^T \mathbf{C}^{-1} \mathbf{x}} \quad (1)$$

      find $p(x_1)$.

(33 points)

Good Luck!