Cosmology

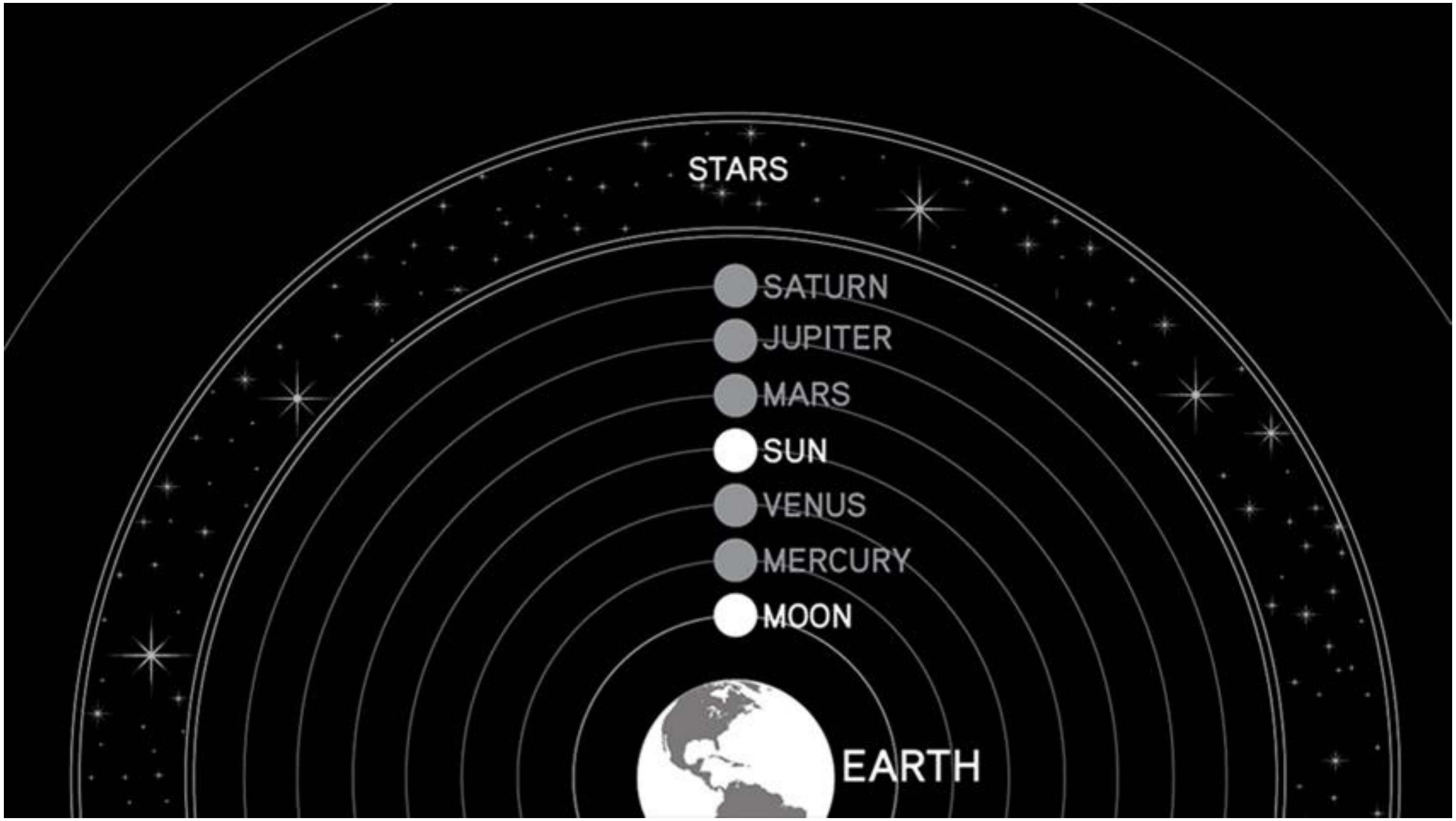
- Teacher: Saleem Zaroubi
- Room 182, phone 34082, email: saleem@astro.rug.nl
- Office hours: You are always welcome to come to my office for short questions. It is better to set an appointment.
- Teaching assistants: Ewoud Wempe and Fabian Gunnink.
  Emails: wempe@astro.rug.nl and f.j.gunnink@student.rug.nl
- Problem sets are mandatory (5 out of 7 at least)
- Written exam at the end of the term
- Lecture slides could be obtained from my website:

  http://www.astro.rug.nl/~saleem

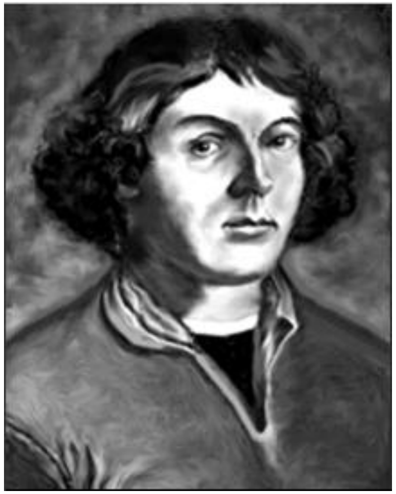
History of Cosmology

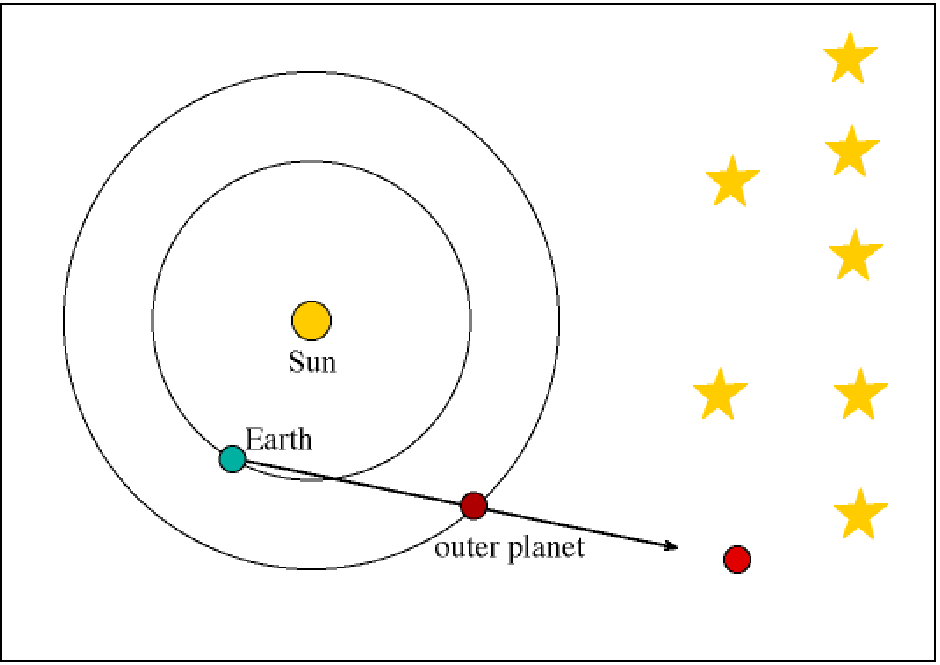
The two sphere model of Asristotle

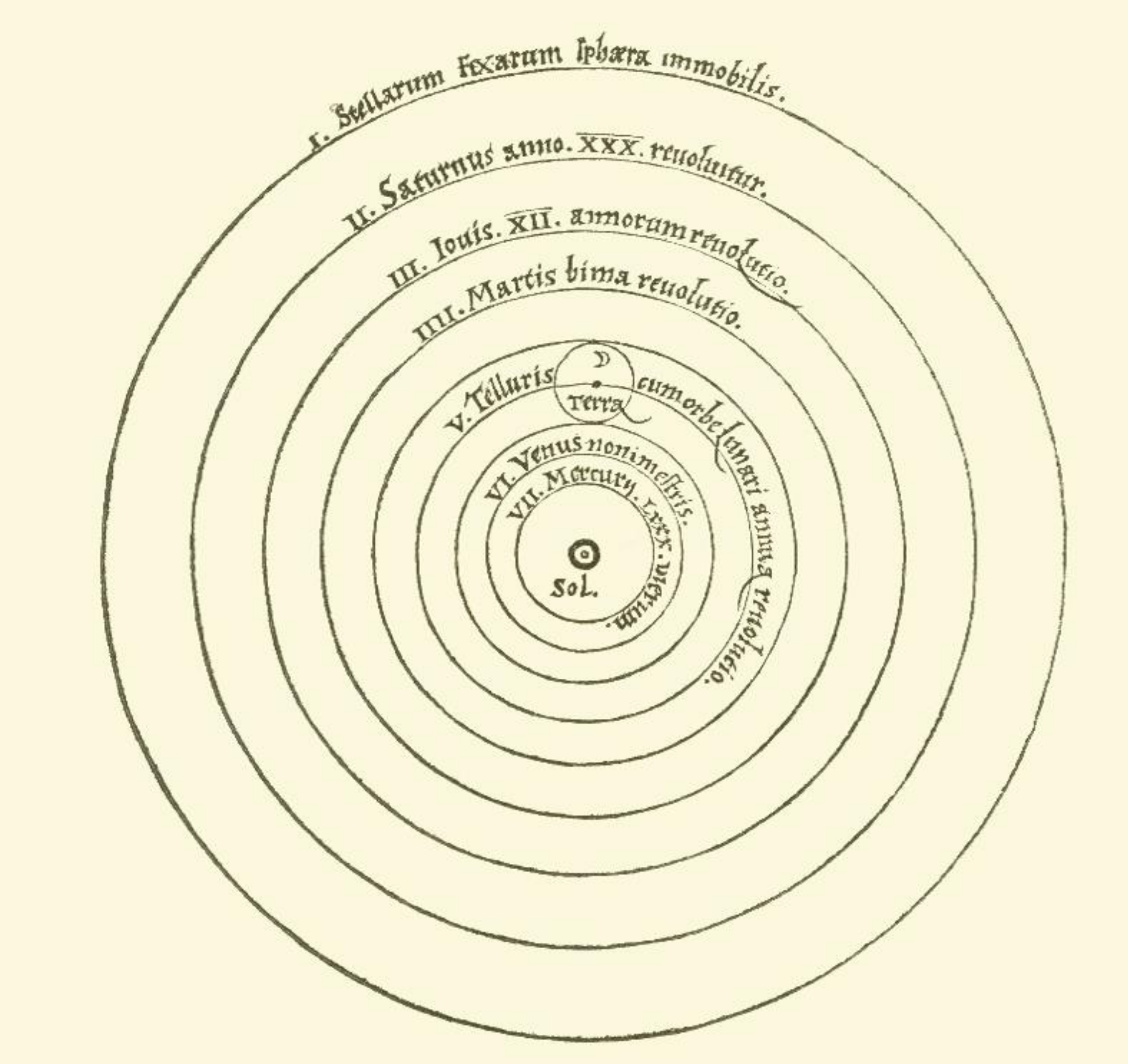
The Universe according to Ptolemy



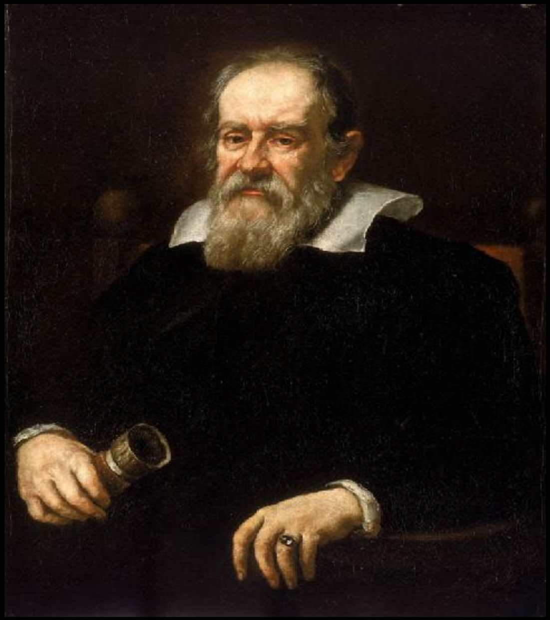
The Copernican Revolution







The Copernican revolution and the Scientific Revolution



Galileo

The first scientist

Newton

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NATURE and Nature's
Laws lay hid in
Night:
God said, "Let Newton
be!" and all was
light.
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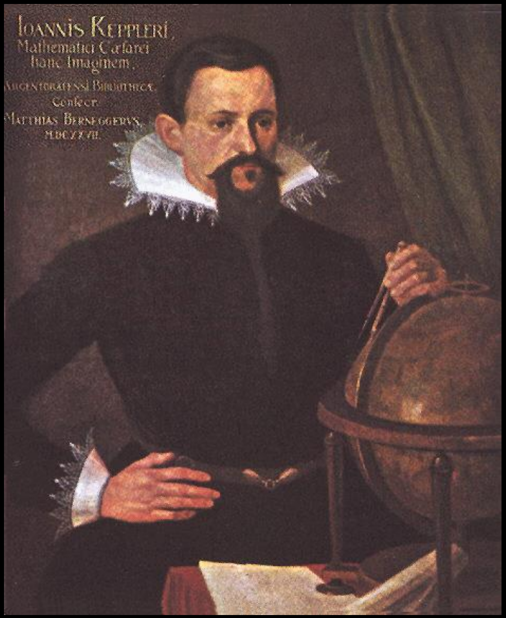


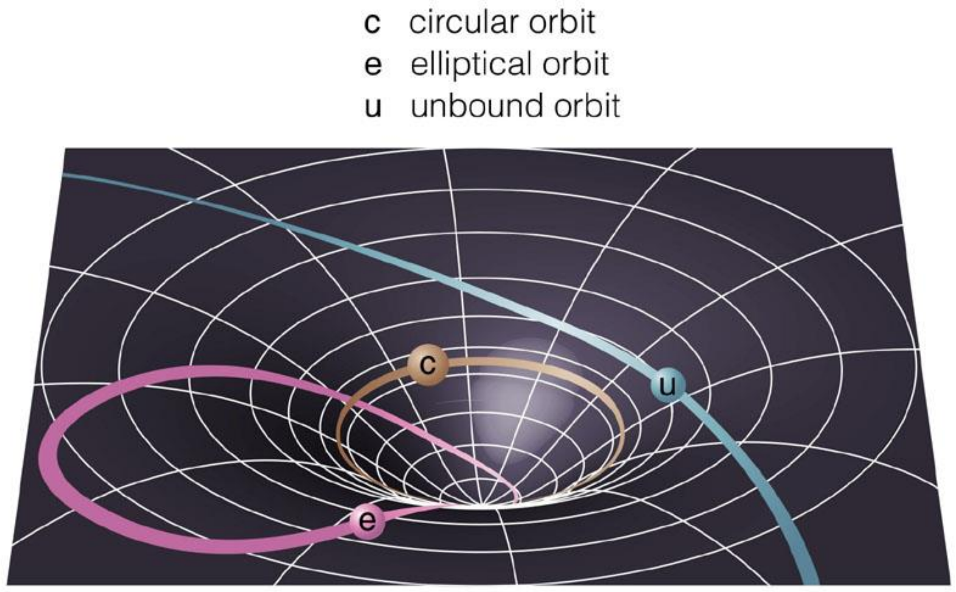
Kepler

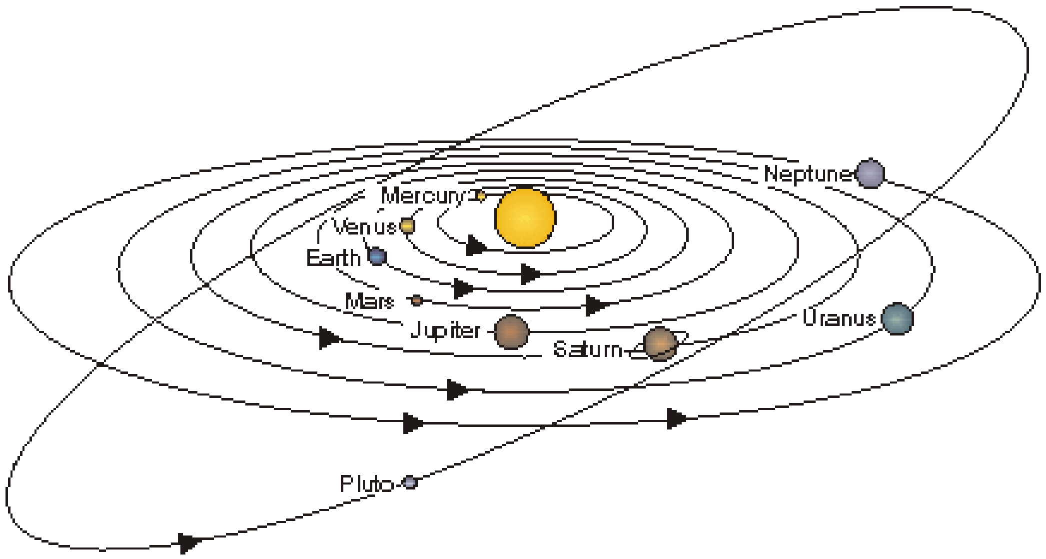
The three laws of planetary motion

General Theory of Relativity 1915

General Relativity
Gravity = Geometry

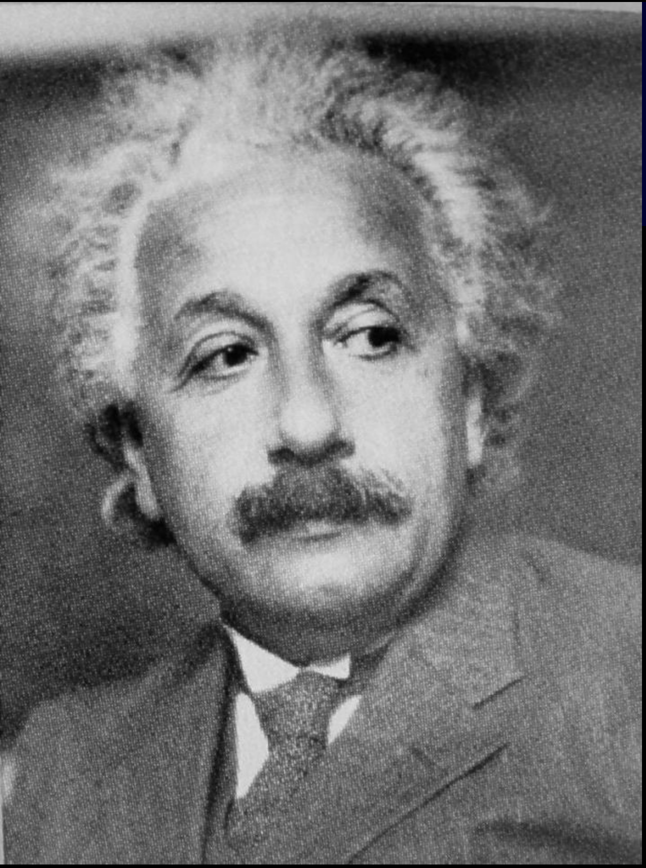
Newtonian Physics
Gravity = Force

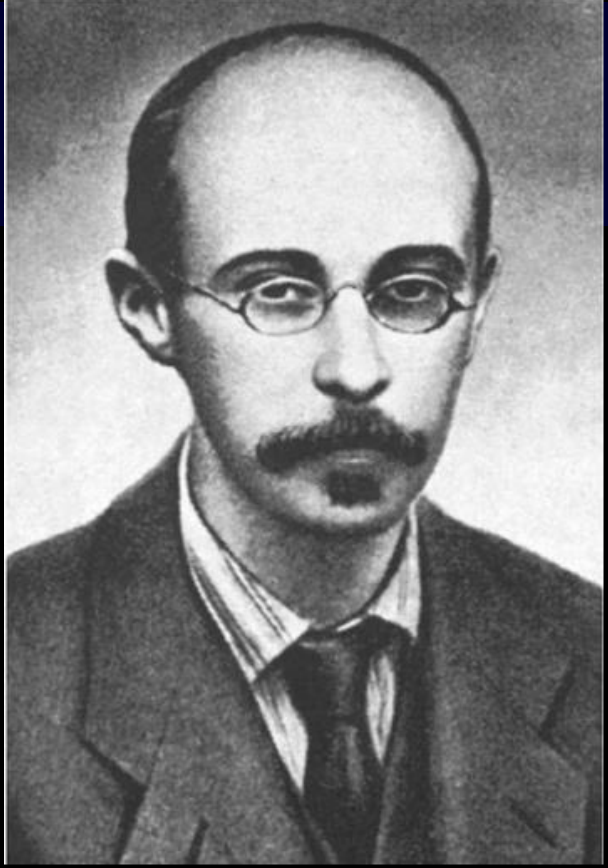




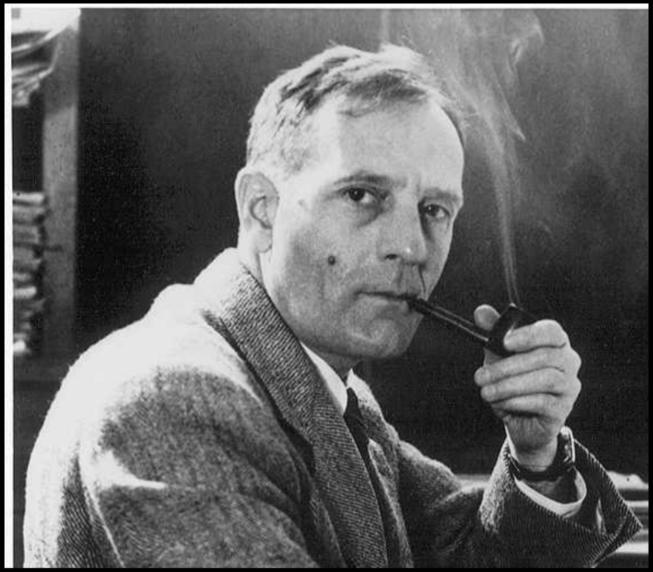
$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\vec{F} = G\frac{m_1 m_2}{r^2}\hat{r}$$





Einstein wanted a static Universe (1917), but the equations always gave an expanding or contracting solutions (Friedmann 1922, Lemaitre 1927,).



The Universe’s Expansion

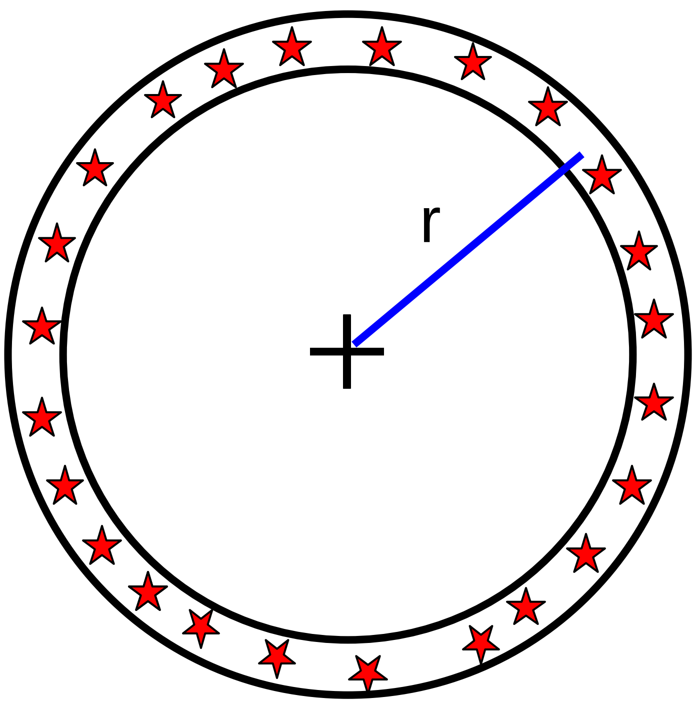
Key Observations in Cosmology

- Olbers' Paradox: why the night sky is dark?
- The Universe on large scales is homogeneous and isotropic.
- Hubble expansion.
- Cosmic Microwave Background Radiation.
- Abundance of elements in the Universe.
- The Universe's content (the dark sector)

Olbers' Paradox

Main assumptions:
1. The Universe is infinite
2. The distribution of stars is uniform throughout the Universe.
3. The Universe has lived for ever.
4. The luminosity of stars does not change with distance.

If that is the case, then the night sky should NOT be dark!



Assume that the number of stars per unit Volume is *n,* and that each radiates with a Fixed Luminosity, *L*. Hence, the contribution of a shell at radius r and with *dr* to the observed intensity is:

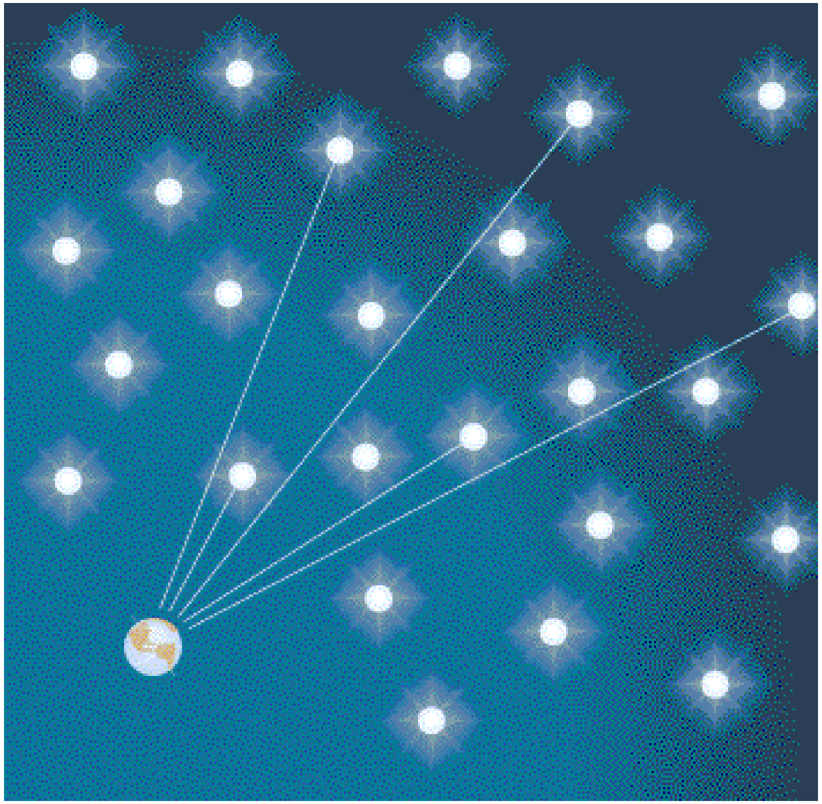
$$dJ = \frac{nL}{4\pi r^2} r^2 dr = \frac{nL}{4\pi}$$

Hence

$$J = \int_0^\infty dJ = \int_0^\infty \frac{nL}{4\pi} dr$$

Olbers' Paradox

The paradox is demonstrated in these two figure!
The question then remains, why is the night sky dark?
Obviously, the night sky is dark which means that one or more of our assumptions is wrong. In what follows we will examine them.



Olbers' paradox

Possible resolution for the paradox:

1. There is too much dust obscuration to see the rest of the stars.

This explanation is wrong since the dust after being heated for long enough time it will start emitting itself (blackbody rules) which we don't see.

1. The Universe has a finite number of stars.

Still, we know that the number of stars is large enough for the sky to be bright.

1. The star are not distributed uniformly.

This is possible (in say a fractal distribution) but against evidence!

1. The Universe expands and redshifts the stars out of the visible.

This is correct (see later) and partially explains the paradox

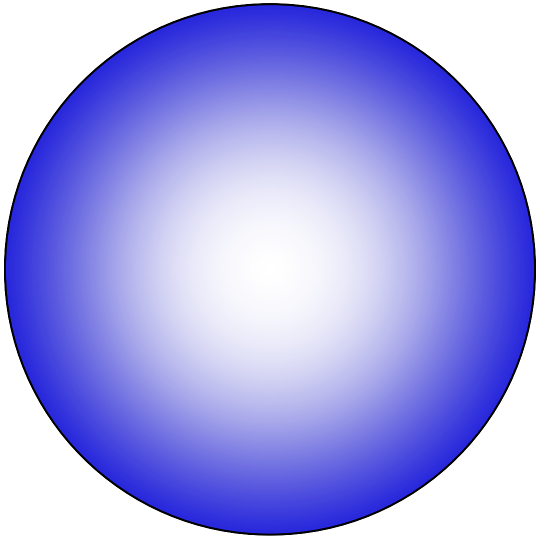
1. The Universe is young, distant stars' light hasn't reached us.

It turns out that this is the most important effect in resolving the paradox.

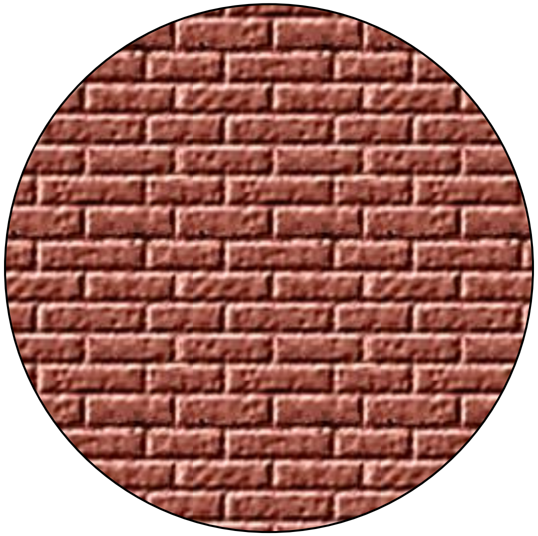
The Cosmological Principle

After the Copernican revolution the idea that we are at the center of the Universe has been discarded once and again. Pushing this notion further one can deduce that indeed there is no preferred point in the Universe nor there is a preferred direction. These types of arguments have lead to a basic assumption in Cosmology which states that the Universe is homogeneous and isotropic. These two assumptions combined constitute what is known as the “Cosmological Principle”.

Note that the two requirements are fundamentally different and having one does not imply the other. Here are two examples of distributions that are either homogeneous or isotropic:



Isotropic pattern with respect to the center

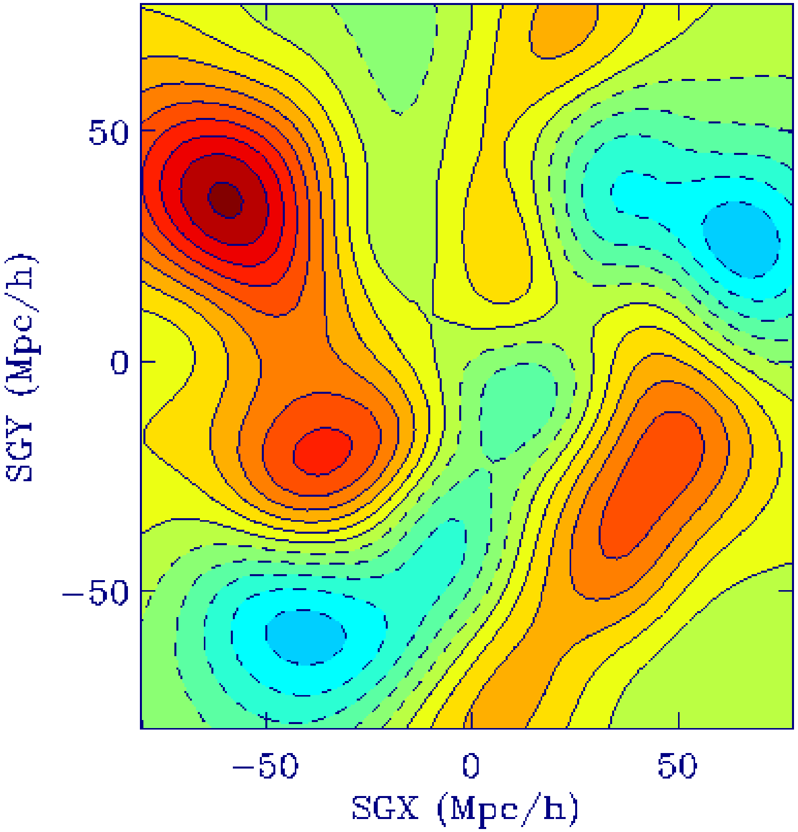


Homogenous but not isotropic

The Cosmological Principle

Though these assumptions are easy to make in theoretically, proving that they hold in the real Universe is a different matter altogether. In fact, the Universe around us is anything but homogeneous and isotropic. This is certainly the case on earth, the solar system and the Galaxy. But even on larger scales it is hard to argue that these two assumptions hold. The Universe actually is not isotropic and homogeneous even on scales of 100 Mpc.

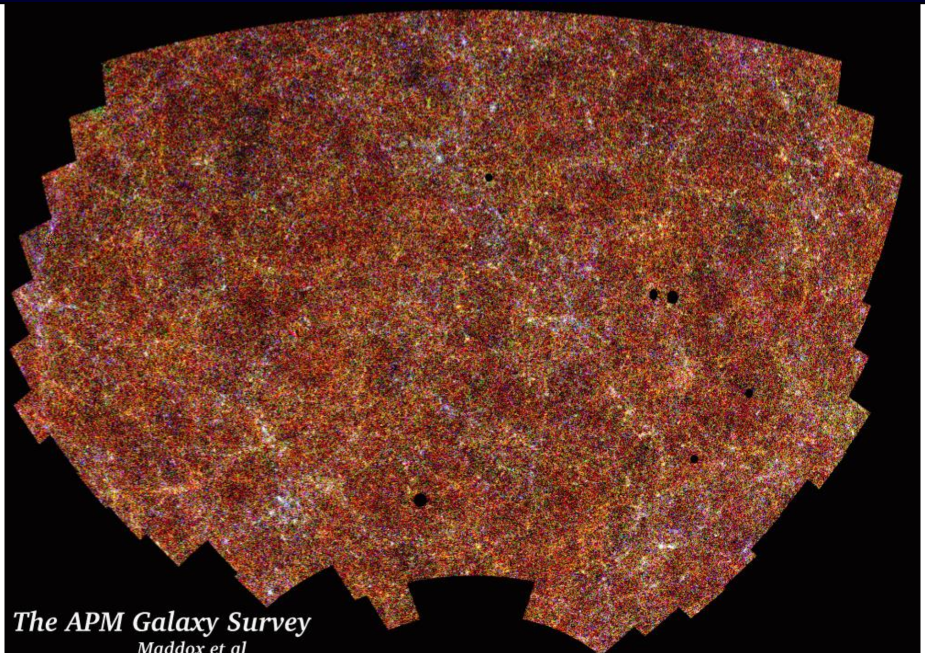
This image shows a reconstruction of the matter density field within a 200 Mpc/h box centered at the Milky way (located at the center). This reconstruction is done using galaxy peculiar velocity data and it clearly show deviation from the Cosmological principle.



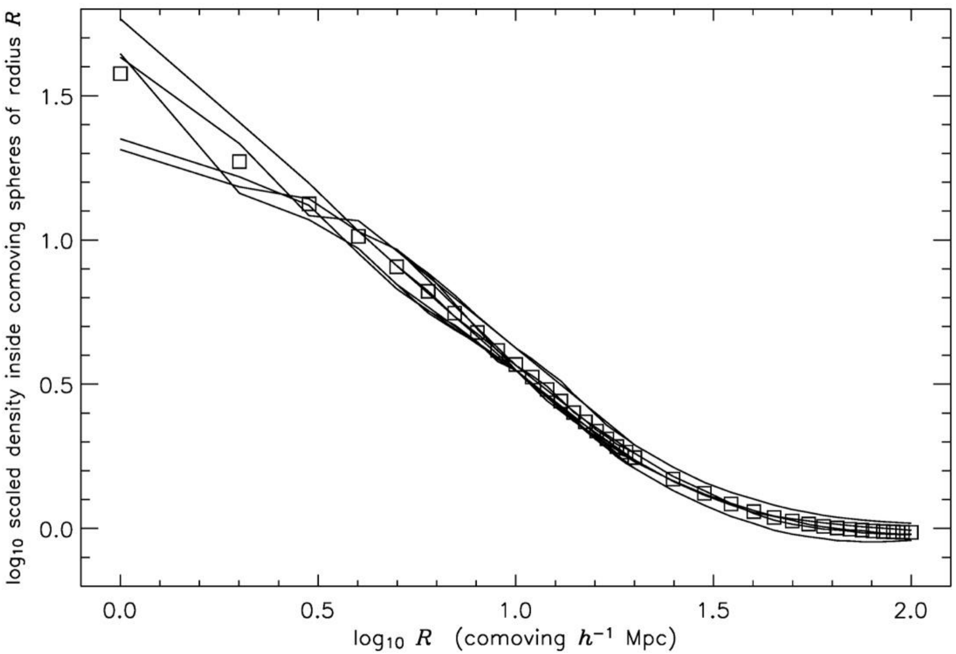
The Cosmological Principle

It is only on very large scales that the homogeneity and isotropy of the Universe becomes apparent.

The APM galaxy survey includes 3 million galaxies and spans very large scales and could be shown consistent with isotropic Universe on very large scales.



Sloan Digital Sky Survey has shown that the Universe also approached homogeneity on very large scales. The result shown in this figure is taken from Hogg et al.



The Cosmological Principle

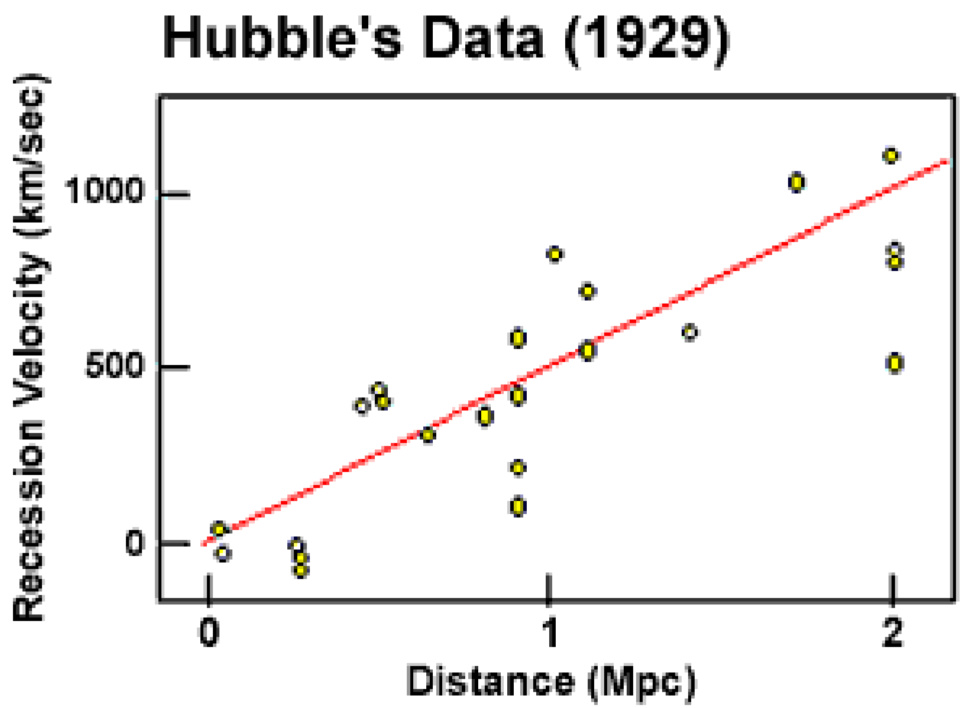
The most striking evidence for the Universe's isotropy comes from the cosmic microwave background measurement which we will discuss shortly.

All of these data have led to rephrasing of the cosmological principle which now states that the Universe is homogeneous and isotropic on sufficiently large scales.

These assumption together with Einstein's general theory of relativity (normally referred to as GR) led to the derivation of the key equations that describe the Universe's geometry given by the metric known as Friedmann–Lemaître–Robertson–Walker metric (sometimes the metric is called after a subset of these names, e.g., in the Ryden book it is called the Robertson-Walker metric).

The Hubble expansion

Evidence for an expanding Universe was first presented by the american astronomer Edwin Hubble who carefully constructed a distance ladder to measure extragalactic distances with the help of Cepheid period measurements. Hubble correlated the measured distances of the Cepheids in other galaxies with their redshifts. The redshift measures the line-of-sight speed of a certain galaxy to the observer. The redshift is defined as $z= (l-l_0)/l_0 = Dl/ l_0$ where l is the measured wavelength and l_0 is the rest-frame wavelength.

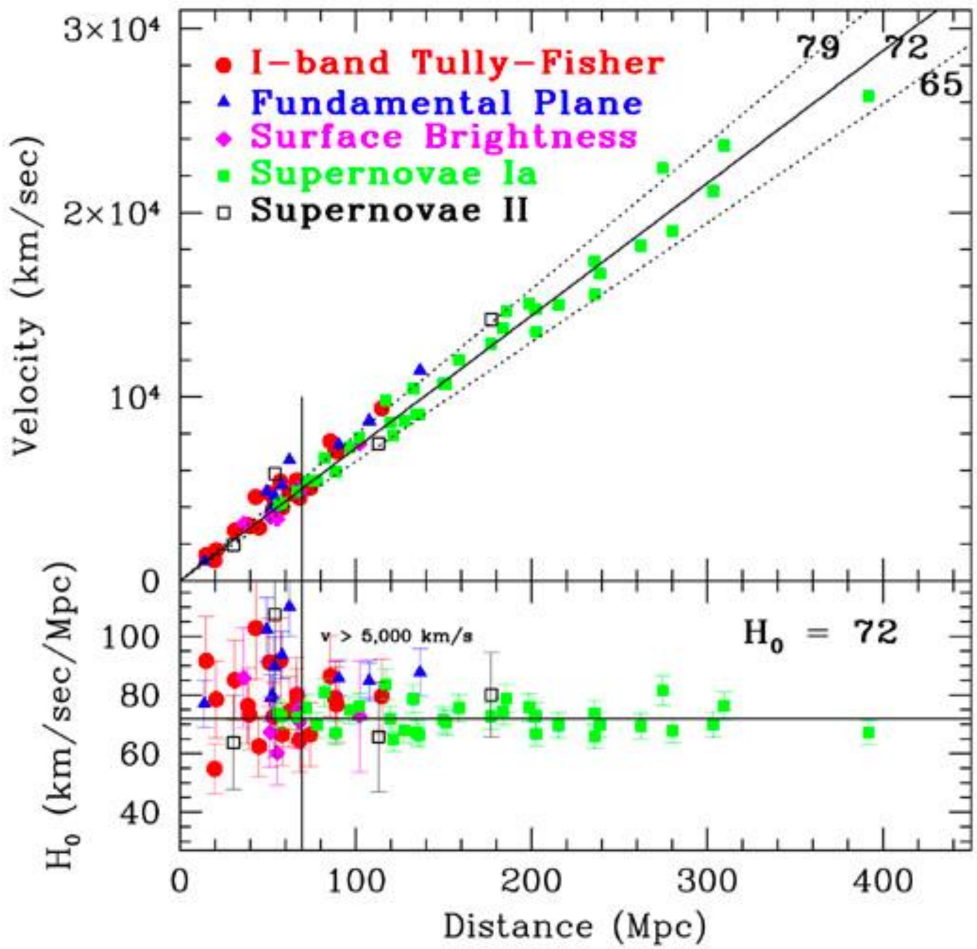


The Hubble expansion

From this correlation Hubble formulated his famous law where he interpreted the redshift as a doppler shift and related it to the line-of-sight velocity of the galaxy,

$$v = cz = H_0 r$$

Where c is the speed of light, and H_0 is a constant known now as the Hubble constant. Note that Hubble obtained a value of 500 km/s/Mpc for his constant about an order of magnitude larger than the modern value of this constant which is 72 km/s/Mpc (see figure).



The Hubble law

To put Hubble's observation more mathematically, the distance between two points increases with time in a manner proportional to their distance at a given time (similarity). Then the distance between two points could be written as:

$$\vec{r}_{12}(t) = a(t)\vec{r}_{12}(t_0)$$

Hence the relative velocity along the vector connecting between the two points is:

$$\dot{\vec{r}}_{12}(t) = \dot{a}(t)\vec{r}_{12}(t_0) = \frac{\dot{a}}{a}\vec{r}_{12} = H(t)\vec{r}_{12}$$

Obviously, since the expansion is homogeneous and isotropic this will be true for any two points in space.

Now, one can use this law to estimate the age of the Universe. Assume that there is no force between the two points in discussion then will be moving in constant velocity, hence the ratio between their distance to their relative velocity at a given time will give the time the two points spent getting away from each other:

$$t = \frac{r}{v} = \frac{1}{H_0}$$

The Hubble law

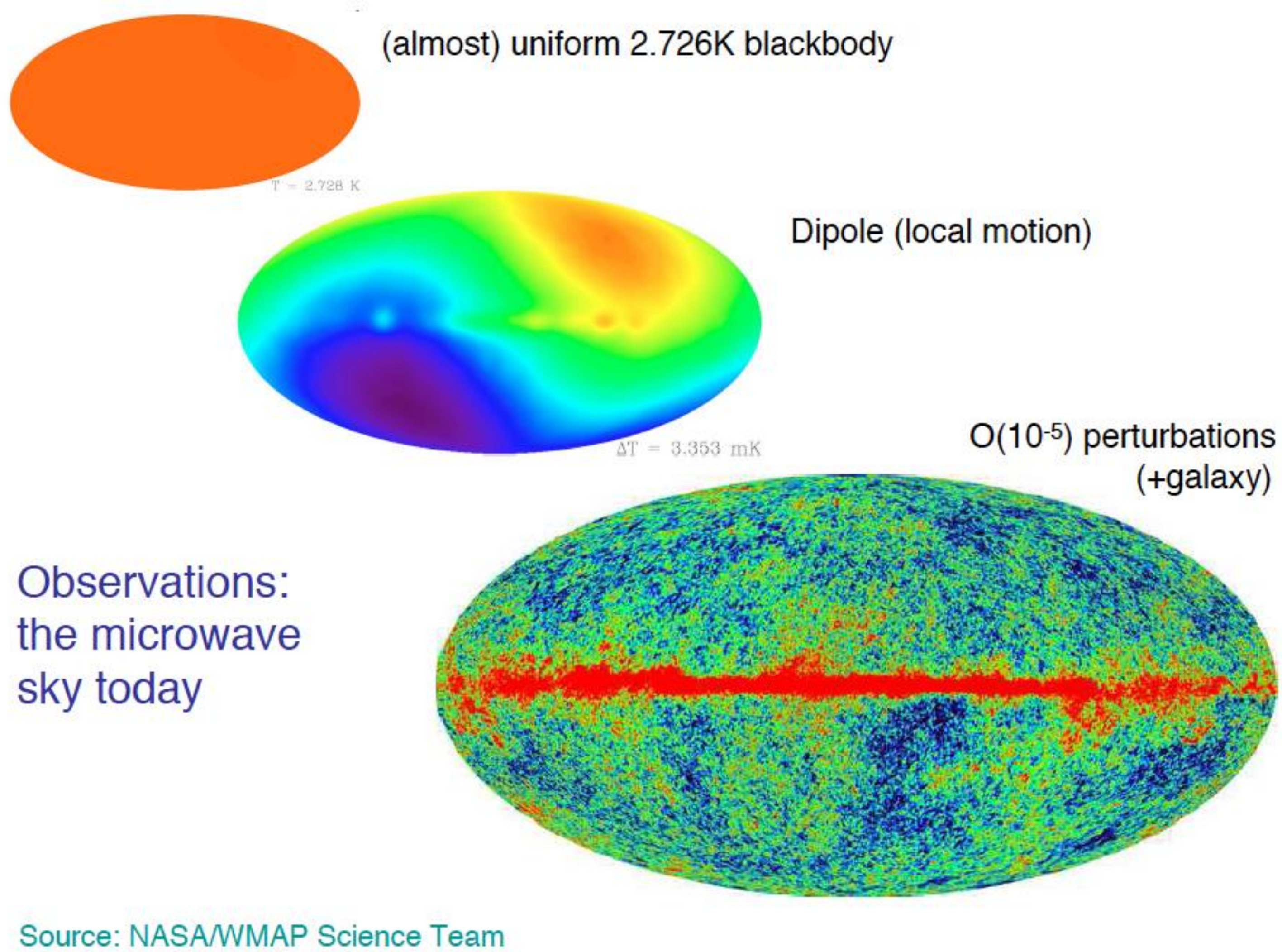
This gives about 14 Gyrs as the age of the Universe. This is roughly the correct age. One should note however that this relation is just an order of magnitude estimate since we know that gravity acts against the expansion of the Universe and the expansion rate of the Universe changes with time, i.e., H(t).

In general, Hubble law comes naturally in Big-Bang models. If one assumes General Relativity and the cosmological principle the result give a geometry of an expanding Universe. Such solutions were disliked by Einstein, and he invented the cosmological constant in order to reach a non-expanding Universe solution. We'll discuss this in detail in the coming lessons.

Hubble law exists also in a non-Big-Bang models such as the steady state model which was proposed by Herman Bondi, Thomas Gold & Fred Hoyle. This model assumes the, so called, perfect cosmological principle in which the Universe is not only symmetric under translations (homogeneous) and rotation transformation (isotropy) but also with time. In such a Universe the properties of the Universe do not change with time (constant density, Hubble constant, etc.).

The Steady State model was one of the most serious contender for a cosmological theory and by the middle sixties it was the preferred model for most astronomers. However, the discovery of the Cosmic Microwave Background radiation, a thermal relic from the early Universe, which clearly tipped the balance in favor of the Big Bang model.

CMB observations



Two Nobel prizes dedicated to this field: discovery, blackbody radiation and fluctuations

CMB: Blackbody radiation

Blackbody radiation arises when an idealized body reaches thermal equilibrium between the incident electromagnetic radiation which is total absorbed, i.e., perfect absorber, and then re-emits for obvious reasons the same amount of energy, that it has just absorbed. The spectrum of the emitted radiation is a very specific spectrum, known as blackbody radiation. In such body the energy density of photons in the frequency range between ν and $\nu + d\nu$ is given by the formula:

$$\varepsilon(\nu;T)d\nu = \frac{8\pi h}{c^3}\frac{\nu^3 d\nu}{\exp h\nu/kT - 1}$$

The total energy density of blackbody radiation is $\varepsilon_\gamma = \alpha T^4$, called Stefan-Boltzmann Law, where

$$\alpha = \frac{\pi^2}{15}\frac{k^4}{\hbar^3 c^3} = 7.56 \times 10^{-16}\,\mathrm{J\,m^{-3}\,K^{-4}}$$

One can also calculate the number density of photons from the Planck distribution which is,

$$n_\gamma = \beta T^3$$

where the numerical factor is:

$$\beta = \frac{2.404}{\pi^2}\frac{k^3}{\hbar^3 c^3} = 2.03 \times 10^7\,\mathrm{m^{-3}\,K^{-3}}$$

It is also useful to remember that photon pressure is: $P_\gamma = \varepsilon_\gamma/3.$

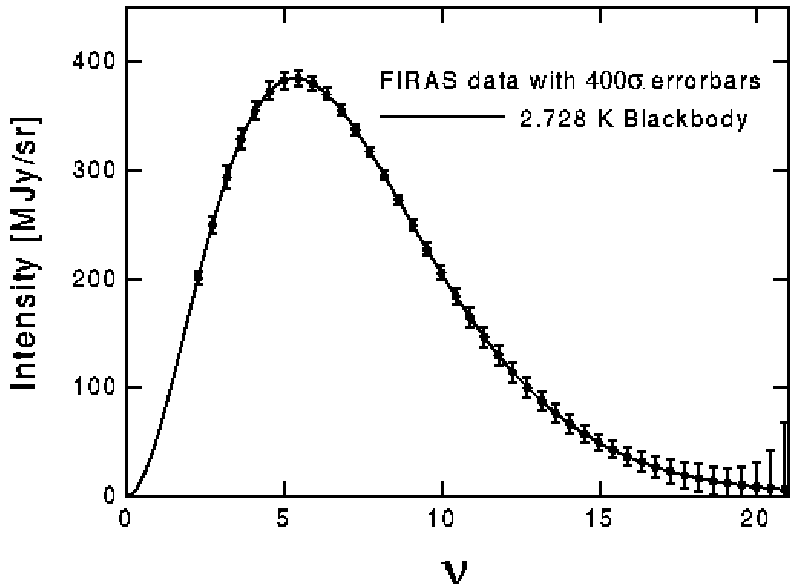
CMB global properties

Assuming no external sources of energy (i.e., heating), then one can use the first law of thermodynamics *dQ=dE+PdV* to obtain *dE= -PdV*. Now remember th $E = \varepsilon_\gamma V = \alpha T^4 V$ and $P = P_\gamma = \alpha T^4/3$

Substituting the two terms in the first law, equation one obtains the following equation:

$$\frac{1}{T}\frac{dT}{dt} = -\frac{1}{3V}\frac{dV}{dt}$$

Remember that V ~ a^3(t) which gives the very nice result $T(t) \propto a(t)^{-1} \ (= 1 + z)$ this could be simply related to (1+redshift) which makes it easy to obtain the photon temperature of the Universe at any redshift provided the local temperature is measured.



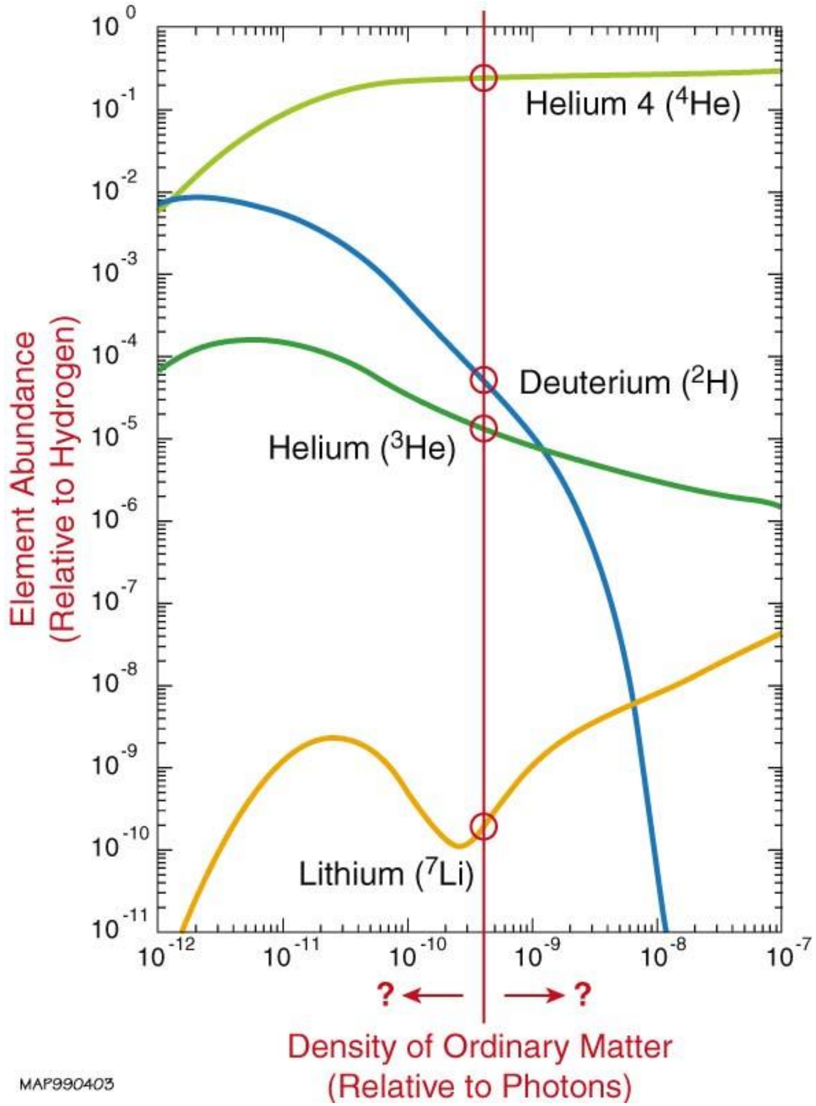
The COBE satellite has indeed measured the CMB temperature to be 2.726K.

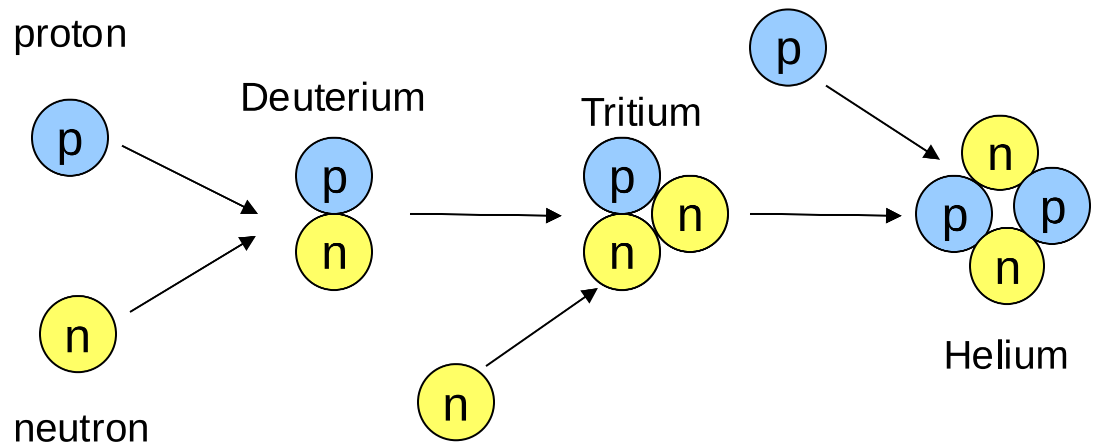
$$\varepsilon_\gamma = 4.17 \times 10^{-14}\,\mathrm{J\,m^{-3}}$$

$$n_\nu = 4.11 \times 10^8 \ m^{-3} = 411 \ cm^{-3}$$

The abundance of elements in the Universe

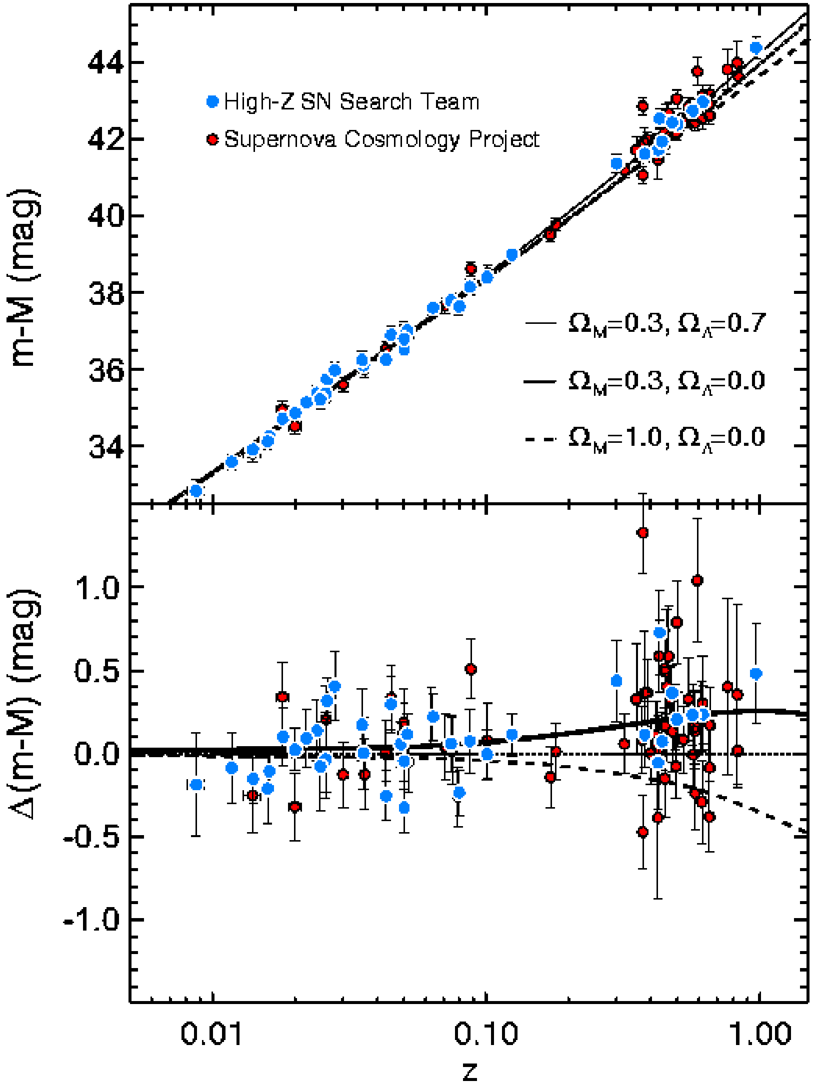
- Abundance of elements (nucleosynthesis)





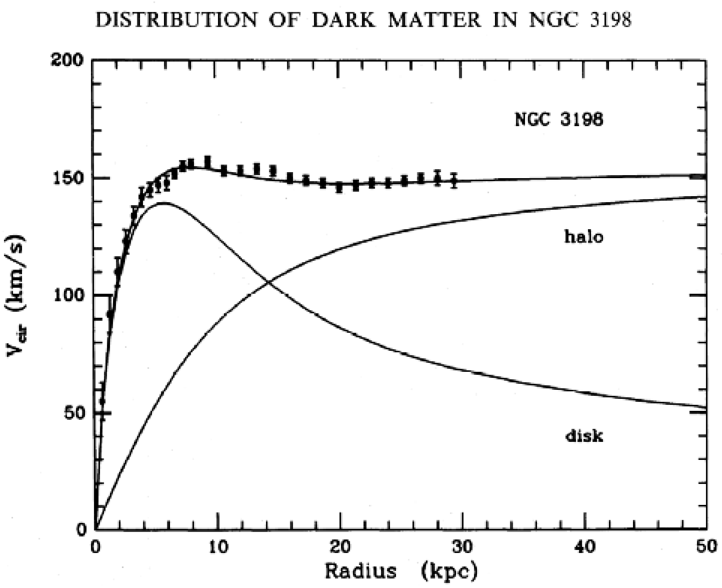
The Universe contains mainly Hydrogen and Helium

Evidence for dark matter and dark energy

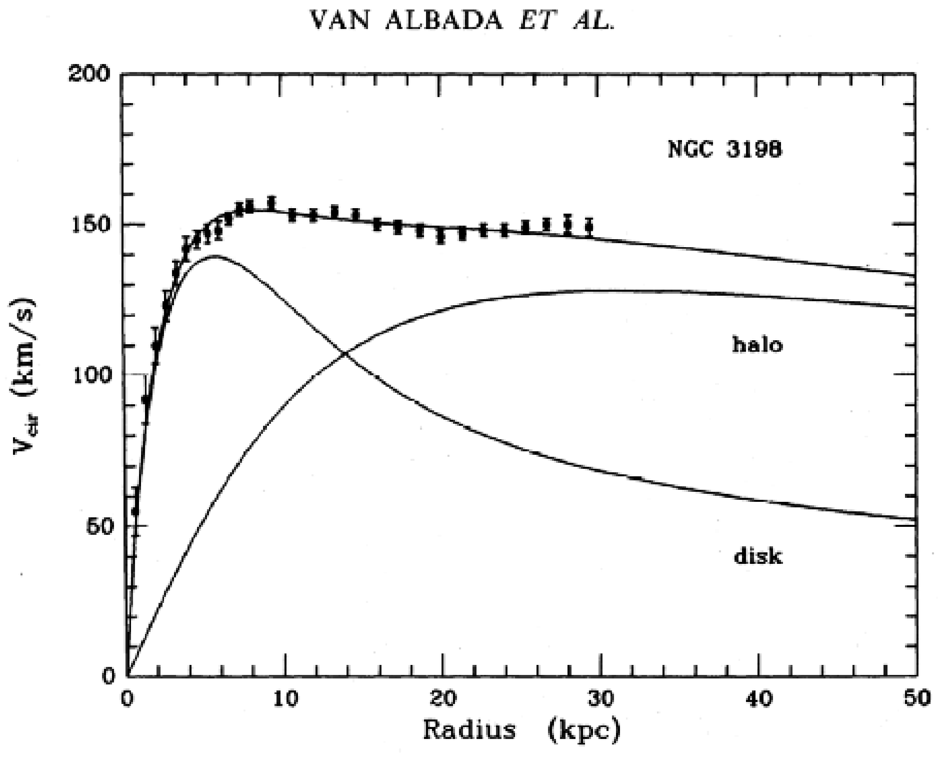


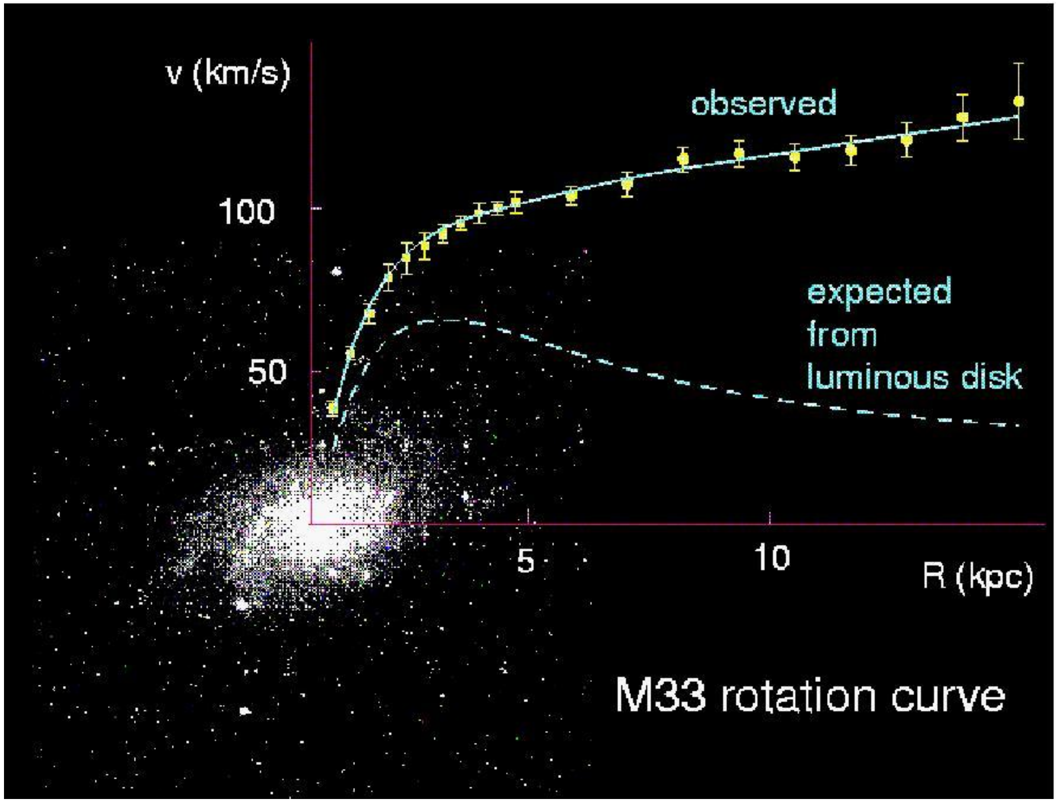
Dark matter

- Galactic rotation curves
- Velocity dispersions of galaxies
- Galaxy clusters and gravitational lensing
- Cosmic microwave background
- Sky Surveys and Baryon Acoustic Oscillations
- Lyman-alpha forest
- Structure formation
- ……



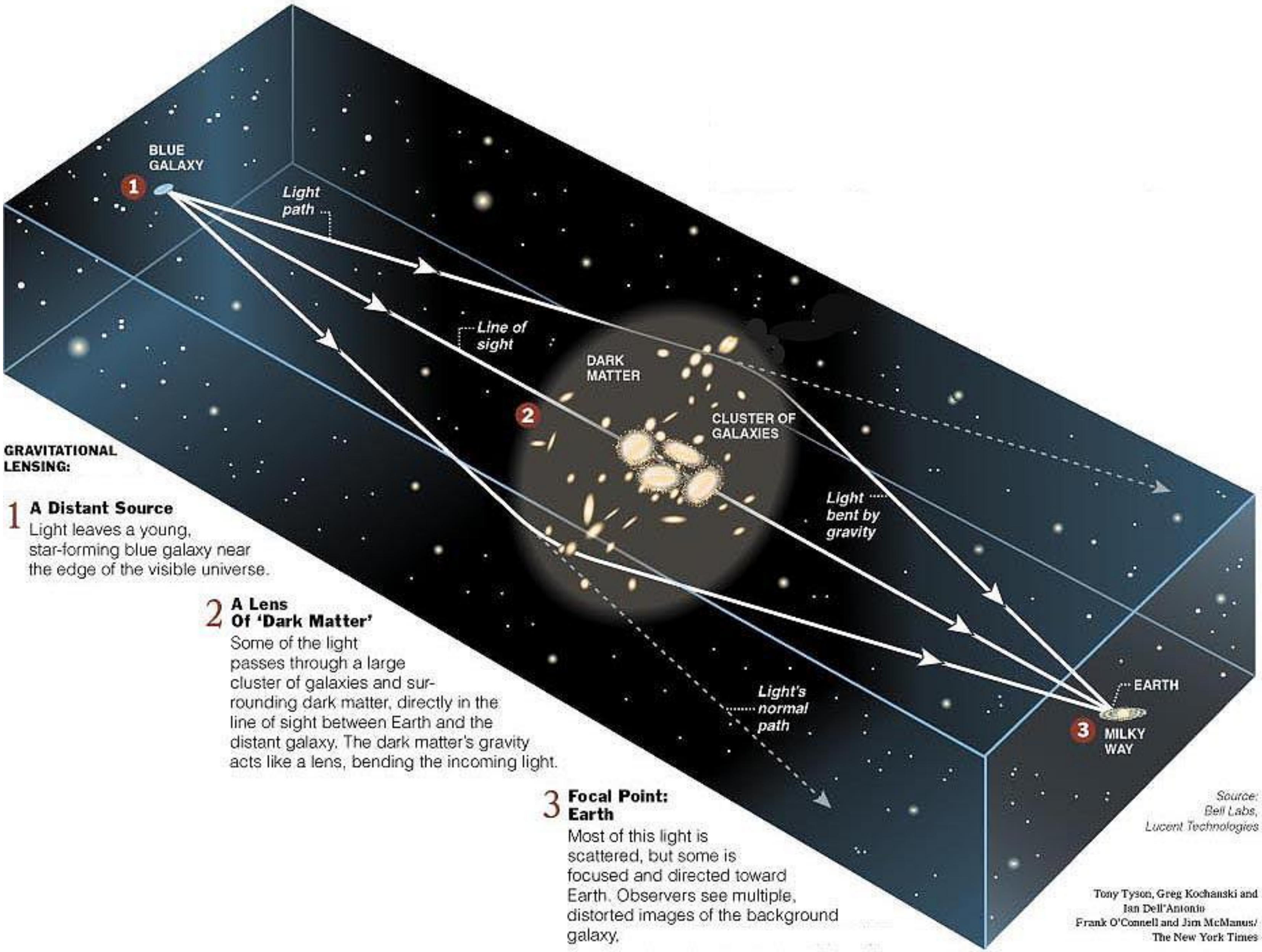
Dark Matter


VAN ALBADA ET AL.
NGC 3198
halo
disk
V_{cir} (km/s)
Radius (kpc)
0
50
100
150
200
0
10
20
30
40
50

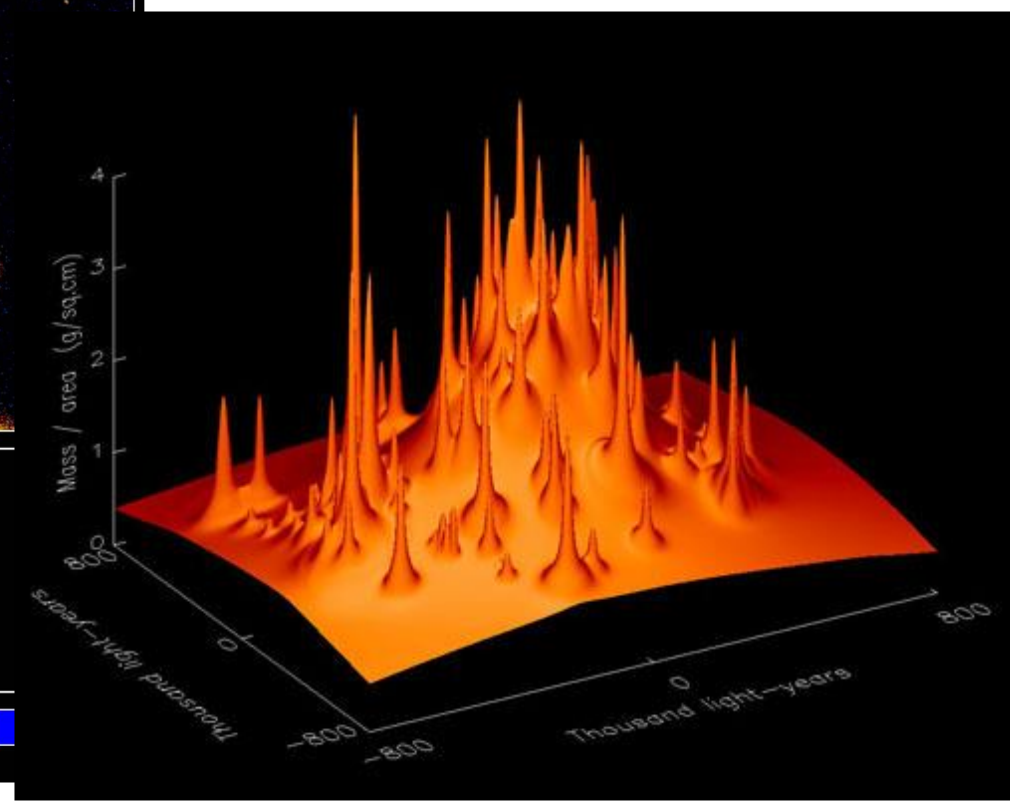

v (km/s)
observed
expected
from
luminous disk
100
50
5
10
R (kpc)
M33 rotation curve

Galaxy Clusters

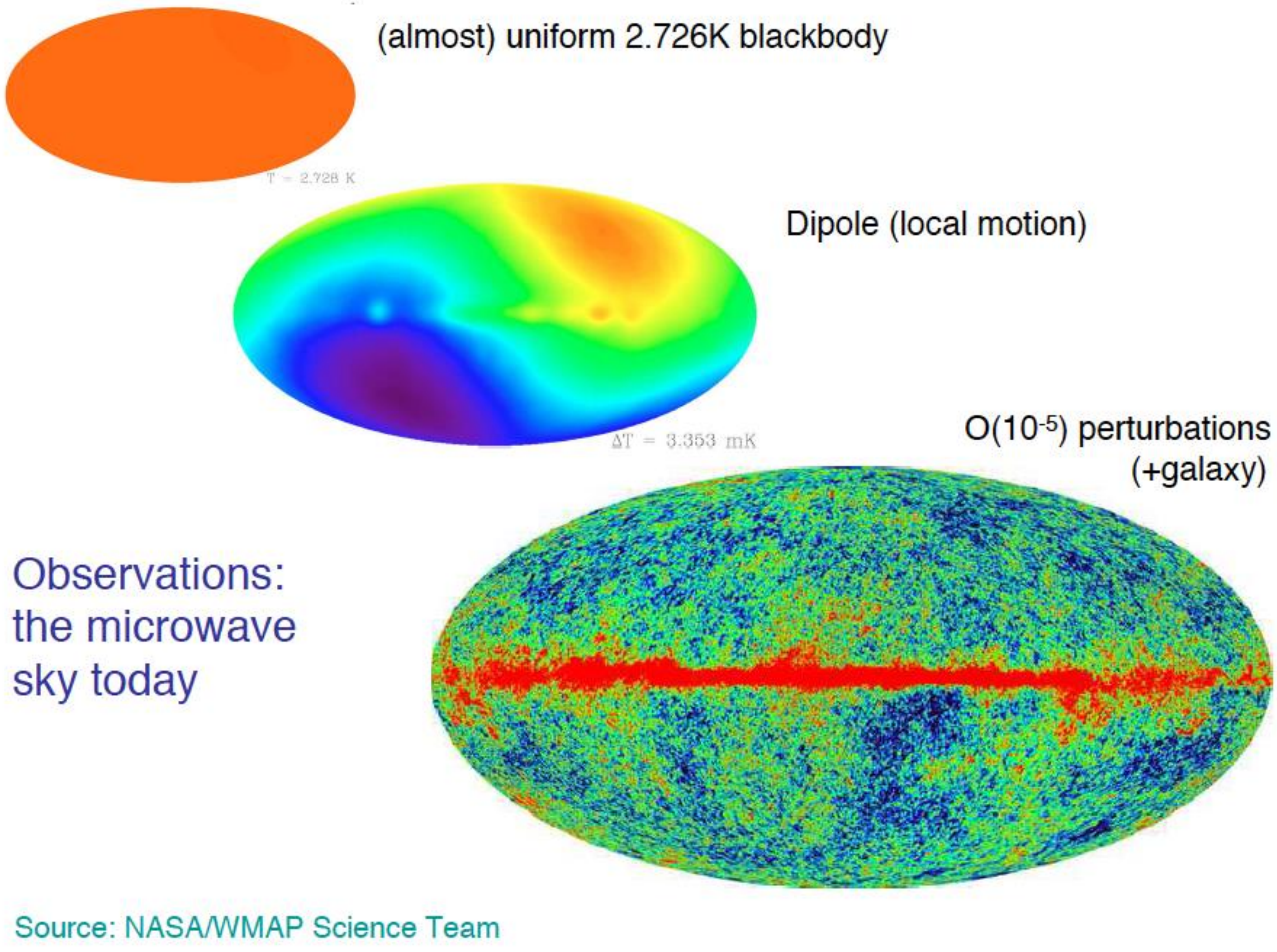
Gravitational lenses



Galaxy Clusters



CMB observations



Two Nobel prices dedicated to this field: discovery, blackbody radiation and fluctuations

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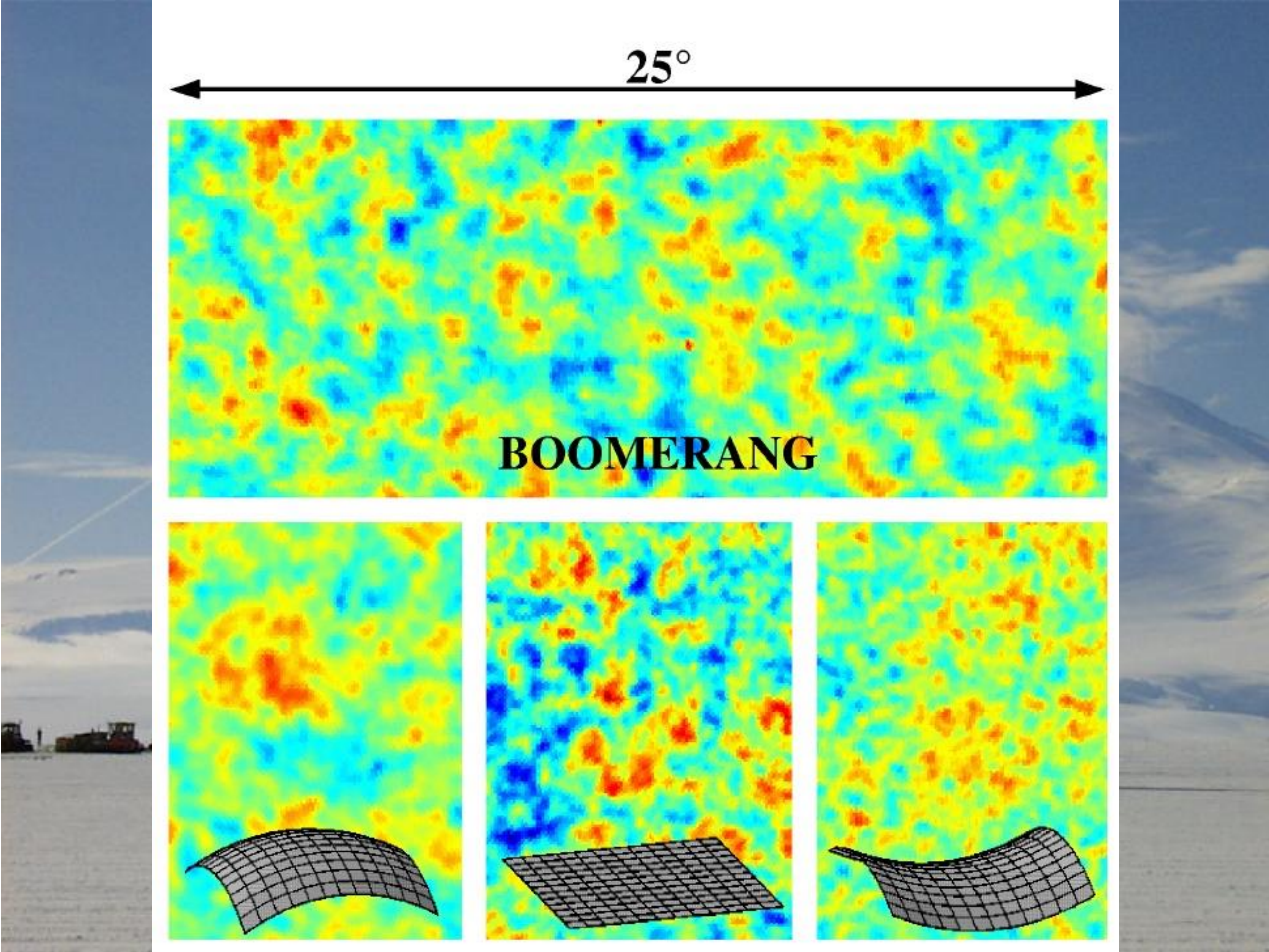
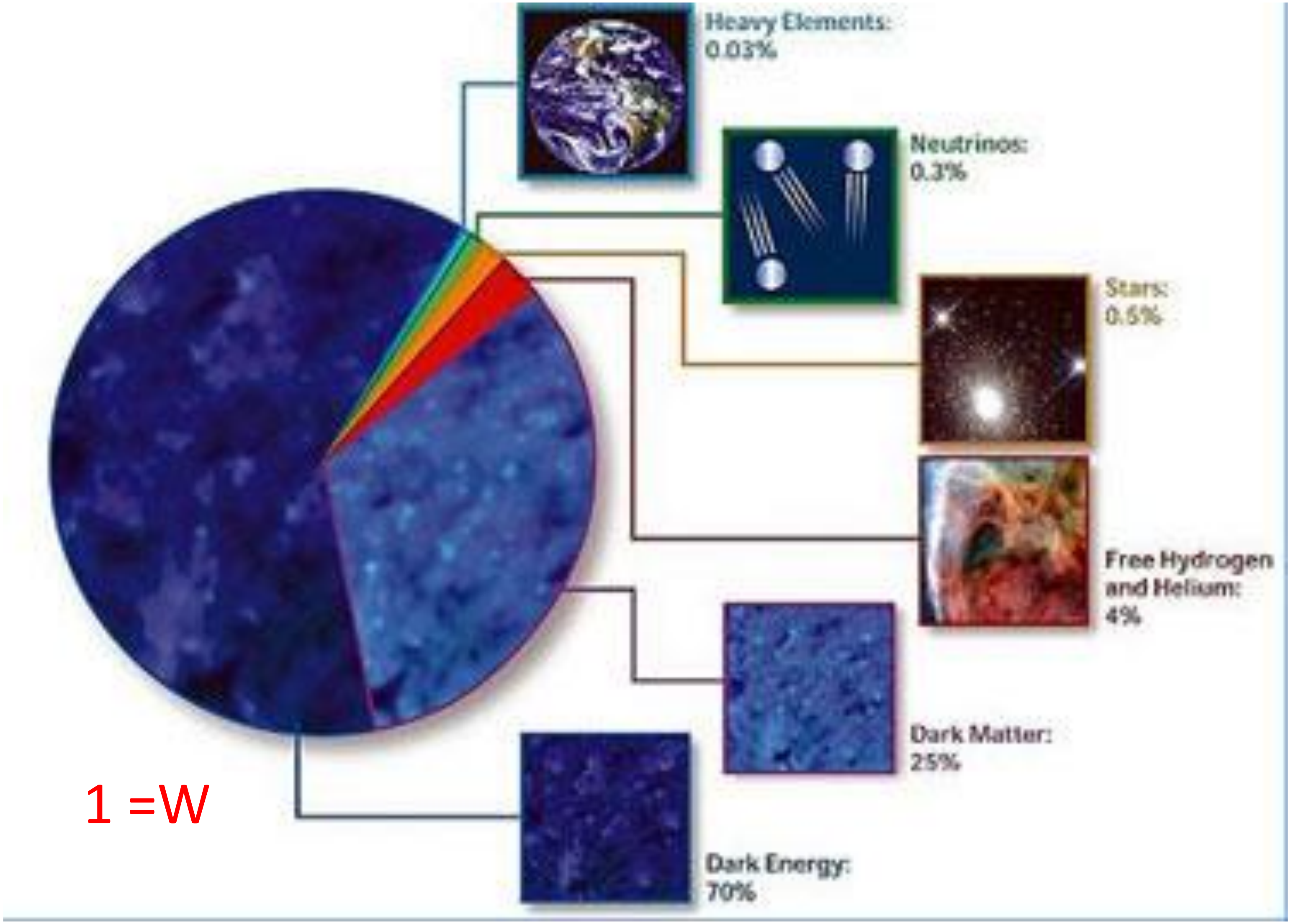

25°
BOOMERANG

$$\Omega = 1$$

The total energy of the Universe is zero

The Cosmic Energy Budget

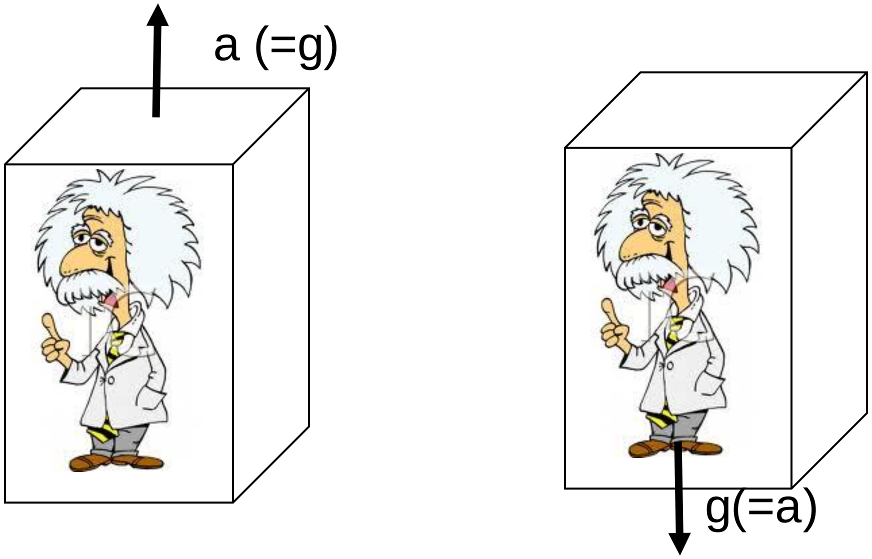


1 =W

Geometry of the Universe

The assumption in this course is that the student did not learn general relativity. Therefore, derivation of the geometric, dynamical and thermal properties of the Universe will not be done rigorously, but rather, presented in an ad hoc manner.

General relativity is a generalization of special relativity to include the effect of gravity. In its formulation Einstein (1915) extended special relativity principles to include the Equivalence principle which states that gravitational mass and inertial mass are equivalent.



Even Einstein can not tell whether the elevator is accelerating upwards or he is pushed downwards by a gravitational force.

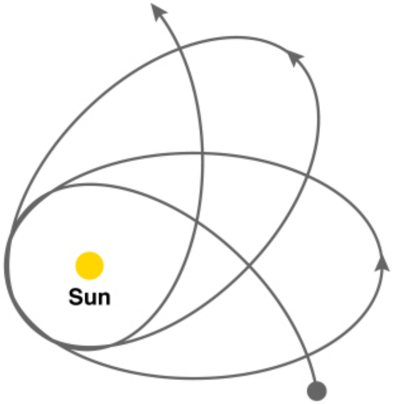
Einstein vs. Newton

As a result of this insight (and after about 7 years of trying) Einstein was able to construct a very powerful theory that describes the behavior of gravity. This theory led to a fundamental change in the interpretation of the laws of dynamics.

In the case of classical mechanics Newton laws allow us to solve for the motion of particle under gravity by:

1- Newton gravity law determines the force felt by a body with mass M; $F=GMm/r^2$.

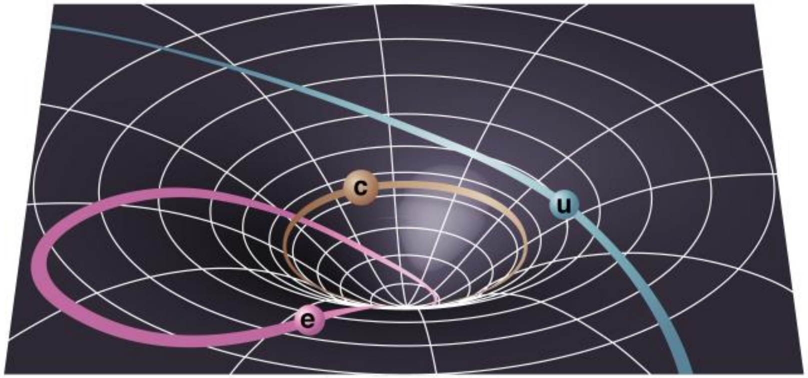
2- This force in turn determines how the body moves in space ($F=ma$).



However, in the case of General Relativity the motion of a certain is determined as follows:

1- the Mass, or more accurate energy, distribution determines the geometry of spacetime, i.e., the curvature of space-time at each point.

2- In turn, this geometry determines how the body is moving.



In this course we will use the Newtonian interpretation of motion and speak of real forces and motions. This is good enough for our purpose in this course, although one should bear in mind that this is not an accurate description of nature, which is described much better with General Relativity and Einstein's interpretation of it.

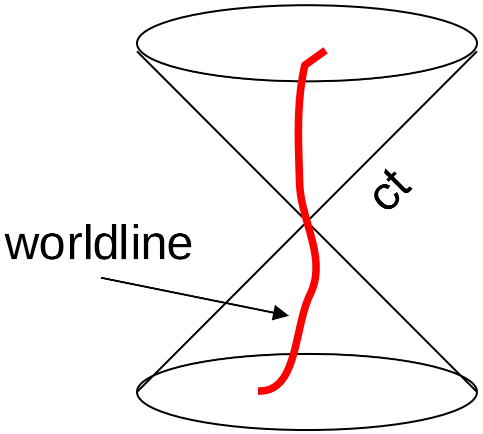
Geometry of the Universe

In special relativity the geometry is described by the so-called Minkowski metric:

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

Minkowski metric is flat, i.e., the spatial coordinates describe a Euclidean (flat) geometry.
A particle in such a geometry has three possible worldlines (trajectories or geodesics):

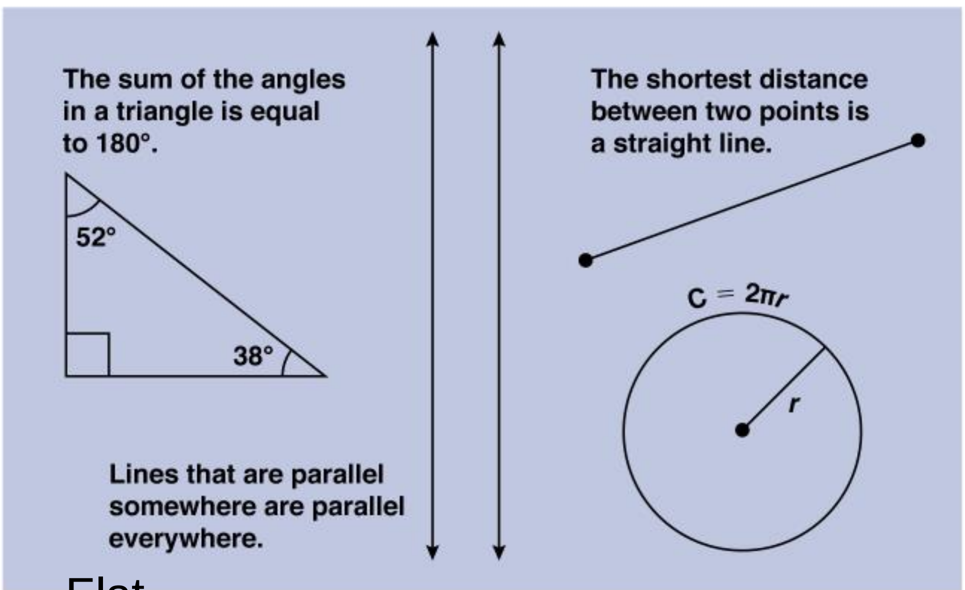
1. $ds^2<0$ time-like
2. $ds^2=0$ light-like (in GR this will be called null geodesy)
3. $ds^2>0$ space-like



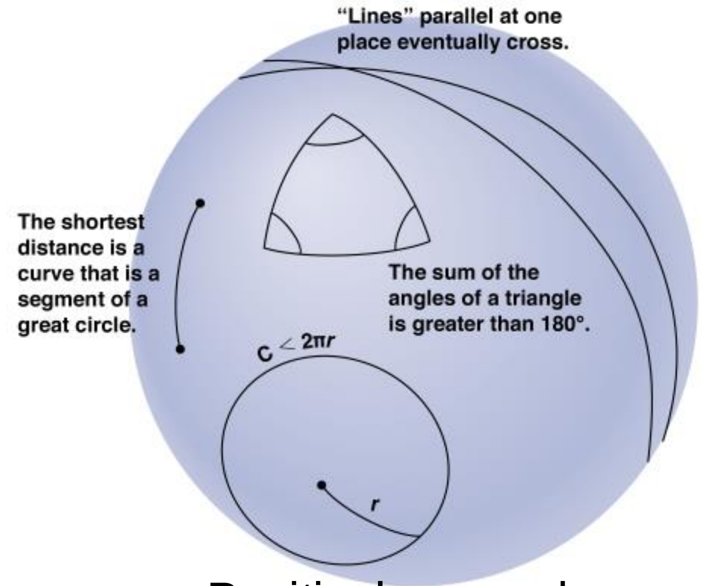
In a spherical coordinates one can write the Minkowski metric as:

$$\begin{aligned} ds^2 &= -c^2dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \\ &= -c^2dt^2 + dr^2 + r^2 d\Omega^2 \end{aligned}$$

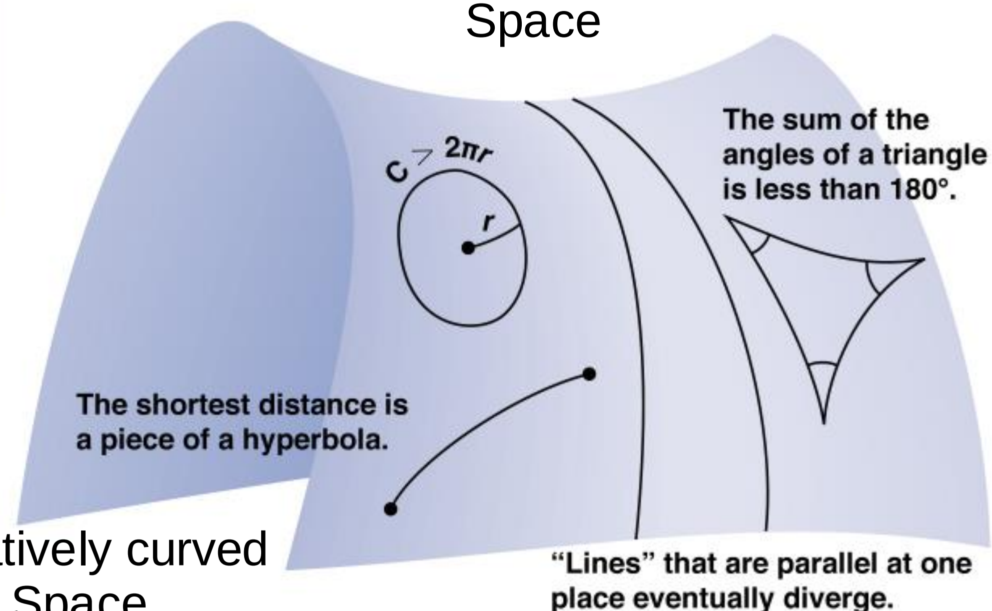
Curved Spaces



Flat Space



Positively curved Space

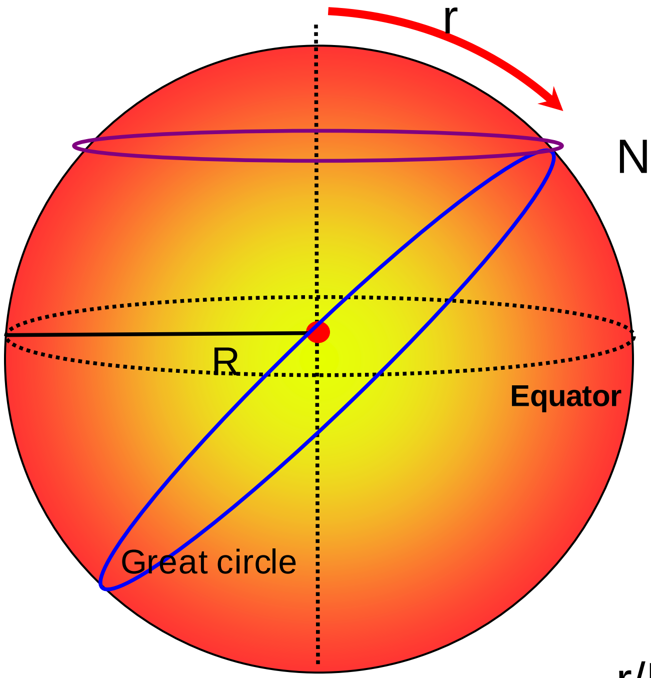


Negatively curved Space

Curved Spaces

The modification of the Minkowski metric in curved spaces requires GR, however, here will try to derive a simple generalization of this metric for a homogeneous and isotropic Universe (remember, the cosmological principle).

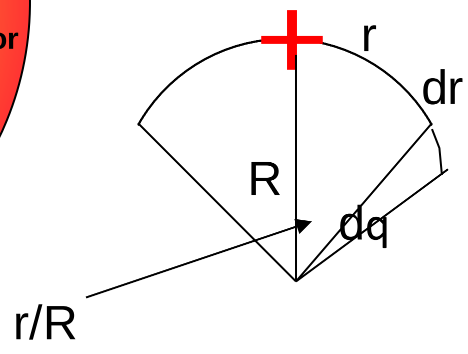
Positively curved surface: Imagine a sphere with radius R. A great circle drawn in the surface of the sphere is a circle that shares the same center of the sphere. For example, on the surface of Earth such a circle would be the equator or any circle which passes through both the south and north poles. Now a line element along a great circle with arc length r from the pole is given by the following metric:

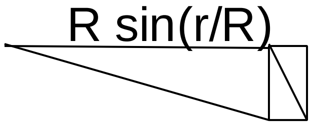


$$ds^2 = dr^2 + R^2 \sin^2(r/R) d\theta^2$$

Now, this could be generalized as follows:

$$ds^2 = dr^2 + R^2 \sin^2(r/R)[d\theta^2 + \sin^2\theta d\phi^2]$$





Curved Spaces

In case the space is negatively curved one can show that a line element could be written as:

$$ds^2 = dr^2 + R^2 \sinh^2(r/R)[d\theta^2 + \sin^2\theta d\phi^2]$$

The three metric for a homogeneous and isotropic three-dimensional space could be simply written as:

$$ds^2 = dr^2 + S_\kappa(r)^2 d\Omega^2$$

Where

$$S_\kappa(r) = \begin{cases} R\sin(r/R) & \text{if} \quad \kappa = +1 \\ r & \text{if} \quad \kappa = 0 \\ R\sinh(r/R) & \text{if} \quad \kappa = -1 \end{cases}$$

and the k is called the curvature constant and it attains a value of 0 for a flat space, +1 for positively curved space and -1 for a negatively curved space. If the space is curved then the quantity R is called the radius of curvature.

One could also express this metric differently by choosing, for example, another radial coordinate, $x = S_\kappa(r)$, then the metric is written as:

$$ds^2 = \frac{dx^2}{1 - \kappa x/R^2} + x^2 d\Omega^2$$

The two forms of writing the metric are equivalent and differ only in the choice of radial coordinate.

Geometry of the Universe: The Friedmann-Lemaître-Robertson-Walker Metric

Generalization of the 3D curved metrics to space-time (4D) metric is very simple. The spatial part is the similar as the 3D case, except that one needs to take into account the expansion of the Universe. Therefore, the metric is either:

$$ds^2 = -c^2dt^2 + a(t)^2\left[\frac{dx^2}{1-\kappa x/R_0^2} + x^2d\Omega^2\right]$$

The addition of the scale factor, *a*(t), here accounts for the expansion of the Universe. Also notice the change in R to R_0 which now indicates that the space is uniformly curved. Obviously, the metric could also be written as:

$$ds^2 = -c^2dt^2 + a(t)^2[dr^2 + S_\kappa(r)^2d\Omega^2]$$

The time coordinate is called cosmological proper time or in short cosmic time; k is the curvature constant mentioned earlier.

Historic remark: The first to come with such a solution for a homogeneous and isotropic space was the russian physicist Alexander Friedmann in 1922-1924, though his work remained unnoticed. In 1927, Georges Lemaître, a Belgian priest and lecturer at the University of Leuven, rediscovered the same solution which he published in a Belgian journal. Eddington noticed this paper and republished it in English in 1930. In the 1930 Robertson (US) and Walker (GB) also found the same metric. The name of the metric is often named after a subset of the 4 names mentioned above. In Ryden's book the metric is called the Robertson-Walker metric.

FLRW-Metric: Basic implications

Proper distance, is the distance between the observer and a certain point in space, say a galaxy, at a fixed time, t:

$$ds^2 = a(t)^2[dr^2 + S_\kappa(r)^2 d\Omega^2]$$

Since for a given galaxy both q and f are constants the equation becomes simply

$$ds = a(t)dr$$

Integrating this equations gives:

$$dp(t) = a(t) \int_0^r dr = a(t)r$$

Which brings immediately to mind, Hubble law:

$$\dot{d}p(t) = \dot{a}(t) \int_0^r dr = \frac{\dot{a}}{a} d_p \equiv H(t) d_p$$

At $t=t_0$ the Hubble parameter becomes Hubble constant, H_0. Notice, that Hubble law is valid at every instance in cosmic time but with varying value of the constant.

The fact that we recover Hubble law is not surprising since we designed our metric to do so. However, in proper GR derivation this results comes out naturally as a genuine prediction of the theory.

FLRW-Metric: Basic implications

Another important implication of this result is that at a large enough distance from us galaxies will recede at a speed faster than that of light. It so happened that this distance for a Hubble constant of 70 km/s/Mpc is about 4300 Mpc.

The fact that the speed of such galaxies is larger than the speed of light in not a major issue as in GR, unlike in Special Relativity, there is no constraint on the relative velocity larger than c except in a local sense. In other words, special relativity applies locally to space time but not globally. There is no restriction in GR in the expansion rate of space itself.

Now we move to derive the null geodesic along a direction connecting the observer to a distant object with the purpose of. In this case we'll take q and f are constants and the condition is that ds=0 (just like light moves on the light cone in special relativity). One can easily show that (see exercise) :

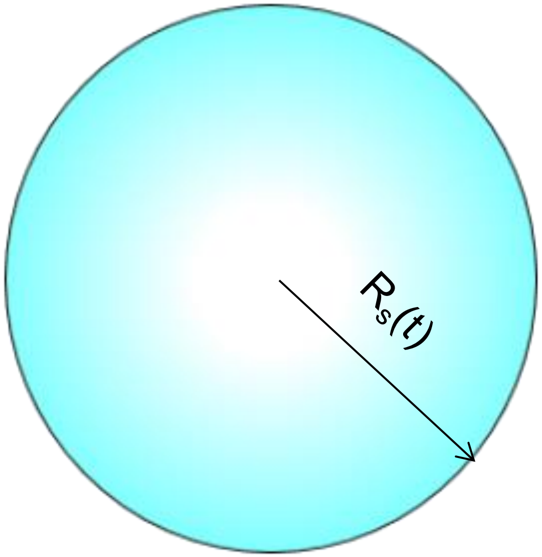
$$1 + z = a(t_0)/a(t) = 1/a$$

where we chose $a(t_0)$ to be one.

Cosmic Dynamics

In order to determine the dynamics of the Universe in full rigor one must bring to bare the full power of GR, specifically using the so-called Einstein equation which describes gravity in space-time that is being curved as a result of the matter and energy distribution in it, much like Poisson equation in classical gravity. However, since here we assume no previous knowledge in GR we will derive the main equations that describe the dynamics of the Universe by using Newtonian description. This approach will get us far but still we need to assume a number of things that arise naturally in the GR derivation of the equations.

The equations that we are going to derive for a homogeneous and isotropic Universe were initially derived by Friedmann himself but were not given enough attention until the discovery of Hubble law 7 years later.



Imagine a uniform sphere of mass M_s and radius $R_s(t)$ and density $\rho(t)$. Conservation of energy gives the following equation for the expansion velocity:

$$\frac{1}{2}\left(\frac{dR_s}{dt}\right)^2 = \frac{GM_s}{R_s(t)} + U$$

Here U is the total energy of the system (constant of integration)

Friedmann's Equation

Now we recall that the distribution of matter within the sphere is uniform and depends only on time, yielding:

$$M_s = \frac{4\pi}{3}\rho(t)R_s(t)^3$$

We could also express the radius of the sphere in relation to its radius at a specific time (comoving radius) which gives,

$$R_s(t) = a(t)r_s$$

Putting everything together gives:

$$\frac{1}{2}r_s^2\dot{a}^2 = \frac{4\pi}{3}Gr_s^2\rho(t)a(t)^2 + U$$

With some manipulations we obtain the so called Friedmann equation in its Newtonian form:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t) + \frac{2U}{r_s^2}\frac{1}{a(t)^2}$$

It is clear that the integration of this equation depends on whether U is positive, zero or negative. For a $U>0$ the sphere will expand for ever, for $U=0$ it will stop at infinity and for $U<0$ it will re-collapse.

Friedmann's Equation

Now we will recast the same equation in GR terms. Friedmann's equation then takes the following form:

$$H(t)^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{\kappa c^2}{R_0^2}\frac{1}{a(t)^2}$$

Where we have substituted $r = e/c^2$ which basically says that one needs to take into account any component in the sphere that contributes to the energy density. The second term that we replaced is due to the proper relativistic treatment and yields that the second RHS term depends on the curvature of the Universe and its “radius”.

Now in the case of a flat Universe (k=0) at z=0 the LHS of this equation becomes H_0^2 (known as the Hubble constant) which gives the density known as the critical density, namely the density of the Universe that will make it flat (assuming no extra-ingredients in the Universe, like cosmological constant which we will discuss later).

Obviously, one can show that the critical energy and density at any time (redshift) is given by

$$\varepsilon_c(t) = \frac{3c^2 H(t)^2}{8\pi G}$$

$$\rho_c(t) = \frac{3H(t)^2}{8\pi G}$$

Friedmann's Equation

The current value of these quantities are given in the book (based on $H_0 = 70\pm7$ km $s^{-1}Mpc^{-1}$)

$$\varepsilon_{c,0} = \frac{3c^2}{8\pi G}H_0^2 = (8.3 \pm 1.7) \times 10^{-10}\,\mathrm{J\,m^{-3}} = 5200 \pm 1000\,\mathrm{MeV\,m^{-3}}$$
$$\rho_{c,0} \equiv \varepsilon_{c,0}/c^2 = (9.2 \pm 1.8) \times 10^{-27}\,\mathrm{kg\,m^{-3}} = (1.4 \pm 0.3) \times 10^{11}\,\mathrm{M_\odot\,Mpc^{-3}}$$

It is also customary to write energy density in units of the critical energy density, namely, in terms of the dimensionless parameters:

$$\Omega(t) = \frac{\varepsilon(t)}{\varepsilon_c(t)}$$

The Friedmann equation could then be written in a very "economical" manner:

$$\Omega + \Omega_\kappa = 1, \quad \text{where} \quad \Omega_\kappa = -\frac{\kappa c^2}{R_0^2 a(t)^2 H(t)^2}$$

This is a very interesting form of the equation as it interprets the curvature as some sort of energy density. It should be emphasized that this is not a energy density term but rather a geometrical term that arises from the Gmn term of Einstein's equation and not the Tmn which gives the real energy density of the Universe that is only given by W.
Another comment that has to be mentioned here is that the we have mentioned W generically without discussing the contribution due to its various components, e.g., matter, radiation, etc. This will be discussed and developed in subsequent lectures.

Thermodynamics and the acceleration equation

As important as Friedmann's equation is, it can not give us a solution of how the Universe will evolve as it has two unknowns, a(t) and e*(t).* Hence another equation is needed. In order to get another equation, we will take the derivative of the Friedmann equation which could be complimented by thermodynamic relations that will allow us to solve for both quantities depending on the Universe's ingredients and their equation of state.

For a homogeneous Universe where there is no exchange of heat between its parts the expansion is adiabatic, namely, the heat flow into a given region dQ is 0. Therefore, from the first law of thermodynamics one concludes that,

$$\dot{E} + P\dot{V} = 0$$

Where E is the internal energy, P the pressure and V the volume. Substituting the volume of our sphere *V*=4p/3 r_s^3 a^3 and energy of the sphere E=V(t) e(t) and some manipulations we reach the following equation about the rate of change in the energy density of the Universe:

$$\dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + P) = 0$$

Acceleration equation and equation of state

Taking the time derivative of Friedmann's equation and substituting the rate of change in the energy density of the Universe we obtained in the last equation yields,

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{3c^2}\left(\varepsilon + 3P\right)$$

This equation gives the acceleration rate of the Universe.

Clearly, if the Universe has normal pressure component, then the it will decelerate. However, if the pressure is negative (also known as tension) then under certain condition (P < -e/3) the Universe can accelerate.

In other words, the fate of the Universe in this model hinges upon the relation between P and e, i.e., the equation of state of the Universe, which in turn depends on its constituents.

For cosmological important substances, the equation of state could be written as simple linear relation, $P = w\varepsilon$ where w is the dimensionless equation of state parameter.
. This parameter is 1/3 for relativistic particles (photons and neutrinos), 0 for dust and around -1 for dark energy (cosmological constant) component. The equation of state parameter has one main restriction and that it should be ≤ 1. This will be proven in the werkcollege!!

Cosmological constant (dark energy)

Notice that we can define and cosmological constant energy density of L as: $\varepsilon_\Lambda \equiv \frac{c^2}{8\pi G}\Lambda$

Dividing by the critical density we obtain e_L in units of e_c, as W_L = L/3H^2 in this case we can also write Friedmann's equation as $W+W_k=W_r+W_m+W_L+W_{k=}1$.

Another interesting observation is that in Einstein static Universe both $\dot{a}$ and $\ddot{a}$ are zero. Therefore, one can deduce the value of radius of curvature

$$R_0 = \frac{c}{2(\pi G\rho)^{1/2}} = \frac{c}{\Lambda^{1/2}}$$

The problem that the introduction of the cosmological constant is that if it is interpreted as the vacuum energy. In such case one can use quantum field theory to calculate its value but this is beyond the scope of this paper. As an alternative we deduce its value from the Planck scale and mass (which are the natural scale and mass that involve G, c, and ħ) and have the following values, $\ell_P \equiv \left(\frac{G\hbar}{c^3}\right)^{1/2} = 1.6\times10^{-35}\,\mathrm{m}$, and $M_P \equiv \left(\frac{\hbar c}{G}\right)^{1/2} = 2.2\times10^{-8}\,\mathrm{kg}$

This gives a value for the vacuum energy density of $3\mathrm{x}10^{133}$ eVm^{-3} which about 124 orders of magnitude larger than we measure.
A number of decades ago the main question with regard to the cosmological constant was why it is so large, especially since what is measured was (about) zero? However now the main question is why it is so small and yet not zero?

Single and Multi-component Universe

In principle one would like to solve the following three equations:

Friedmann's equation,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon - \frac{\kappa c^2}{R_0^2 a^2} + \frac{\Lambda}{3}$$

The fluid equation,

$$\dot{\varepsilon} + 3\frac{\dot{a}}{a}\left(\varepsilon + P\right) = 0$$

The equation of state,

$$P = w\varepsilon$$

We then obtain the solution for the scale factor, a, the energy density, e, and the pressure, P.

Notice that the acceleration equation has been replaced with the fluid equation.

Solution of these equations for all components (radiation, matter and dark energy) has to be done numerically therefore we first focus on several simple cases and then put everything together.

The energy density evolution

In the case of a number of components that contribute to the energy density of the Universe one has to make the following modifications to the previous equations:

- Friedmann's equation preserve its form except that the energy density is the some of the contribution of all components. Namely,

$$\varepsilon = \sum_w \varepsilon_w$$

- The total pressure will also attain a similar form, namely,

$$P = \sum_w P_w$$

Where w is the equation of state constant for each component. This mean that, in principle, the equation of state and the fluid equation need to be modified.

Assuming that the various components do not interact with each other the equation of state and the fluid equation could be then written component by component. It is worth noting that with the exception of the interaction of radiation with the baryonic component of the matter (up to recombination or z ≈1100) the non-interaction assumption is a very good one. To recast what we said mathematically for each component:

$$\dot{\varepsilon}_w + 3\frac{\dot{a}}{a}\left(\varepsilon_w + P_\varepsilon\right) = \dot{\varepsilon}_w + 3\frac{\dot{a}}{a}\left(1 + w\right)\varepsilon_w = 0$$

The energy density evolution

For a full solution of the last equation, we have to assume some boundary conditions. Here we assume that we know the energy density of the energy density component at z=0 and assume that the scale factor at z=0 is 1. These assumptions yield a very simple solution for the energy density evolution of each component,

$$\varepsilon_w = \varepsilon_{w,0} a^{-3(1+w)}$$

As we mentioned earlier, we typically have three components: Radiation, Matter (dark & baryonic) and Cosmological constant (or dark energy). For each of these components the solution is given in the following table:

Component	EoS parameter	Energy density evolution
Radiation	$w = 1/3$	$\varepsilon_{r,0} a^{-4}$, $\varepsilon_{r,0}(1+z)^4$
Matter (dust)	$w = 0$	$\varepsilon_{m,0} a^{-3}$, $\varepsilon_{m,0}(1+z)^3$
Cosmological constant	$w = -1$	$\varepsilon_{\Lambda,0}$

Obviously, once one knows the manner in which each of the energy density components evolves then one can solve Friedmann's equation. In the following few slides this is exactly what we will do for various assumptions about these components (which we can solve analytically). Generally, however, the solutions can be only obtained numerically.

Solutions of Friedmann's equation: Special cases

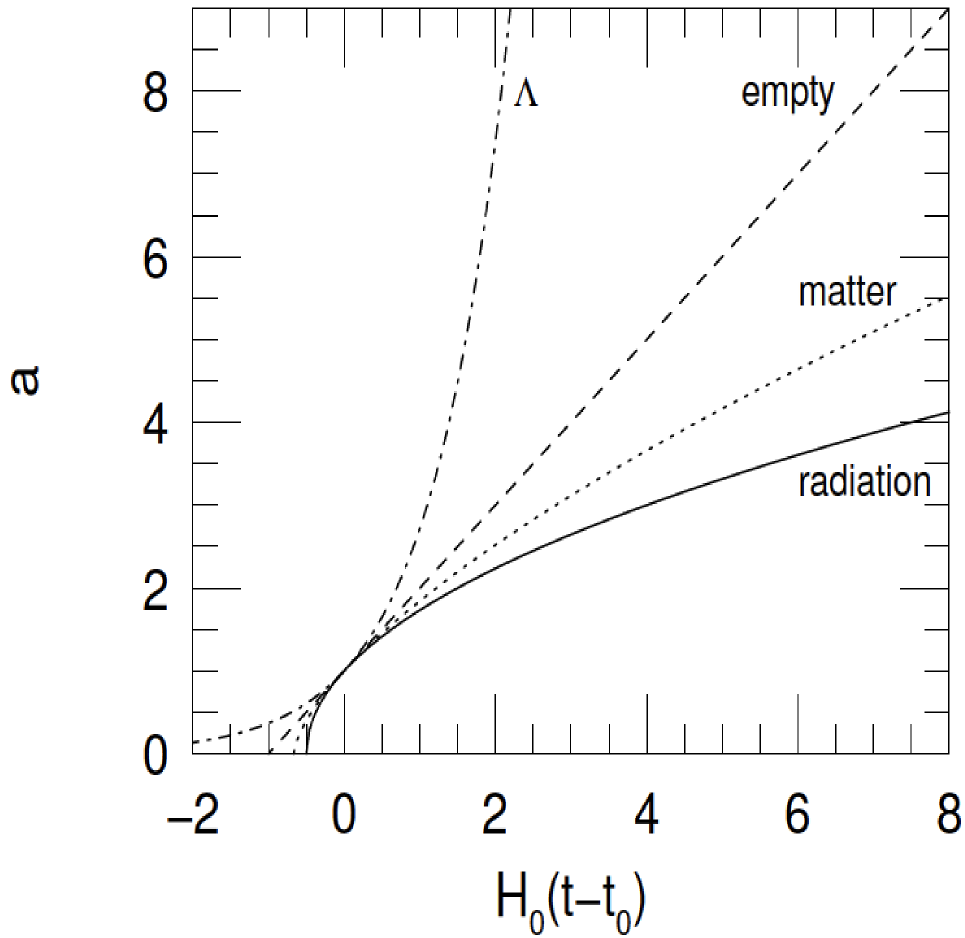
Here we will show solutions of Friedmann's equation for a number of special cases

Curvature only Universe:

If the Universe has no matter nor radiation then the equations allow two types of solution a solution with k=0 and $\dot{a}=0$, namely, an empty static Universe whose geometry is described by Minkowski metric.

The other type of Universe that is allowed is one with k=-1 (open Universe). In this case, $\dot{a}=\pm c/R_0 \equiv \pm 1/t_0$, or $a(t)=\pm t/t_0$. Notice, that with the lack of gravity the expansion rate is constant. Solve for the proper distance!

We will also show the scale factor of the Universe for the case of radiation dominated, matter dominated, and Cosmological constant dominated Universes. This figure show the evolution of the scale factor for each case.



Spatially Flat Universe

The case of a flat Universe with a single component has the following solutions:

$$\text{Matter alone } (w=0): \quad a(t) = \left(\frac{t}{t_0}\right)^{2/3} \quad \text{where} \quad t_0 = \frac{2}{3H_0},$$

$$\varepsilon(t) = \varepsilon_0 \left(\frac{t}{t_0}\right)^{-2} \quad \text{and} \quad d_p(t_0) = \frac{2c}{H_0}\left[1 - \frac{1}{\sqrt{1+z}}\right]$$

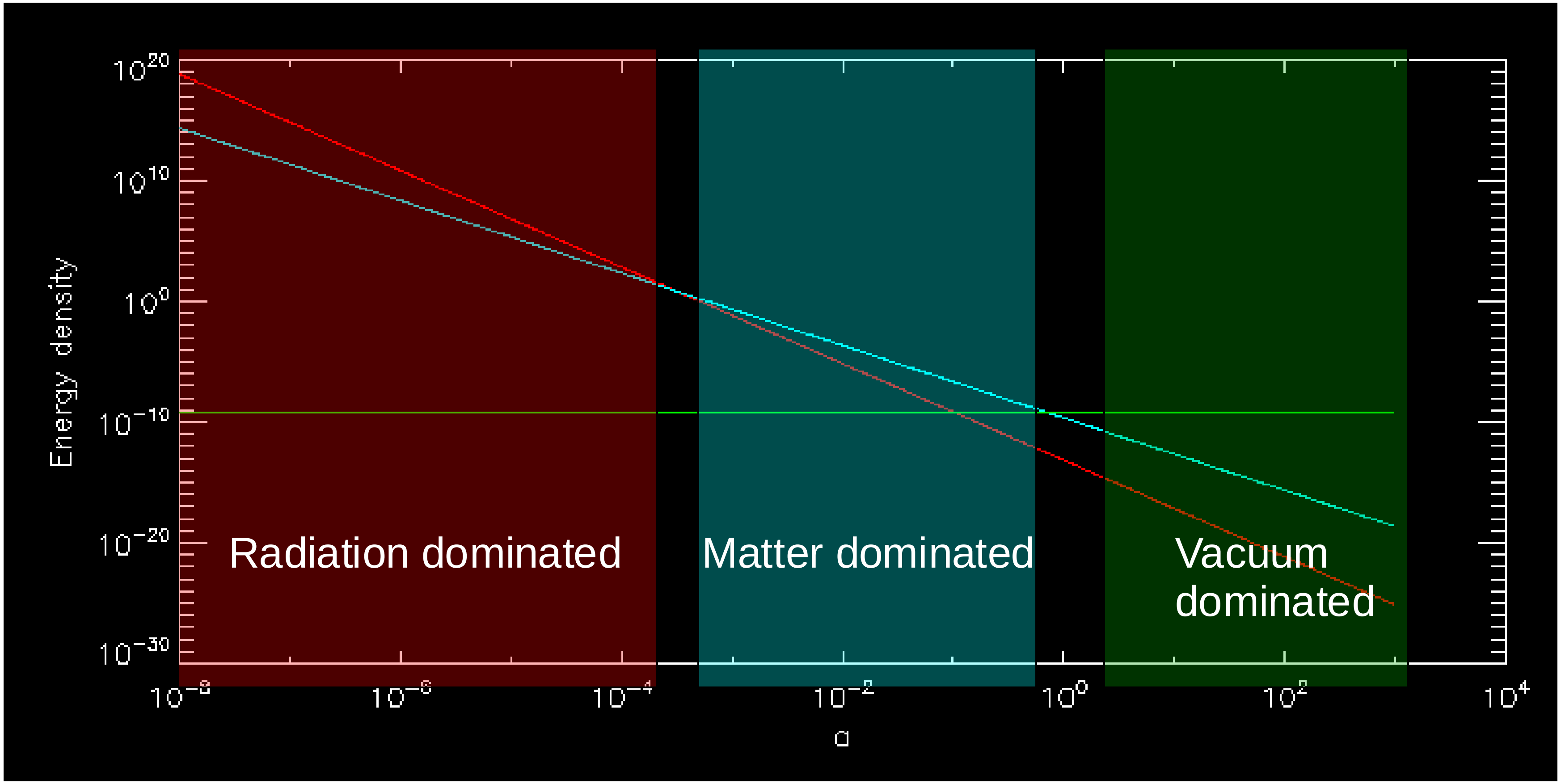
$$\text{Radiation alone } (w=\frac{1}{3}): \quad a(t) = \left(\frac{t}{t_0}\right)^{1/2} \quad \text{where} \quad t_0 = \frac{1}{2H_0},$$

$$\varepsilon(t) = \varepsilon_0 \left(\frac{t}{t_0}\right)^{-2} \quad \text{and} \quad d_p(t_0) = \frac{c}{H_0}\frac{z}{1+z}$$

$$\Lambda \text{ alone } (w=-1): \quad a(t) = e^{H_0(t-t_0)} \quad \text{where} \quad H_0 = \left(\frac{8\pi G \varepsilon_\Lambda}{3c^2}\right)^{1/2},$$

$$\varepsilon(t) = \varepsilon_\Lambda \quad \text{and} \quad d_p(t_0) = \frac{c}{H_0} z$$

The Energy density of our Universe



Multiple-Component Universe

Since we know the various components of the Universe we can write Friedmann's equation in generic terms, where we absorb the dark energy term in the first term of the RHS of the equation.

$$H(t)^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{\kappa c^2}{R_0^2}\frac{1}{a(t)^2}$$

As an aside remember that this equation could be written as:

$$\Omega + \Omega_\kappa = 1, \quad \text{where} \quad \Omega_\kappa = -\frac{\kappa c^2}{R_0^2 a(t)^2 H^2}$$

and W is the total energy density in units of the critical energy density at any given time
We should distinguish between three cases here:

1.If the Universe is flat at any given time, then it always remains so.

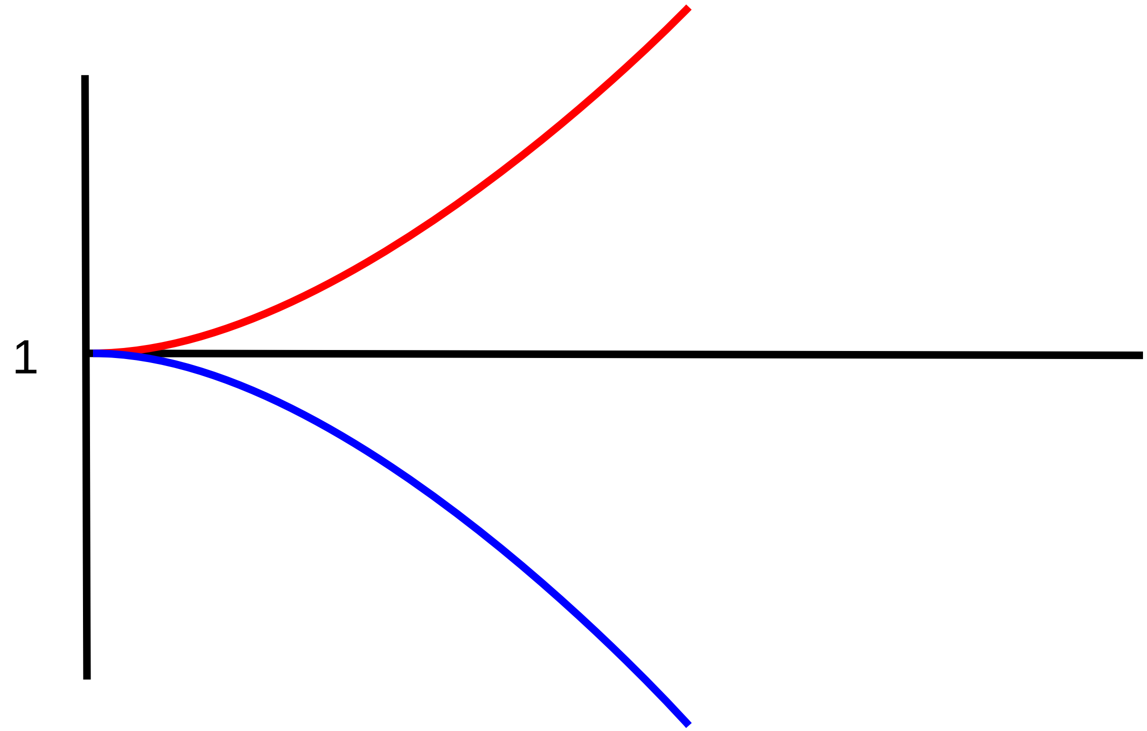
2.If the Universe is closed (k =+1) then there are two cases to consider.
   i. Matter or radiation dominated Universe. The Universe's W increases with time.
   ii. Cosmological constant dominated Universe where W asymptotically reaches unity.

1.If the Universe is open (k =-1) then there are two cases to consider.
i.Matter or radiation dominated Universe. The Universe's W decreases with time.
ii.Cosmological constant dominated Universe. The Universe W asymptotically reach 1.

Multiple-Component Universe

Given that in the past the Universe was radiation and matter dominated it means that the Universe was incredibly close to flat at the early Universe (remember we measure the Universe as flat down to a certain accuracy but whatever our accuracy is in the past the Universe flatness was even greater).

So, the big question is why out of all the values that energy density of the Universe could have it has the critical value?



Multiple-Component Universe

Now we remember that the second term of the RHS of this equation could be actually written as

$$H(t)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{H_0^2}{a(t)^2}\left(\Omega_0 - 1\right)$$

Where we used the relation:

$$\Omega_{\kappa,0} = 1 - \Omega_0 = -\frac{\kappa c^2}{H_0^2 R_0^2}$$

Dividing by H_0^2, Friedmann's equation obtain the following form:

$$\frac{H(t)^2}{H_0^2} = \frac{\varepsilon(t)}{\varepsilon_{c,0}} + \frac{1-\Omega_0}{a(t)^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} + \frac{1-\Omega_0}{a(t)^2}$$

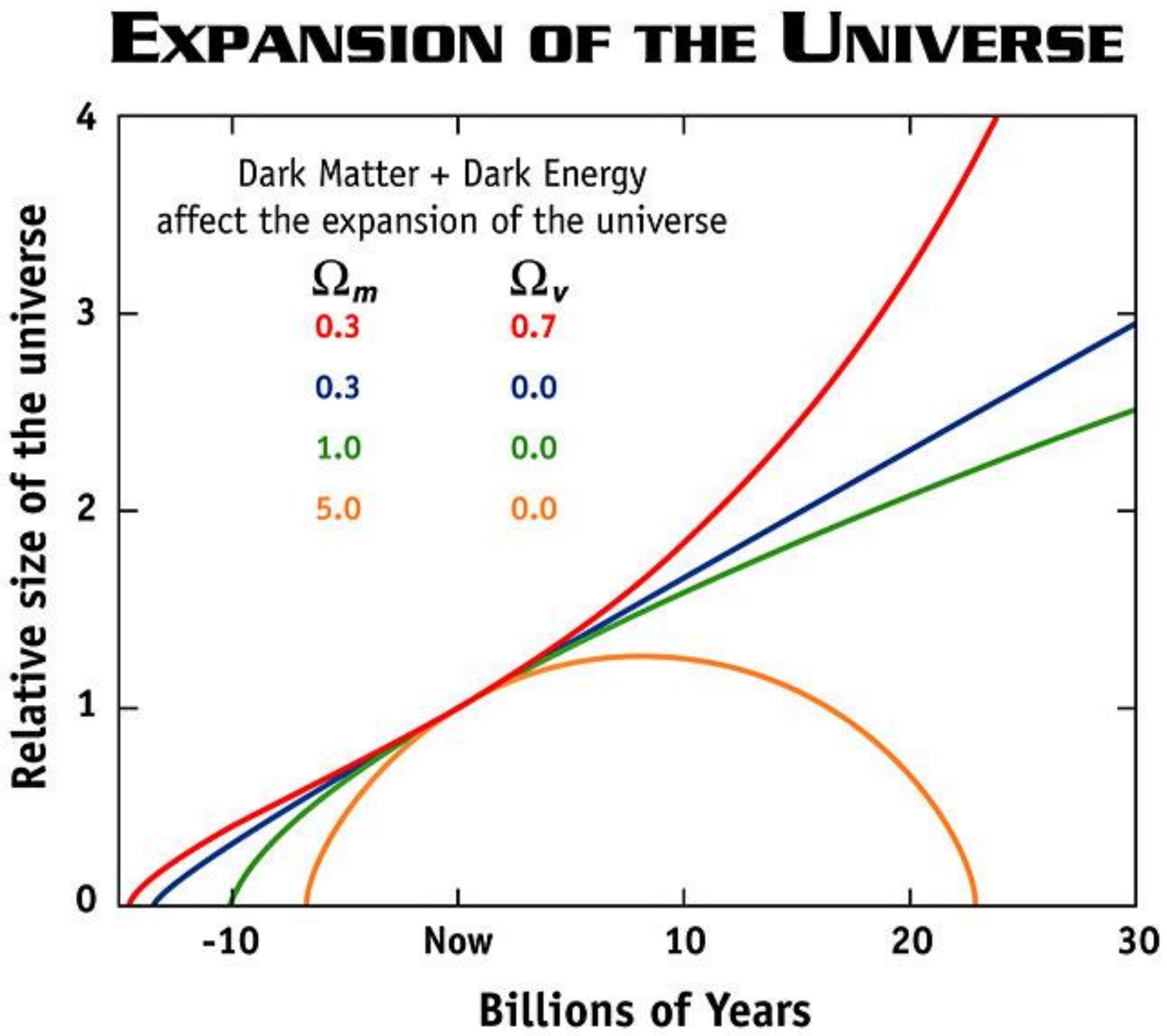
where, $W_0 = W_{r,0}+W_{m,0}+W_{L,0}$

This equation leads to the integral:

$$\int_0^a \frac{da}{\left[\Omega_{r,0}/a^2 + \Omega_{m,0}/a + \Omega_{\Lambda,0}a^2 + (1-\Omega_0)\right]^{1/2}} = H_0 t$$

Multiple-Component Universe

This integral could not be performed analytically. However, an analytical solution could be obtained in certain cases which we will discuss briefly here.



Matter and Curvature

Friedmann's equation in this case is given as,

$$\frac{H(t)^2}{H_0^2} = \frac{\Omega_0}{a^3} + \frac{1-\Omega_0}{a^2}$$

It is easy to see that in case of a closed Universe (W_0 > 1) there is a max scale factor after which the Universe will recollapse in a so called big-crunch. This max scale factor is simply

$$a_{\max} = \frac{\Omega_0}{\Omega_0 - 1}$$

In such case we could obtain the scale factor evolution with the integral

$$H_0 t = \int_0^a \frac{da}{[\Omega_0/a + (1-\Omega_0)]^{1/2}}$$

Which could solved parametrically with q (which runs over 0-2p with a=0 at the two borders),

$$a(\theta) = \frac{1}{2}\frac{\Omega_0}{\Omega_0 - 1}(1-\cos\theta) \qquad t(\theta) = \frac{1}{2H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}(\theta - \sin\theta)$$

Curvature and matter

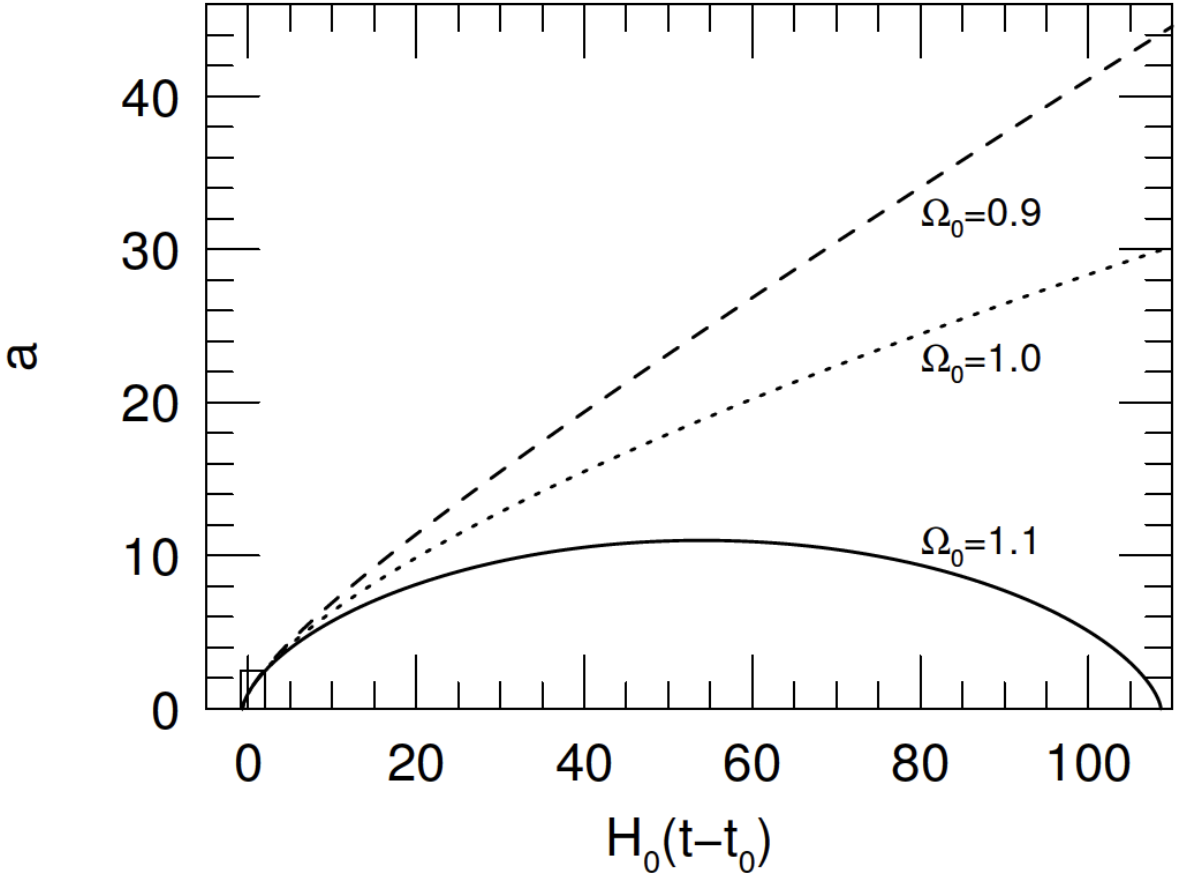
We obtain a big bang at t=0 and a big crunch at, $$t_{\mathrm{crunch}} = \frac{\pi}{H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}$$

For an open Universe then there is also a parametric solution for the integral and it has the form:

$$a(\eta) = \frac{1}{2}\frac{\Omega_0}{1-\Omega_0}(\cosh\eta - 1)$$

$$t(\eta) = \frac{1}{2}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}(\sinh\eta - \eta)$$

With h running from 0 to infinity.



Matter and Radiation

As we mentioned earlier at the early Universe radiation was the dominant energy density component. However, as the Universe expands the contribution to the energy density gradually changes from radiation dominated to matter dominated. Such transition occurs at the scale factor,

$$a_{rm} = \Omega_{r,0}/\Omega_{m,0} = 2.8 \times 10^{-4}$$

which corresponds to z_{rm} ~ 3600. Therefore, a flat cosmology with radiation and matter components alone is worth exploring. In such a Universe Friedmann's equation takes the form,

$$\frac{H^2}{H_0^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3}$$

This equation yields the analytical solution,

$$H_0 t = \frac{4a_{rm}^2}{3\sqrt{\Omega_{r,0}}}\left[1 - \left(1 - \frac{a}{2a_{rm}}\right)\left(1 + \frac{a}{a_{rm}}\right)^{1/2}\right]$$

We could easily verify the two limits of this equation, namely, $a \ll a_{rm}$ and $a \gg a_{rm}$.

Find the time of matter density equality.

Matter and Lambda

The other case that is worth exploring is that of a flat cosmology with matter and cosmological constant. This is obviously the second mixture that is important for our Universe. Friedmann's equation in this case is simply given by,

$$\frac{H^2}{H_0^2} = \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} = \frac{\Omega_{m,0}}{a^3} + (1 - \Omega_{m,0})$$

In the book they explore the case of negative cosmological constant which provides an attractive force instead of a repulsive force. In such case the Universe will recollapse in a big crunch exactly like a Universe without cosmological constant but with matter density parameter > 1. We will not explore this option here if you are interested, please look it up in the book.

Here we will focus on the case with a positive cosmological constant (like the one we measure). In such a case one can define a matter-L equality scale factor, a_{mL}~0.72 which corresponds to z_{mL}~0.4. In this case also an analytical solution of Friedmann's equation exists,

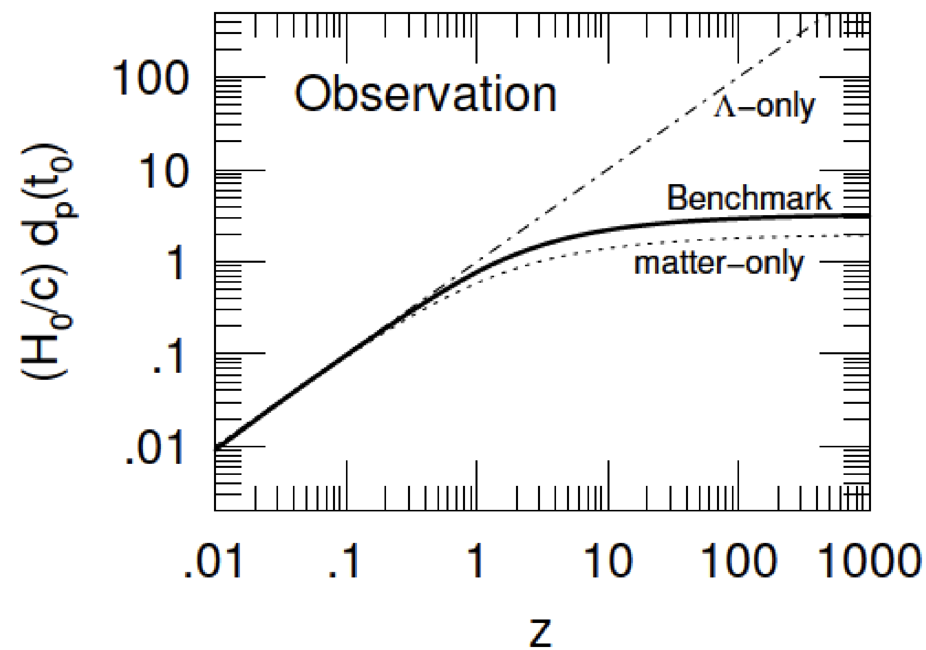
$$H_0 t = \frac{2}{3\sqrt{1 - \Omega_{m,0}}} \ln\left[\left(\frac{a}{a_{m\Lambda}}\right)^{3/2} + \sqrt{1 + \left(\frac{a}{a_{m\Lambda}}\right)^3}\right]$$

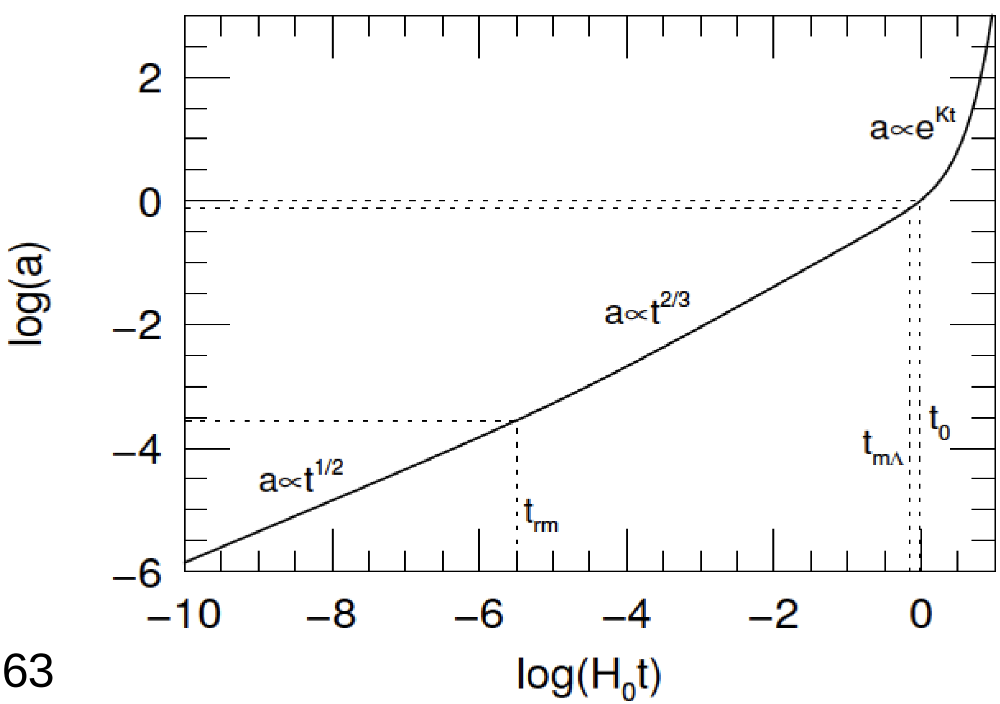
Verify limits and find the equality time (about 10 Gyrs).

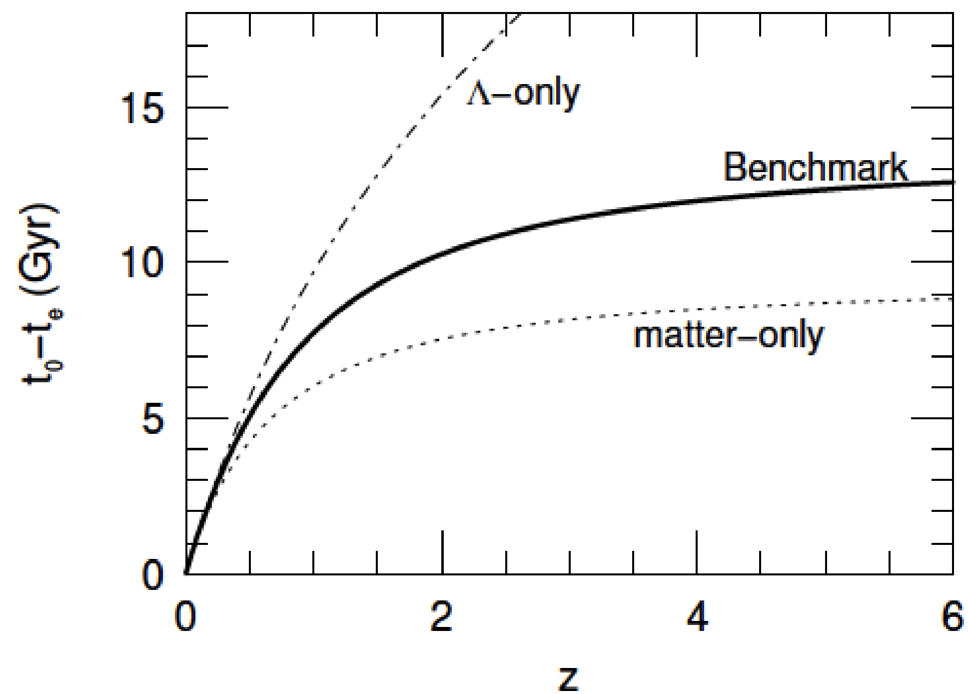
The current standard model

List of Ingredients	
photons:	$\Omega_{\gamma,0} = 5.0 \times 10^{-5}$
neutrinos:	$\Omega_{\nu,0} = 3.4 \times 10^{-5}$
total radiation:	$\Omega_{r,0} = 8.4 \times 10^{-5}$
baryonic matter:	$\Omega_{\text{bary},0} = 0.0486$
nonbaryonic dark matter:	$\Omega_{\text{dm},0} = 0.2588$
total matter:	$\Omega_{m,0} = 0.30$
cosmological constant:	$\Omega_{\Lambda,0} \approx 0.70$

Important Epochs		
radiation-matter equality:	$a_{rm} = 2.8 \times 10^{-4}$	$t_{rm} = 4.7 \times 10^{4}$ yr
matter-lambda equality:	$a_{m\Lambda} = 0.75$	$t_{m\Lambda} = 9.8$ Gyr
Now:	$a_0 = 1$	$t_0 = 13.5$ Gyr







Current Best Parameters From Planck Surveyor + ...

Parameter	TT+lowP+lensing 68% limits	TT,TE,EE+lowP+lensing+ext 68% limits
n_s	0.9677 ± 0.0060	0.9667 ± 0.0040
H_0	67.81 ± 0.92	67.74 ± 0.46
Ω_Λ	0.692 ± 0.012	0.6911 ± 0.0062
Ω_m	0.308 ± 0.012	0.3089 ± 0.0062
$\Omega_b h^2$	0.02226 ± 0.00023	0.02230 ± 0.00014
$\Omega_c h^2$	0.1186 ± 0.0020	0.1188 ± 0.0010
σ_8	0.8149 ± 0.0093	0.8159 ± 0.0086
z_{re}	$8.8^{+1.7}_{-1.4}$	$8.8^{+1.2}_{-1.1}$
Age/Gyr	13.799 ± 0.038	13.799 ± 0.021

The deceleration parameter

Measuring the scale factor of the Universe as a function of time allows us to determine a number of cosmological parameters. The simplest parameter that it allows to measure is the Hubble constant. A simple Taylor expansion of the scale factor around t_0 to second order give,

$$a(t) \approx a(t_0) + \left.\frac{da}{dt}\right|_{t=t_0} (t - t_0) + \frac{1}{2} \left.\frac{d^2a}{dt^2}\right|_{t=t_0} (t - t_0)^2$$

This could be written as

$$a(t)/a_0 = a(t) \approx 1 + H_0(t - t_0) - \frac{1}{2} q_0 H_0^2 (t - t_0)^2$$

where the parameter q_0 is the so-called deceleration parameter, defined as,

$$q_0 \equiv -\left.\left(\frac{\ddot{a}a}{\dot{a}^2}\right)\right|_{t=t_0} = -\left(\frac{\ddot{a}}{aH^2}\right)_{t=t_0}$$

Now we use the acceleration equation to obtain this relation between q_0 the Universe's energy density components:

$$-\left.\left(\frac{\ddot{a}a}{\dot{a}^2}\right)\right|_{z=0} = \frac{1}{2} \left[\frac{8\pi G}{3c^2 H^2}\right] \sum_w \varepsilon_w (1 + 3w) \Big|_{z=0} = \Omega_{r,0} + \frac{1}{2}\Omega_{m,0} - \Omega_{\Lambda,0}$$

Hence measuring the evolution of the scale factor with time gives us important constraints on these components.

The deceleration parameter

The proper distance to a certain galaxy is, $d_p = a(t)\int_0^r dr = a(t)r$

For light, $c\int_{t_e}^{t_0}\frac{dt}{a(t)} = \int_0^r dr = r$, hence $d_p(t_0) = c\int_{t_e}^{t_0}\frac{dt}{a(t)}$

If the galaxy is not far away then one can use Taylor expansion to obtain,

$$\frac{1}{a(t)} \approx 1 - H_0(t-t_0) + (1+q_0/2)H_0^2(t-t_0)^2$$

Use z and invert

Hence the proper distance close to us to second order is,

$$d_p(t_0) \approx c(t_0 - t_e) + \frac{cH_0}{2}(t_0 - t_e)^2$$

We can connect this to redshift by remembering the z-a relation which gives

$$t_0 - t_e \approx H_0^{-1}\left[z - (1+q_0/2)z^2\right]$$

Hence the proper distance to a galaxy at redshift z in such case is,

$$d_p(t_0) \approx \frac{c}{H_0}\left[z - (1+q_0/2)z^2\right] + \frac{cH_0}{2}\frac{z^2}{H_0^2} = \frac{c}{H_0}z\left[1 - \frac{1+q_0}{2}z\right]$$

Distances in the Universe

As mentioned earlier in the course within the frame of work of Robertson-Walker metric one can define a number of distances. This is due to the fact that distances are measured between objects that have different cosmic times, different expansion rates (along or across the line of sight) etc. It is this mixing of time and proper coordinates in the metric that gives rise to very different possibilities of defining distances in the Universe.

In this lecture I will define the various distances normally invoked in Cosmology. I will also show how different observations are sensitive to different types of distances. Furthermore, I will discuss how these measurement provide information on the basic cosmological parameters.

But first I'll start by reminding you with the so-called proper distance that we have defined in one of the previous lectures. This distance is defined as the distance between the observer and a certain point in space, say a galaxy, at a fixed time, t:

$$ds^2 = a(t)^2 \left[dr^2 + S_\kappa(r)^2 d\Omega^2\right]$$

Since for a given galaxy both q and f are constants, the equation becomes simply

$$ds = a(t)dr$$

Integrating this equations gives:

$$d_p = a(t) \int_0^r dr = a(t)r$$

Event Horizon distance

We want to calculate the horizon size of the Universe at time t (not necessarily t_0). In order to calculate the horizon distance, we integrate over *cdt* but we recall that the Universe expands as the light passes through it hence the integration should take into account by simply taking the ratio between the scale factor at the time we calculate the distance and the time *cdt* is calculated giving the following horizon distance:

$$d_{hor}(t) = a(t) \int_0^t \frac{cdt'}{a(t')}$$

the generic distance *c* d*t* travelled by a light ray between *t* and *t'*=*t*+d*t'* has been multiplied by a factor *a(t)/a(t').*

The horizon of a particle at any time t divides the set of all points into two classes: those which can, in principle, have been observed by an observer at time t (inside the horizon), and those which cannot (outside the horizon). Notice however, that in case *a(t)* converges to zero quickly enough then a particle could have seen the whole Universe at early times.

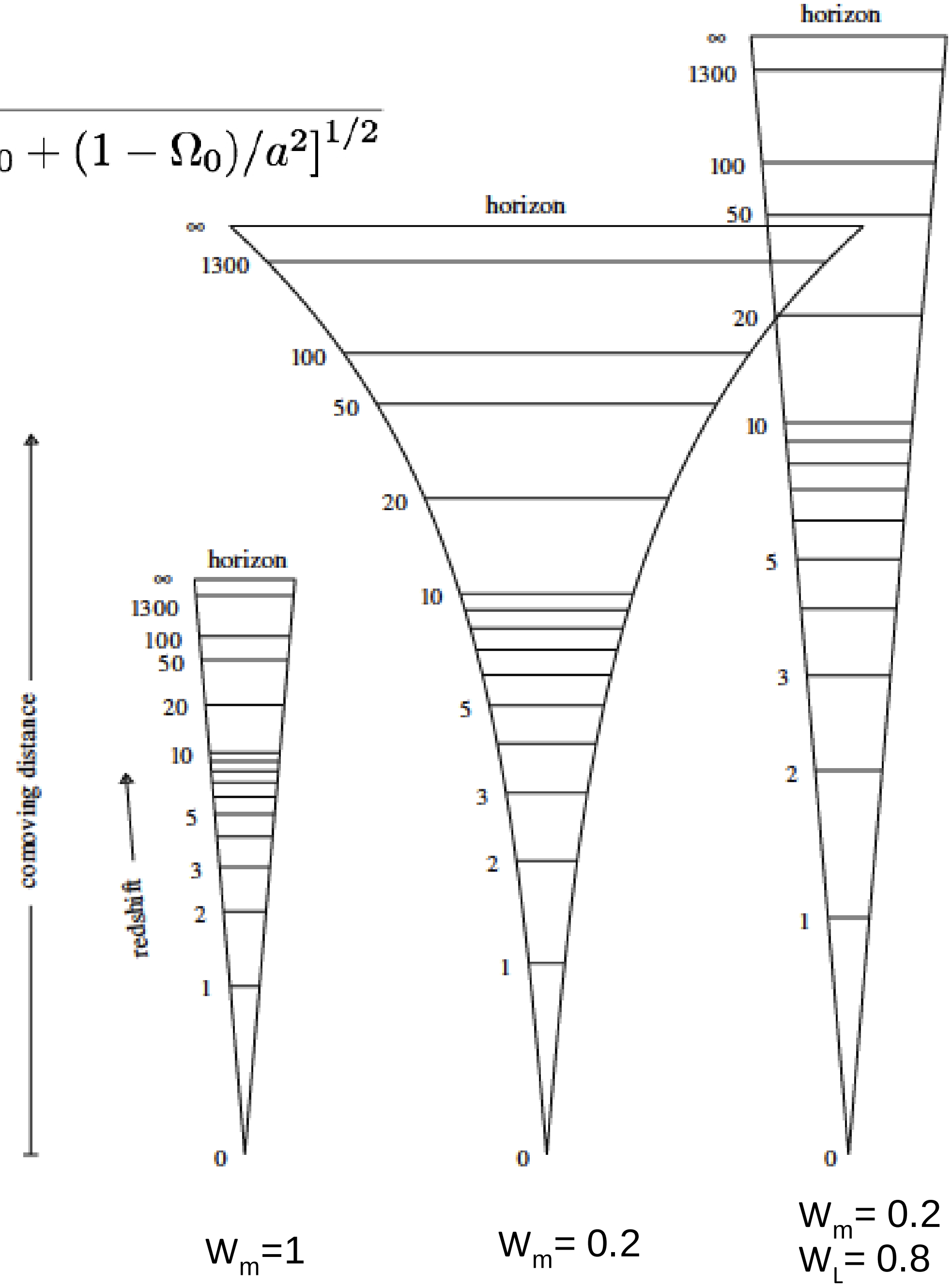
One can show that this is not the case in general, though with inflationary theories which we will discuss later, the comoving size of the horizon at very early time is much larger than the current horizon scale, namely, at some stage in the Universe's history the scale factor increased faster than the speed of light and regions that were causally connected (within the horizon of each other) grow apart so fast that they are currently outside each other horizon!

Event Horizon distance

$$d_{hor} = \frac{a(t)c}{H_0} \int_0^a \frac{da}{a\left[\Omega_{r,0}/a^4 + \Omega_{m,0}/a^3 + \Omega_{\Lambda,0} + (1-\Omega_0)/a^2\right]^{1/2}}$$

Here we used Friedmann's equation to get the horizon distance as a function of *a.* Once that is done one can use the scale factor-redshift relation to explore how this distance evolves with redshift.

On the right, we see a solution for the horizon scale as a function of redshift for various cosmological models.



Luminosity distance

The Luminosity distance is defined in such a way as to preserve the Euclidean inverse-square law for the dimming of light with distance from a point source. Let *L* denote the power emitted by a certain source which observed by us to have a flux *f*, then the luminosity distance is given by,

$$d_L \equiv \left(\frac{L}{4\pi f} \right)^{1/2}$$

Now we remember that the FLRW metric which gives,

$$ds^2 = -c^2 dt^2 + a(t)^2 \left[dr^2 + S_\kappa(r)^2 d\Omega^2 \right]$$

The surface area of a sphere now is simply $4pS_k(r)^2$ which means that the luminosity drops as inverse of this expression due to geometry. Furthermore, the energy per photon also drops by 1+z. Another 1+z comes from the fact the time between two successive photons emitted in our direction will be observed with (1+z) factor between their arrival time. Therefore, the combination of all of these factors gives,

$$f = \frac{L}{4\pi S_\kappa(r)^2 (1+z)^2}$$

Hence,

$$d_L = S_\kappa(r)(1+z)$$

Luminosity distance

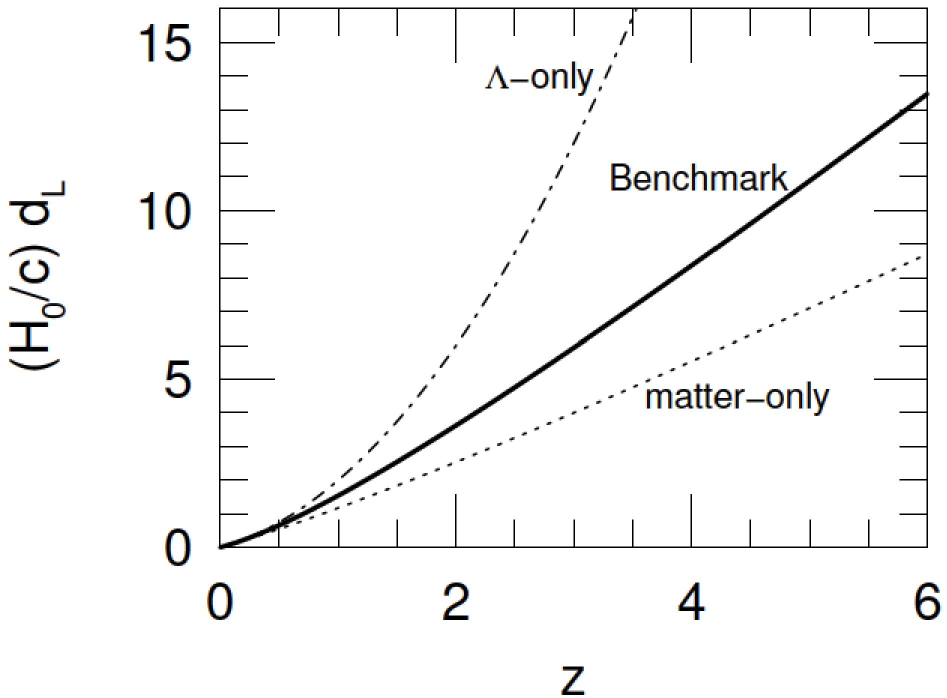
Notice that even is a flat Universe the distance to a standard candle will be overestimated by a factor of 1+z if one take the naive inverse square distance for dimming of light.

The luminosity distance for small z and flat Universe can be approximated by $d_L \sim d_p\,(1+z)$ recalling the discussion about proper distance at low redshifts one obtains,

$$d_L \approx \frac{c}{H_0} z \left(1 - \frac{1+q_0}{2} z\right)(1+z) \approx \frac{c}{H_0} z \left(1 + \frac{1-q_0}{2} z\right)$$

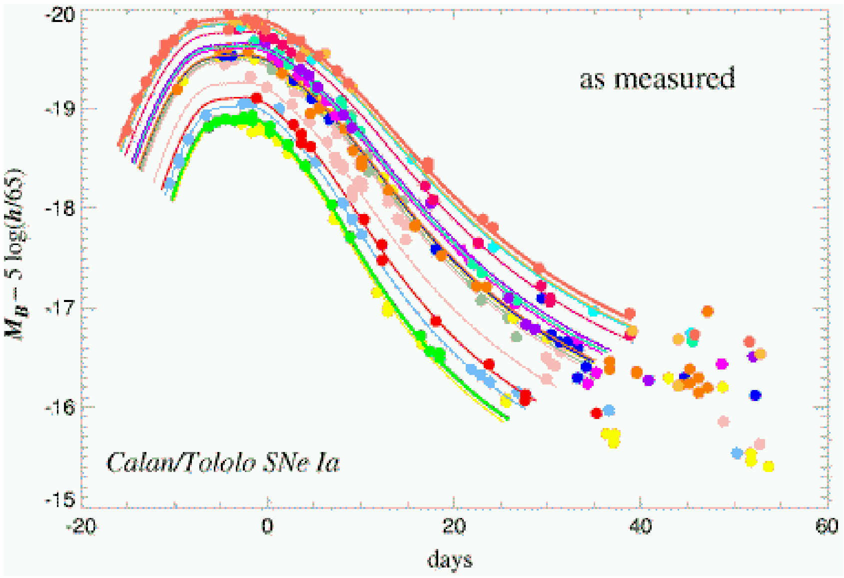
Which for standard candles allows the measurement of both Hubble constant and q_0.

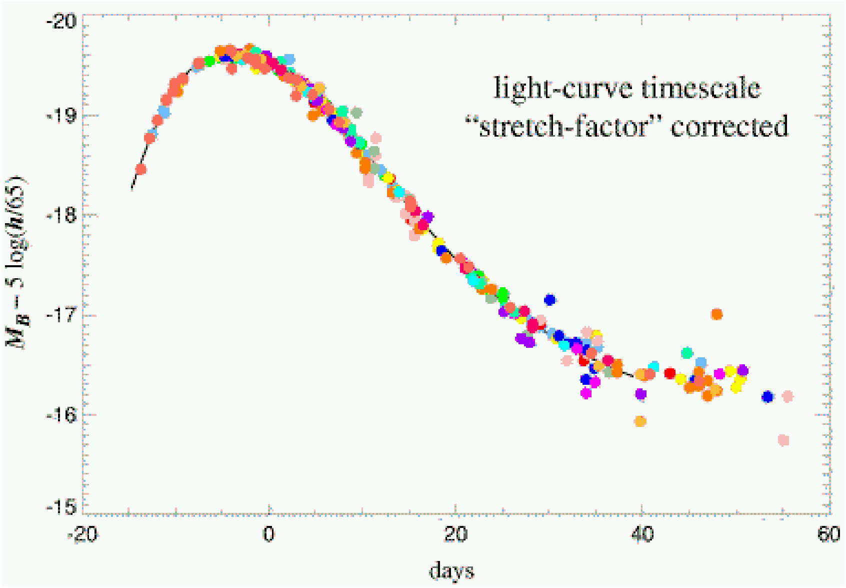
The figure here shows an accurate solution of d_L as a function of z for three models.



Measuring Ω_m and Ω_L

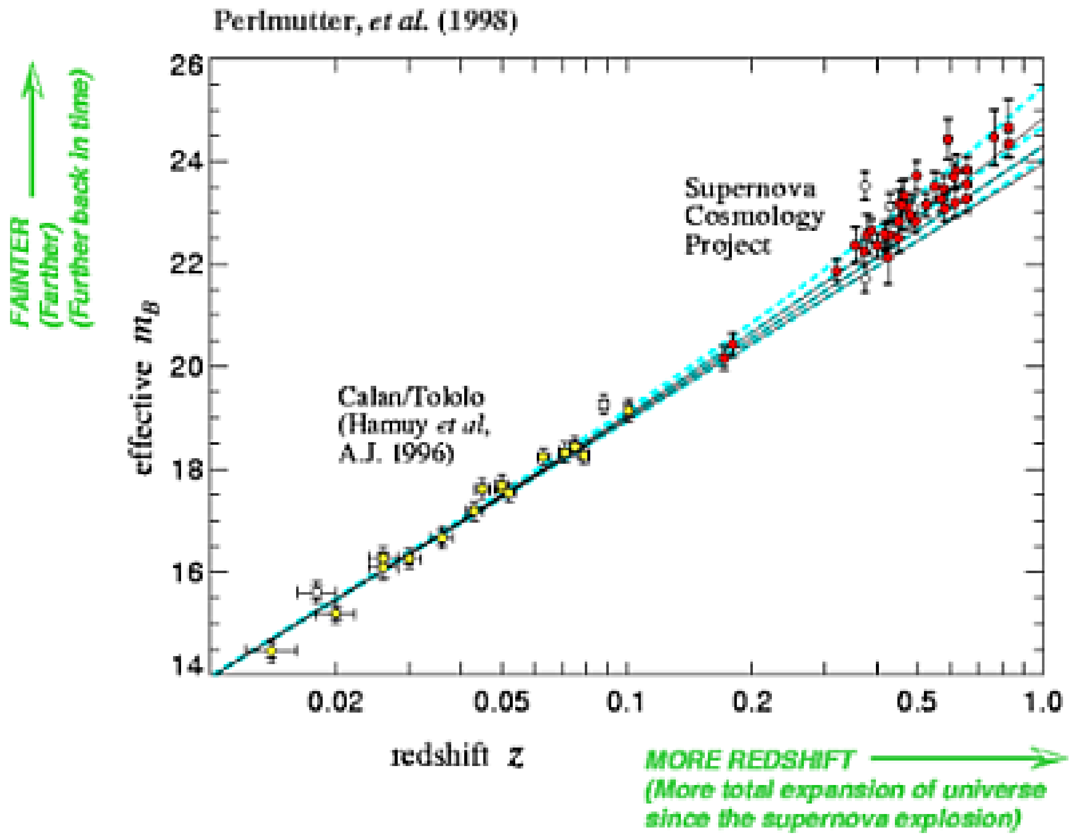
Measuring supernovae type Ia, which is a thermonuclear explosion supernovae (and not core collapse supernovae) gives a light curve that can be standardized.

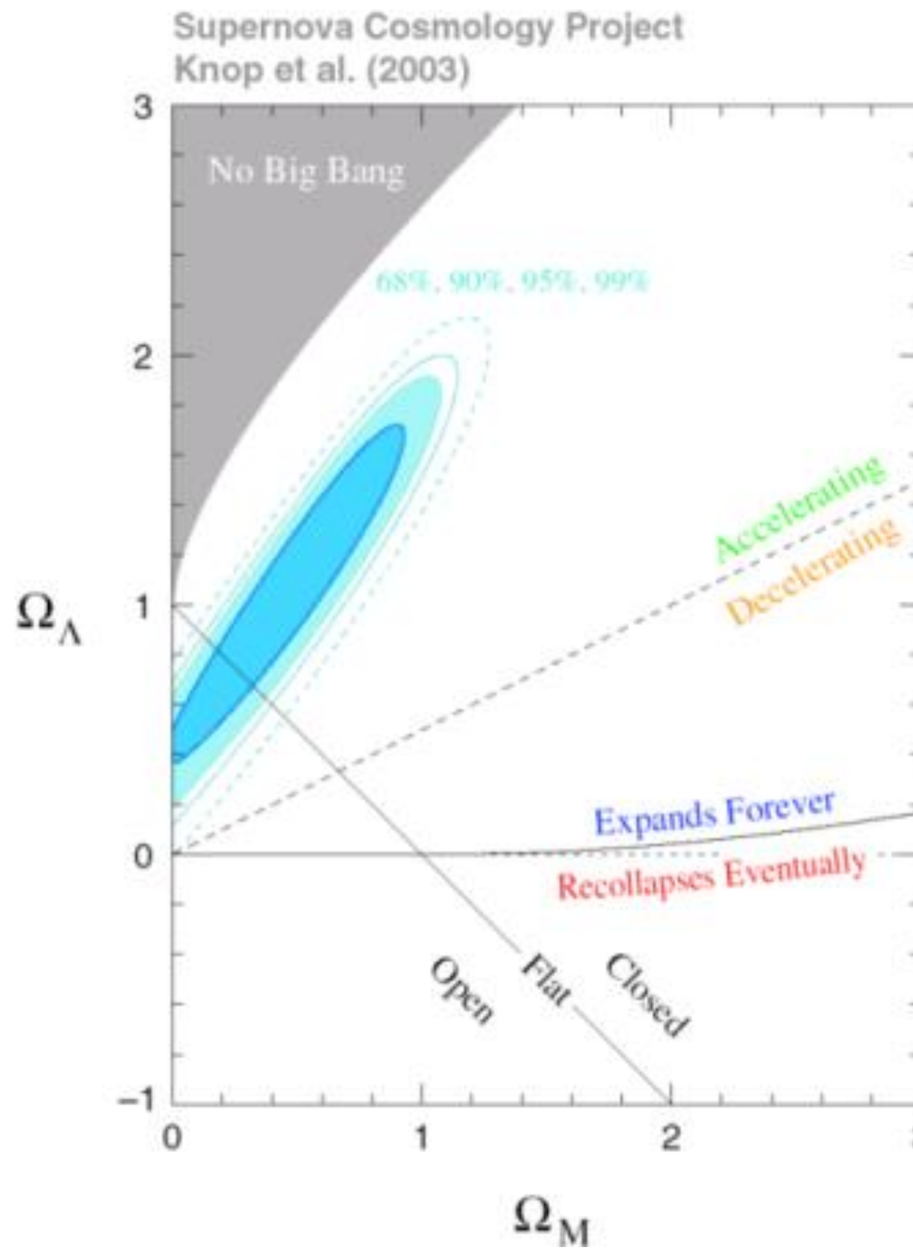




This allows the use of this type of SNe as standard candles which could then be used to measure the luminosity distance and the cosmological constants.

Measuring W_m and W_L





Angular diameter distance

Suppose we have a set of standard rulers, objects that we know are all the same size l, then the question is the angular size, dq*(z)* of these rulers as a function of redshift. In Euclidean geometry the answer to this is simply dq=*l/d* where *d* is the distance to the object. In FLRW geometry, however, the distance that is defined through this equation is called the Angular Diameter Distance and is normally marked as d_A (≡*l*/dq). In order to calculate this distance, we go back again to FLRW metric and obtain,

$$ds = a(t_e)S_\kappa(r)\delta\theta$$

For a standard yardstick with a given length, this gives:

$$\ell = a(t_e)S_\kappa(r)\delta\theta = \frac{S_\kappa(r)\delta\theta}{1+z}$$

Hence, we define the angular diameter distance as,

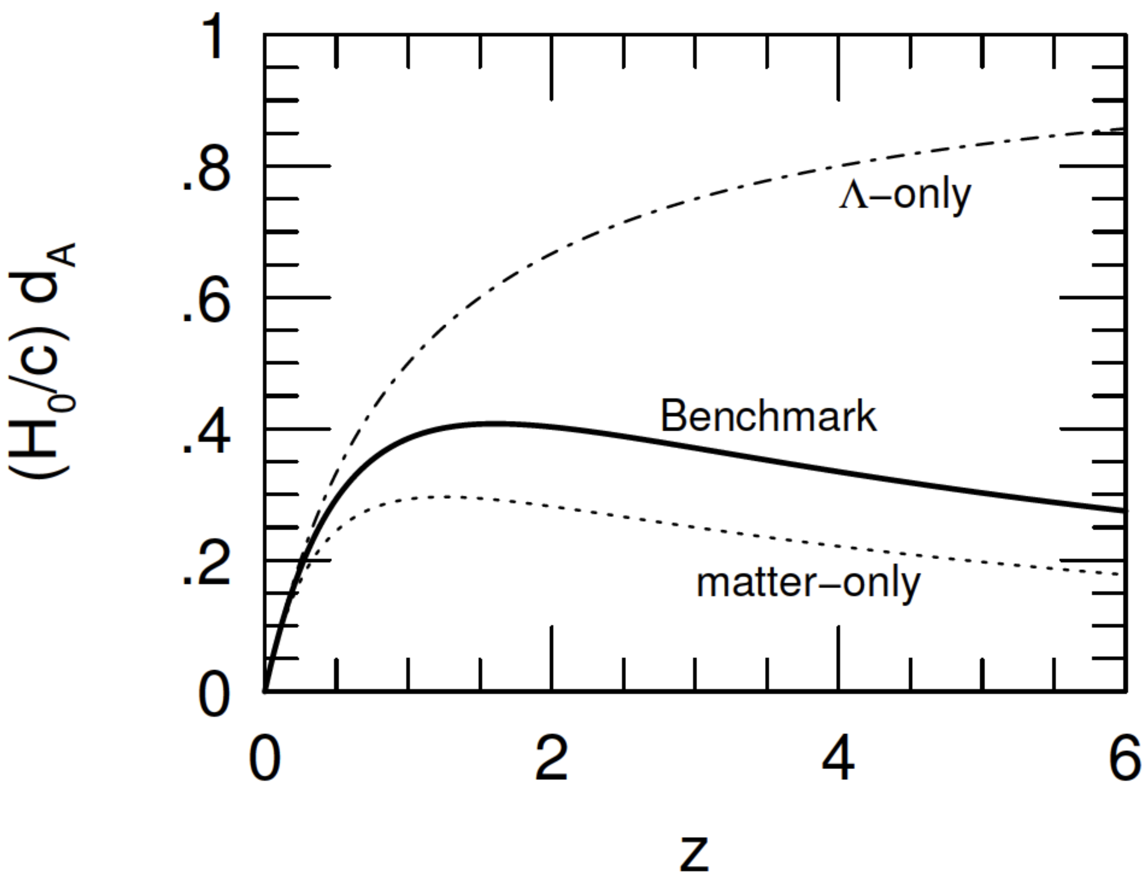
$$d_A \equiv \frac{\ell}{\delta\theta} = \frac{S_\kappa(r)}{1+z}$$

Which means

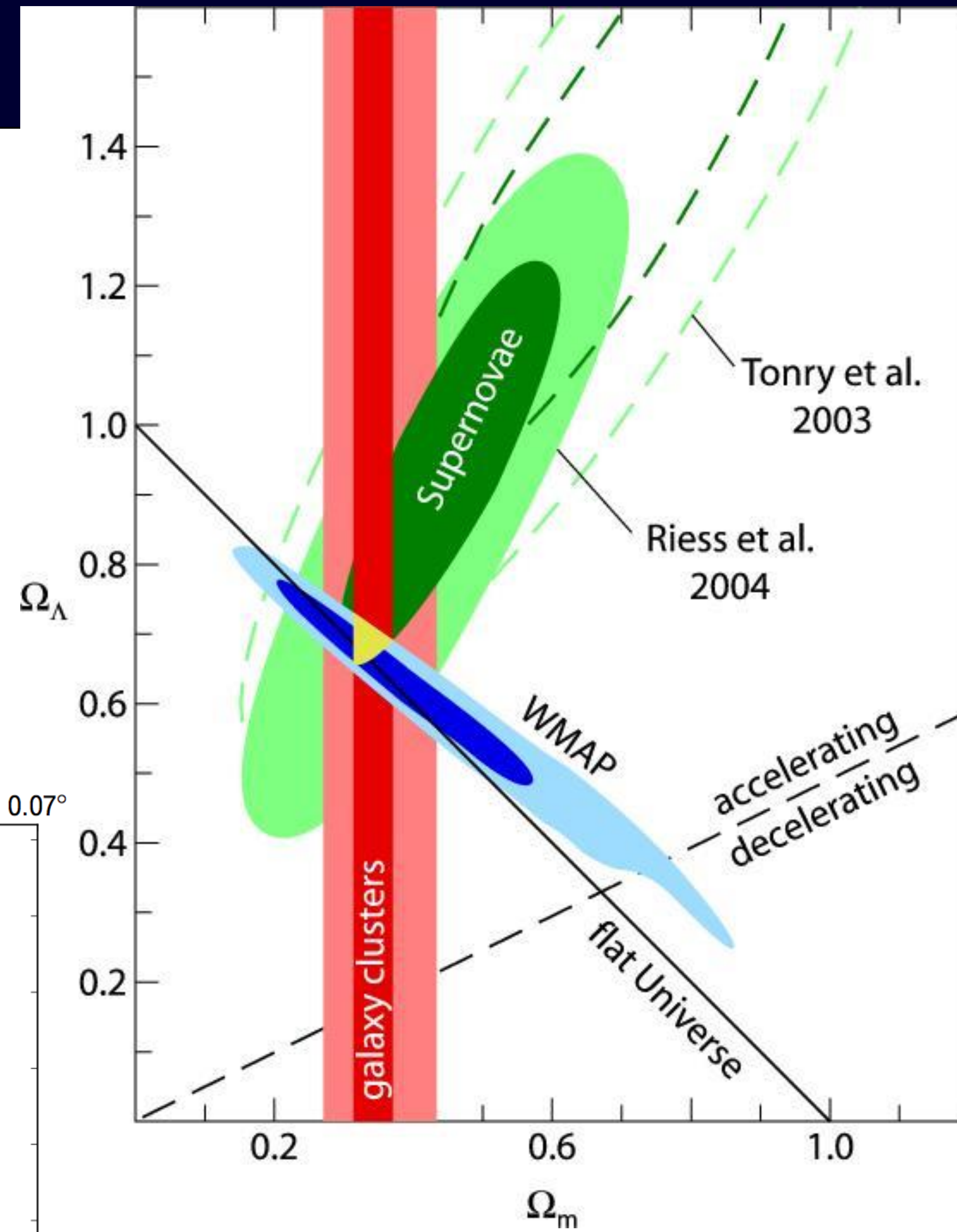
$$d_A = \frac{d_L}{(1+z)^2}$$

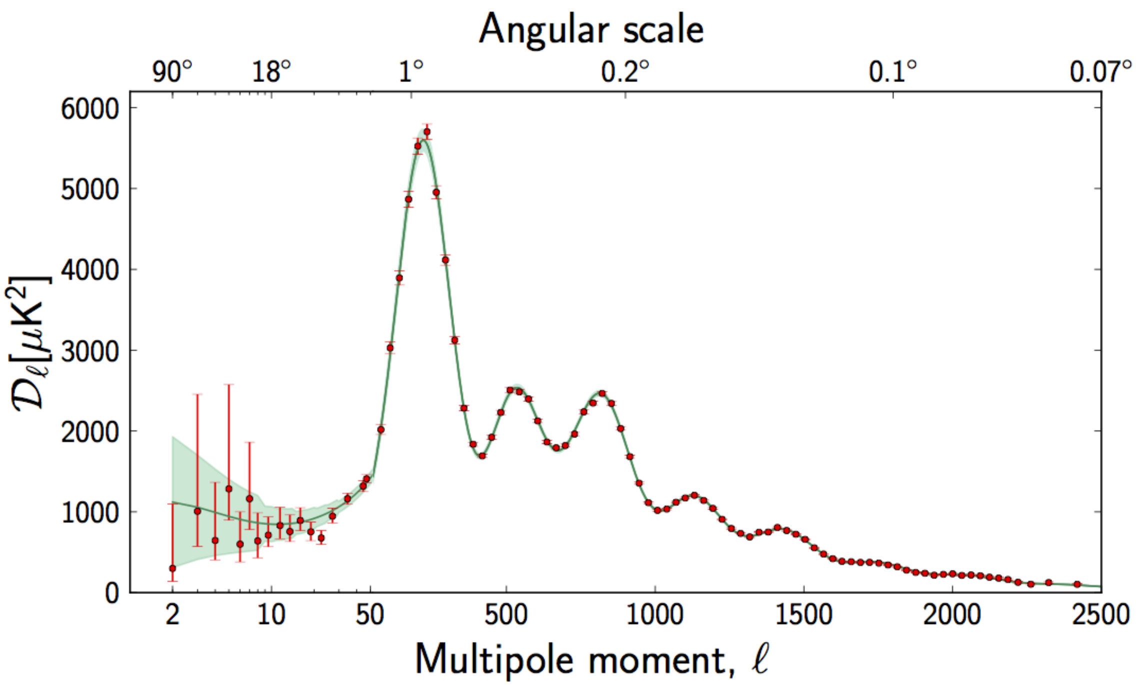
Angular diameter distance

If you have an object that is a standard candle and at the same time a standard yardstick then the angular distance is measured to be much smaller than its luminosity distance. This gives rise to an unusual effect in the measurement of d_A as we see in the figure shown here, at high redshift the distance to objects in certain models actually decreases?

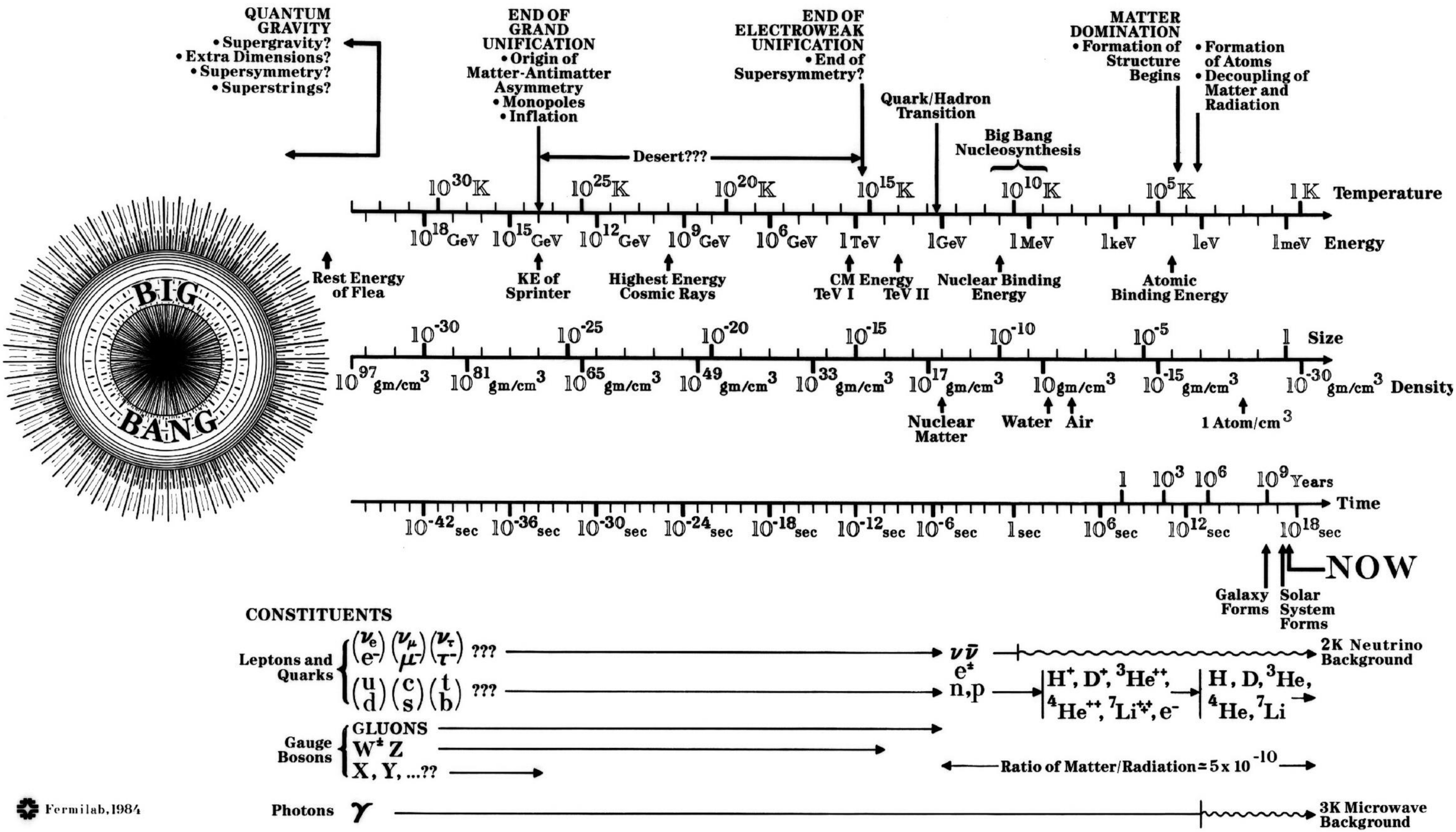


Measuring W_m and W_L





The thermal history of the Universe



Decoupling of particles

In a strict mathematical sense the Universe could not ever be in thermal equilibrium because of its expansion (FLRW metric). In practical terms however the Universe has been in most of its history very near thermal equilibrium. As we mentioned earlier, it is clear also that departure from such equilibrium is essential to explain many features of what we observe around us. Departure from equilibrium often (not always) results in a decoupling of a certain species from some others.

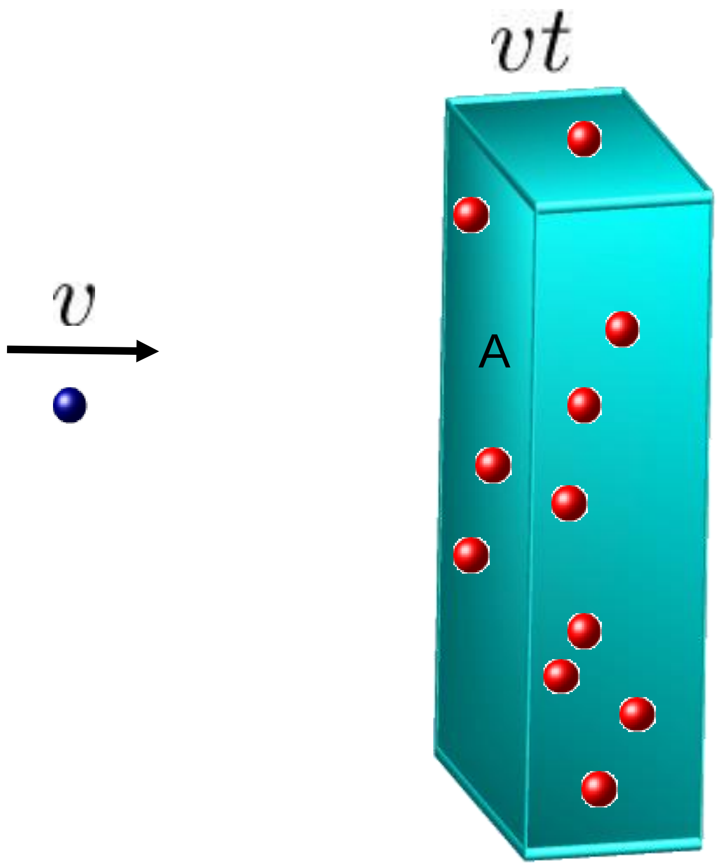
The key to understanding the thermal history of the Universe lies in the relation between the expansion rate of the Universe (which translates to the rate of change in the Universe's temperature) and the interaction rate between the particle in question and the other particles in the Universe. The expansion rate of the Universe at any redshift is given by the Hubble function and the interaction rate is given by $G = ns|v|$.

If $G > H$ then the interaction rate is faster that of the expansion rate particle stays in thermal equilibrium, whereas if $G < H$ then the species gets out of thermal equilibrium with the rest.

Notice, that even if $G < H$ and the particle is no longer with thermal equilibrium with the other species it does NOT mean that its distribution in not thermal anymore. The departure from thermal distribution requires an interaction that gets it out of such equilibrium.

Example of the last two points are: 1- the CMB, which has decoupled from everything else but maintained thermal distribution and 2- Gas at low redshifts – which gets heated inhomogeneously by astrophysical processes.

Interaction rate



Area covered by particles in this volume is given by:

$$nV\sigma = nAvt\sigma$$

There will be an interaction if the area covered by these particles is of order A. Namely,

$$\begin{aligned} nAvt\sigma &\approx A \\ nvt\sigma &\approx 1 \end{aligned}$$

Which gives an interaction rate:

$$1/t \approx nv\sigma$$

The thermal history of the Universe

The isotropy and the perfect black body spectrum of the CMB show that matter was in thermal equilibrium in the early Universe (at least until the CMB photon were released). Also, it is clear that prior to decoupling the Universe was not transparent. Therefore, we can use the laws of thermodynamics to obtain the thermal history of the early Universe. As we shall see, including both the thermodynamics and the properties of the fundamental particles in the Universe several predictions can be made about the Universe (e.g., the big-bang nucleosynthesis – BBN, recombination and decoupling).

Let's start with a reminder of the number of photon per baryon, the so-called h. Using the current radiation energy density (from the CMB) one can easily conclude that there is 411 CMB photons per cm^3. Similarly, using the baryonic mass energy density (assuming the baryonic matter mass is dominated by protons and neutrons) one can calculate the number of baryons per cm^3 to be, $\sim 0.22 \times 10^{-6}$. Hence $h \sim 5 \times 10^{-10}$. The small value of this number has implications to all kinds of issues that will be discussed in this and the coming few lectures.

The small value of η is a strong indicator for the baryon antibaryon asymmetry in the Universe. (we might discuss this issue later).

Calculate how the baryonic mass (gas) temperature varies with redshift in the case in which baryons and photons are decoupled.

Thermodynamics

In order to follow the thermodynamic evolution of the Universe we remember that elementary particles can be divided into two categories, Bosons and Fermions and we would like to obtain their evolution with time in the case of thermal equilibrium.

An example that we have seen before is that of photons where the distribution is given by the Planck function:

$$n(\nu)\,d\nu = \frac{8\pi}{c^3}\frac{\nu^2\,d\nu}{\exp(h\nu/kT)-1}$$

Which yields the following number of photons per cubic cm,

$$\begin{aligned} n_\gamma &= \int n(\nu)\,d\nu = \left(\frac{kT}{hc}\right)^3 8\pi \int_0^\infty \frac{y^2\,dy}{e^y-1} \\ &= \left(\frac{kT}{hc}\right)^3 16\pi\zeta(3) \\ &\simeq 411.\ \mathrm{cm}^{-3} \end{aligned}$$

Here the z is the so-called Riemann zeta function z(3)@ 1.202 obviously the number of photons per unit comoving volume does not change with redshift, which is a result we noted earlier.

Thermodynamics

More generally we should discuss massive particles and here the general distribution that such particles will follow for a given momentum, ***p***, is the Bose-Einstein or Fermi-Dirac distributions:

$$n(\mathbf{p})d^3p = \frac{1}{e^{(\epsilon(p)-\mu)/kT} \pm 1}\frac{d^3p}{h^3}$$

where the plus sign is for fermions and the minus sign is for bosons. The energy ϵ is related to the momentum through the relativistic form $\epsilon^2 = m^2c^4 + p^2c^2$ and m is the chemical potential.

For photons to convert this relation to Planck distribution (for Bosons) we remember that J=hn=*pc* and that g=2, where g is the quantum weight of each state is 2 for polarization (helicity).

Now in the relativistic limit these particles behave like photons. That is to say, the mass contribution to the energy is negligible, and one can calculate the total number of particles through simple integration. Here we also assume that the chemical potential is negligible for such temperatures (m << ϵ). In such case we find,

$$\varepsilon = g\frac{\pi^2}{30(\hbar c)^3}(kT)^4 \times \begin{cases} 7/8 & \text{Fermions} \\ 1 & \text{Bosons} \end{cases}$$

$$n = g\frac{\zeta(3)}{\pi^2(\hbar c)^3}(kT)^3 \times \begin{cases} 3/4 & \text{Fermions} \\ 1 & \text{Bosons} \end{cases}$$

Thermodynamics

The average energy per particle is then given by:

$$\langle E \rangle = \frac{\rho}{n} = \begin{cases} \dfrac{7\pi^4}{180\zeta(3)} T \approx 3.151 T & \text{fermions} \\ \dfrac{\pi^4}{30\zeta(3)} T \approx 2.701 T & \text{bosons}\,. \end{cases}$$

One can also show that in this case *p*=e/3.

The total energy density requires summing for e over all relativistic particle species i, which may each have a different temperature T_i, giving

$$\varepsilon_r = g_*(T) \frac{\pi^2}{30(\hbar c)^3} (kT)^4$$

Where we define the effective degrees of freedom as,

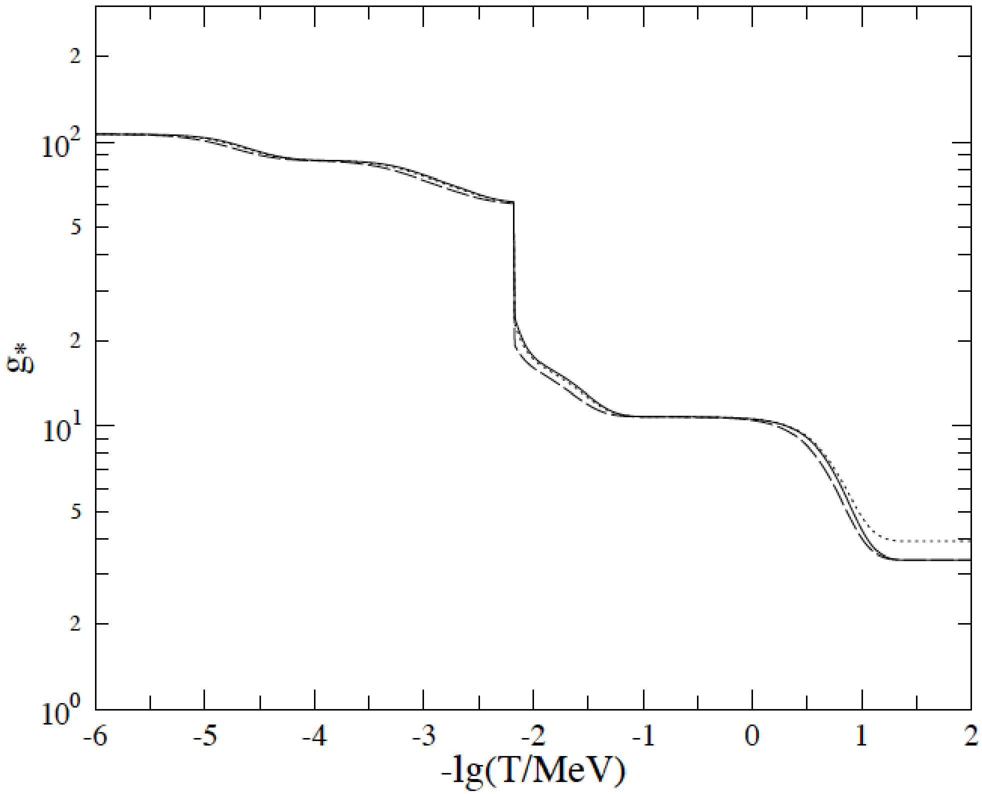
$$g_* = \sum_{i \in \text{bosons}} g_i \left(\frac{T_i}{T} \right)^4 + \frac{7}{8} \sum_{i \in \text{fermions}} g_i \left(\frac{T_i}{T} \right)^4$$

Thermodynamics

Quarks	t	173.0 ± 1.6GeV	$\bar{t}$	spin=$\frac{1}{2}$	$g = 2 \cdot 2 \cdot 3 = 12$	
	b	4.19 ± 0.18GeV	$\bar{b}$	3 colors		
	c	1.27 ± 0.09GeV	$\bar{c}$			
	s	101 ± 29MeV	$\bar{s}$			
	d	4.1–5.8MeV	$\bar{d}$			
	u	1.7–3.3MeV	$\bar{u}$			
						72
Gluons	8 massless bosons			spin=1	$g = 2$	16
Leptons	τ^-	1776.82±0.16MeV	τ^+	spin=$\frac{1}{2}$	$g = 2 \cdot 2 = 4$	
	μ^-	105.658MeV	μ^+			
	e^-	510.999keV	e^+			
						12
	ν_τ	< 2eV	$\bar{\nu}_\tau$	spin=$\frac{1}{2}$	$g = 2$	
	ν_μ	< 2eV	$\bar{\nu}_\mu$			
	ν_e	< 2 eV	$\bar{\nu}_e$			
						6
Electroweak gauge bosons	W^+	80.403 ± 0.029GeV		spin=1	$g = 3$	
	W^-	80.399 ± 0.023GeV				
	Z^0	91.1876 ± 0.0021GeV				
	γ	0 $(< 1 \times 10^{-18}$eV)			$g = 2$	
						11
Higgs boson (SM)	H^0	125.35 ± 0.15GeV		spin=0	$g = 1$	1

$$g_f = 72 + 12 + 6 = 90$$
$$g_b = 16 + 11 + 1 = 28$$

Thermodynamics



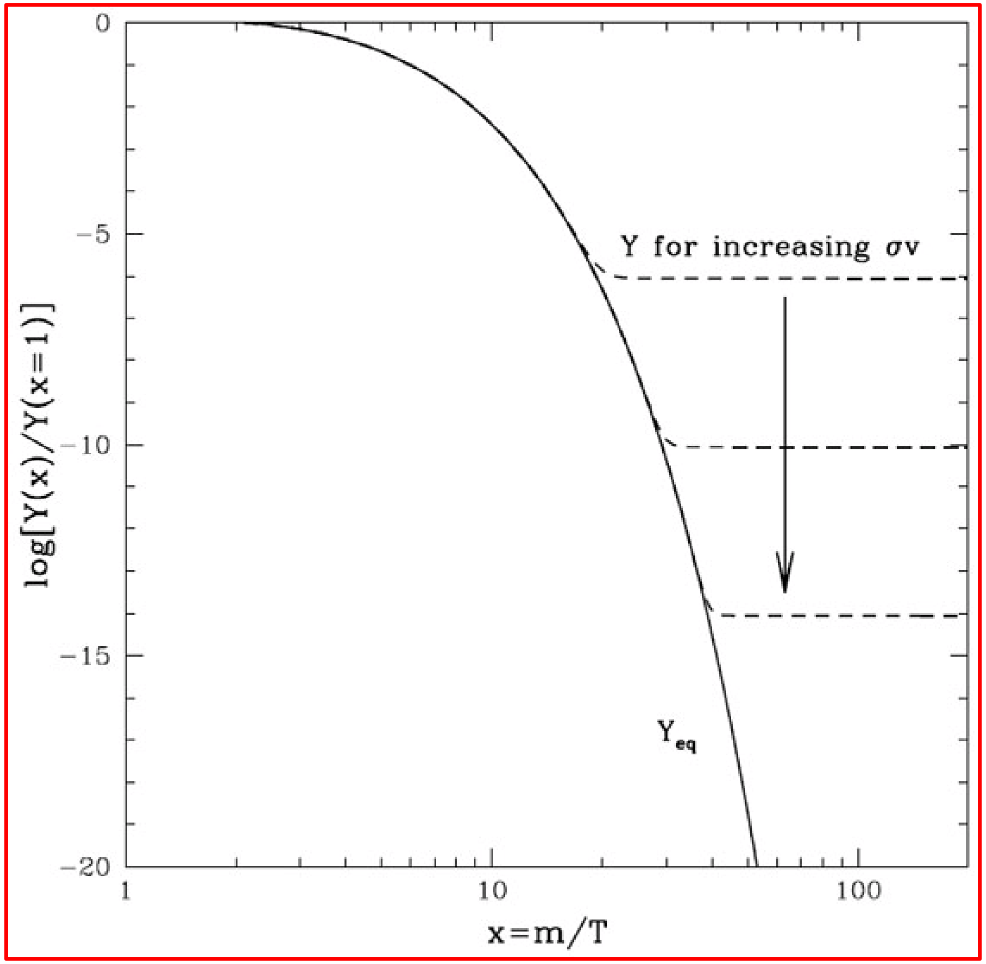
The evolution of the effective g with as a function of temperature (time) for the standard model for particle physics. Notice that at low temperature the only relativistic species are the photons and 3 neutrino families. Since the neutrino temperature is different from the photon temperature (Tn=(4/11)$^{1/3}$ Tg one obtains a value of 3.36 for g_* at the low temperature end. For T>100 MeV the electron and proton become relativistic, and they should be taken into account giving rise to next plateau in this figure (g_* ~ 10). And finally at T> 300 MeV all fundamental particles become relativistic and g_* rises to about a 107.

Thermodynamics

Once the temperature in the Universe drops much below the particle mass ($kT \ll mc^2$), then we get to the other limits in which gives:

$$n_P = g\left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2} e^{-(mc^2-\mu)/kT}$$

In this case the number density of particles drops exponentially to very small numbers (try to substitute the mass of hydrogen atoms at z=0 you will get around $10^{-10^{12}}$). This sudden drop to unrealistic numbers is obviously not correct and the reason for it is that we assume thermodynamic equilibrium even when we shouldn't. Actually, this turns to be the most important aspect of the determination of the number density of various particles as a function of redshift. We will see later how to deal with the non-equilibrium case which is essential to obtain elements abundance, deal with recombination, etc.



Chemical potentials

In rough terms, it is the rate of change of the internal energy of the system with the change of the number of particles, provided everything else is the same. Formally speaking it is the partial derivative of the internal energy with respect to the change of the number of particles of a certain species, where the entropy, density and number of all other species remain constant. Similar definitions could be achieved for the other thermodynamical energy functions (enthalpy, Gibbs and Helmholtz energies).

In some of our discussion we will assume that m=0 which reflects the conservation of the charge, baryon and lepton numbers after the GUT era. This obviously is a wrong assumption in many cases, say, if you have annihilating particles out of equilibrium, etc.

In particular, this assumption will not be correct in the case of species are changing say for example in the case formation of neutral hydrogen out of electrons and proton where the number of particles clearly changes during the interaction. In such cases however, it is possible to calculate the chemical potential explicitly or show that it does not change due to equilibrium.

For photons the chemical potential is always zero.

Neutrino decoupling

Before the annihilation of ⌈+□⌈ pairs at T ~10^{12} K, the Universe is composed mainly of $e^-, e^+,$ $\mu^-, \mu^+, \nu_r, \nu_e, \bar{\nu}_e, \nu_\mu, \bar{\nu}_\mu, \nu_\tau, \bar{\nu}_\tau$ and γ.
The neutrinos are still in thermal equilibrium through scattering reactions of the form

$$\begin{array}{llll} e^- + \mu^+ \rightleftharpoons \nu_e + \nu_\mu, & e^- + p \rightleftharpoons \nu_e + n, & \mu^- + p \rightleftharpoons \bar{\nu}_\mu + n, & \cdots \\ e^+ + e^- \rightleftharpoons \nu_e + \bar{\nu}_e, & e^\pm + \nu_e \rightleftharpoons e^\pm + \nu_e, & \cdots, & \end{array}$$

For the weak interaction processes relevant for neutrinos, the cross section is given by,

$$\sigma_{wk} \simeq g^2 \left[\frac{k_B T}{(\hbar c)^2}\right]^2$$

where g_{wk}~ 1.4x10^{-49} erg cm^3. From this we obtain that G=n s_{wk} $v \sim G_F^2\, T^5$ where G_F is the so-called Fermi constant,

$$\frac{G_F}{(\hbar c)^3} = \frac{\sqrt{2}}{8}\frac{g^2}{m_W^2} = 1.16637(1) \times 10^{-5} \text{GeV}^{-2} .$$

$$= 4.54376 \times 10^{14}\ \text{J}^{-2}$$

when the ratio between the interaction rate and expansion rate reaches unity, the neutrinos decouple from the other particles. This happens when G/H ~(k_BT/ MeV)^3~1. Hence the neutrino decoupling occurs at a temperature of ~ 3x10^{10} K.

Neutrino Temperature

When neutrinos decouple from the plasma their temperature changes (notice, after they decouple formally, we can't talk about their temp.). To describe their behavior, it is appropriate to approximate the expansion of the Universe as adiabatic namely, $d(sa^3)=0$.

Now, we also need to define:

$$g_{*s} = \sum_{i\in\text{bosons}} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_{i\in\text{fermions}} g_i \left(\frac{T_i}{T}\right)^3$$

As the Universe cools and the number of degrees of freedom reduce the energy density and entropy are transformed (through electron positron annihilations process) from electrons, proton and neutrons to photons and not to neutrinos which means that radiation is heated up relative to neutrinos. It is also worth noting that the neutrinos temperature will evolve after they decouple as inverse the scale factor (like photons).

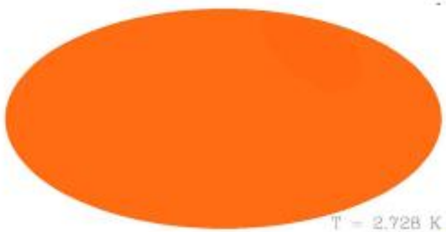
Now connecting the entropy before and after electron-positron annihilation gives:

$$2T_2^3 a_2^3 + 5.25 T_{\nu 2}^3 a_2^3 = 10.75 a_1^3 T_1^3$$

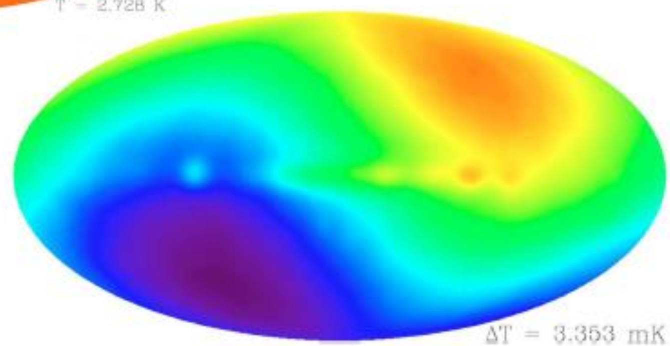
Where the indices 1 and 2 refer to the situation before and after the annihilation. Remember, before e^+-e^- annihilation g_{*s} = 2 (g's) + 3.5 (e^+& e^-) + 5.25 (n's) = 10.75 whereas after it is 2 for g's and 5.25 for n's. Since the neutrinos are not heated during annihilation and they were in equilibrium before this the following equation holds $a_1^3 T_1^3 = a_1^3 T_{n1}^3 = a_2^3 T_{n2}^3$.

Putting all of this together will then gives that after e^+-e^- annihilation, $T_n = (4/11)^{1/3}$ T.

CMB observations

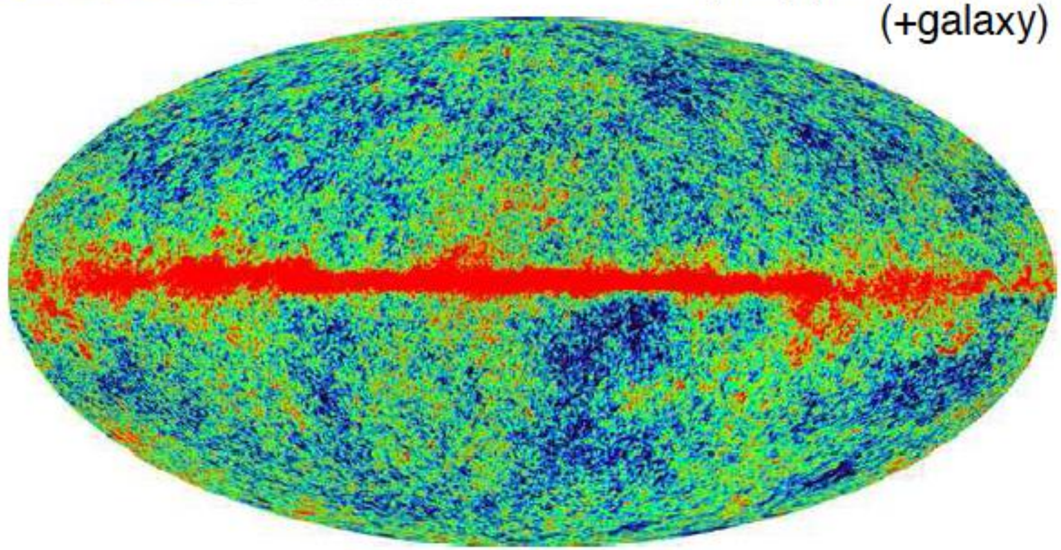


(almost) uniform 2.726K blackbody



Dipole (local motion)

O(10^{-5}) perturbations
(+galaxy)



Observations:
the microwave
sky today

Source: NASA/WMAP Science Team

Recombination and Decoupling

Here we will discuss how the Universe transformed from being ionized plasma to neutral gas (mostly Hydrogen). In this discussion we'll ignore Helium and other atoms that the Universe might have; these components will complicate the calculation but will not alter the basic physical picture that we will obtain. The main interaction that we will be concerned about is the following: $p + e \leftrightarrows H + g$, where the binding energy of hydrogen Q=13.6 eV (1 Ry).

Naively, one should expect that the recombination would freeze out when the Universe's temperature is of the order of the hydrogen binding energy. This would translate to a redshift of the Universe of the order of z~60000 and not 1100 (3000 K) as we observe today.
The reason for this is that the photon spectrum is a black body one and a certain fraction of its photons will have energy equal or larger than the hydrogen atom binding energy even at lower temperature. This fact coupled with very large number of photons-per-baryon result in a significant delay of the freeze out process until the Universe reaches 3000 K.

The other important interaction that one should consider is Thomson scattering which will change significantly as a result of hydrogen recombination due to the fact that this process is not efficient with neutral hydrogen. Thomson scattering interaction is: e + g → e + g which has a cross section of:

$$\sigma_T = \frac{8\pi}{3}\frac{\alpha\hbar}{m_e c} \simeq 6.7 \times 10^{-25}\ \mathrm{cm}^2$$

The Saha Equation

The Saha eq., which is derived from the Maxwell-Boltzmann distribution, quantifies the atoms in different ionization states. Here we will derive it for hydrogen ionization.

Recall that the Maxwell-Boltzmann distribution is given by:

$$n_i = g_i \left(\frac{m_i k T_i}{2\pi\hbar^2} \right)^{3/2} \exp\left[-E_i / kT_i \right]$$

We now take the ratio between the number of atoms in the ground state n_H and the number of atoms in the ionized state n_p. Since the the ionized state involves electrons we define $dn_p(v)$ which is the number of protons with electrons in the velocity range $v - v+dv$. This ratio is given by:

$$\frac{dn_p(v)}{n_H} = \frac{g_p g_e}{g_H} \left(\frac{m_p}{m_H} \right)^{3/2} \exp\left(-\frac{Q + \frac{1}{2} m_e v^2}{kT} \right) \left(\frac{m_e^3}{n_e h^3} \right) d^3 v$$

The factor of $m_e^3/(n_e h^3)$ comes from the phase space volume occupied by free electrons. Now we integrate over the velocities to obtain the following relation (Saha equation):

$$\begin{aligned} \frac{n_p n_e}{n_H} &= \frac{g_p g_e}{g_H} \left(\frac{4\pi m_e^3}{h^3} \right) \left(\frac{m_p}{m_H} \right)^{3/2} \exp\left(-\frac{Q}{kT} \right) \int \exp\left(-\frac{m_e v^2}{2kT} \right) v^2 dv \\ &= \frac{g_p g_e}{g_H} \left(\frac{4\pi m_e^3}{h^3} \right) \left(\frac{m_p}{m_H} \right)^{3/2} \exp\left(-\frac{Q}{kT} \right) \left(\frac{2kT}{m_e} \right)^{3/2} \frac{\pi^{\frac{1}{2}}}{4} \\ &= \frac{g_p g_e}{g_H} \left(\frac{m_p m_e}{m_H} \right)^{3/2} \left(\frac{2\pi kT}{h^2} \right)^{3/2} \exp\left(-\frac{Q}{kT} \right) \end{aligned}$$

The Saha Equation

The Saha eq., which is derived from the Maxwell-Boltzmann distribution, quantifies the atoms in different ionization states. Here we will ... tion.

Recall that the Maxwell-Boltzmann d...

$$n_i = \ldots$$

We now take the ratio betw... of atoms in the ionized state... $dn_p(v)$ which is the number o... is given by:

$$\frac{dn_p(v)}{\ldots} = \frac{g_p g \ldots}{g_H} \ldots$$

The factor of $m_e^3/(n_e h^3)$ comes fro...
Now we integrate over the velocities ...

In order to account for the number of possible electronic states one need to look at the phase space density:

$g_e/h^3\, dx_1 dx_2 dx_3 dp_1 dp_2 dp_3 = g_e m_e^3/h^3\, dV dv_1 dv_2 dv_3$

This finally gives the statistical weight:

$g_e m_e^3/(h^3 n_e)\, dv_1 dv_2 dv_3$

$$\begin{aligned}
\frac{n_p n_e}{n_H} &= \frac{g_p g_e}{g_H}\left(\frac{4\pi m_e^3}{h^3}\right)\left(\frac{m_p}{m_H}\right)^{3/2} \exp\left(-\frac{Q}{kT}\right)\int \exp\left(-\frac{m_e v^2}{2kT}\right) v^2 dv \\
&= \frac{g_p g_e}{g_H}\left(\frac{4\pi m_e^3}{h^3}\right)\left(\frac{m_p}{m_H}\right)^{3/2} \exp\left(-\frac{Q}{kT}\right)\left(\frac{2kT}{m_e}\right)^{3/2}\frac{\pi^{\frac{1}{2}}}{4} \\
&= \frac{g_p g_e}{g_H}\left(\frac{m_p m_e}{m_H}\right)^{3/2}\left(\frac{2\pi kT}{h^2}\right)^{3/2} \exp\left(-\frac{Q}{kT}\right)
\end{aligned}$$

Recombination and Decoupling

Now we will consider equilibrium between ionized and neutral hydrogen. Here we will have a number of assumptions that we'll mention at the stage we use each. The first assumption is that each of the component of the interaction (except the photons) follows a Maxwell-Boltzmann's distribution:

$$n_i = g_i \left(\frac{m_i k T_i}{2\pi \hbar^2} \right)^{3/2} \exp\left[-E_i / k T_i\right]$$

If we ignore Helium then the equilibrium is given by

$$n_H = \frac{g_H}{g_p g_e} n_p n_e \left(\frac{m_H}{m_e m_p} \right)^{3/2} \left(\frac{kT}{2\pi \hbar^2} \right)^{-3/2} \exp\left[\frac{Q}{kT} \right]$$

The statistical weights of protons, electrons and hydrogen atoms are 2, 2 and 4 respectively. We then can define the fraction of free protons relative to total baryon number as

$$X_e = \frac{n_p}{n_B} = \frac{n_p}{n_p + n_H}$$

Recombination and Decoupling

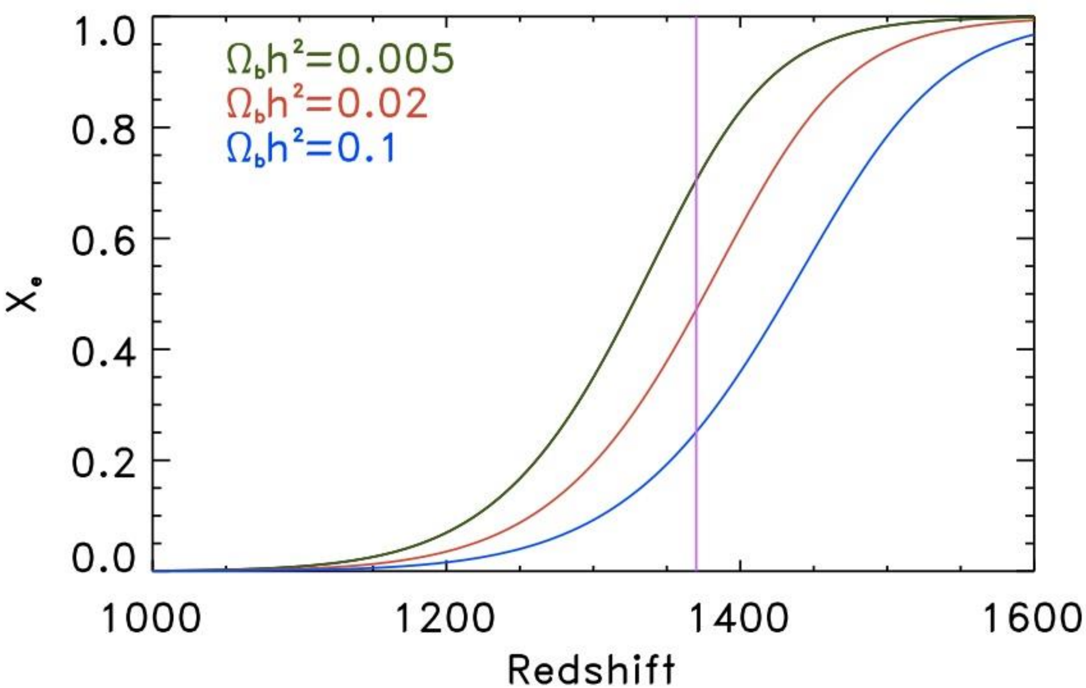
Substituting hn_g for the baryon number density one gets,

$$\frac{1-X_e}{X_e^2} = \frac{4\sqrt{2}\zeta(3)}{\sqrt{\pi}}\eta\left(\frac{kT}{m_ec^2}\right)^{3/2}\exp\left[\frac{Q}{kT}\right]$$

Remember that the photon number density is,

$$n_\gamma = \left(\frac{kT}{hc}\right)^3 16\pi\zeta(3)$$

Now defining the epoch of recombination as that when X_e is half we get z~1300



A more accurate calculation that includes the other species as well as uses not equilibrium calculations (Boltzmann's equation) one get z~1200.

Recombination and Decoupling

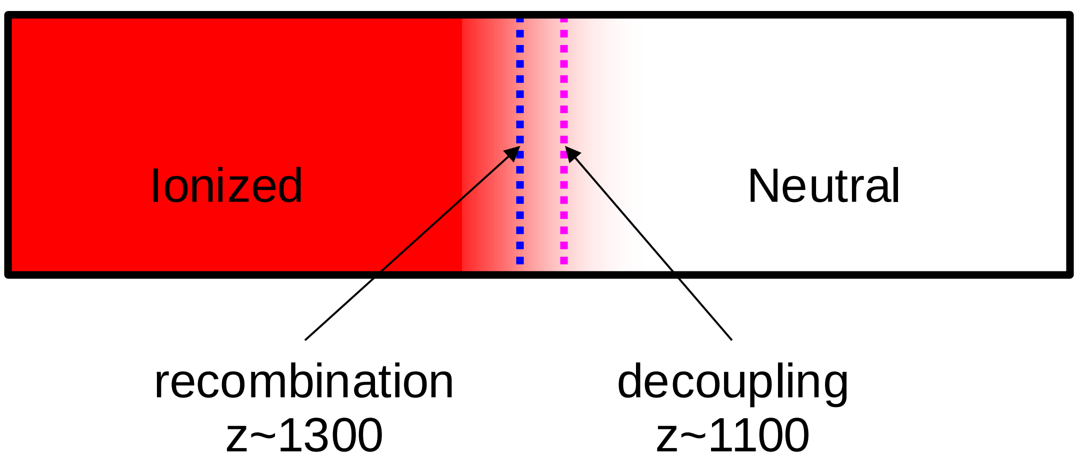
Now we will calculate the freeze out time for Thomson scattering, namely, the interaction:

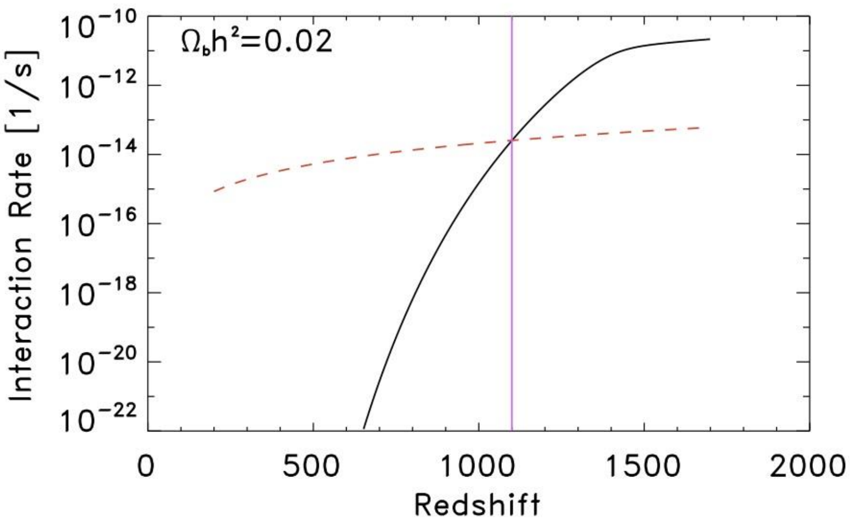
$$e^- + \gamma \rightleftharpoons e^- + \gamma$$

For that we should compare the scattering rate $\Gamma \simeq n_e \sigma_T c$ with the expansion rate, *H*. The only thing to calculate then is the electron number density which we could calculate from X_e as,

$$n_e = X_e n_B = X_e \eta \frac{3\zeta(3)}{2\pi^2} \left(\frac{kT_0}{\hbar c} \right)^3 (1+z)^3$$

Going through the numbers give that the two rates become equal roughly at z_{dec}~1100. Then the ionized fraction, X_e is of the order of 10^{-4}, namely, the freeze out happens when the Universe is largely neutral. The following picture hence emerges:





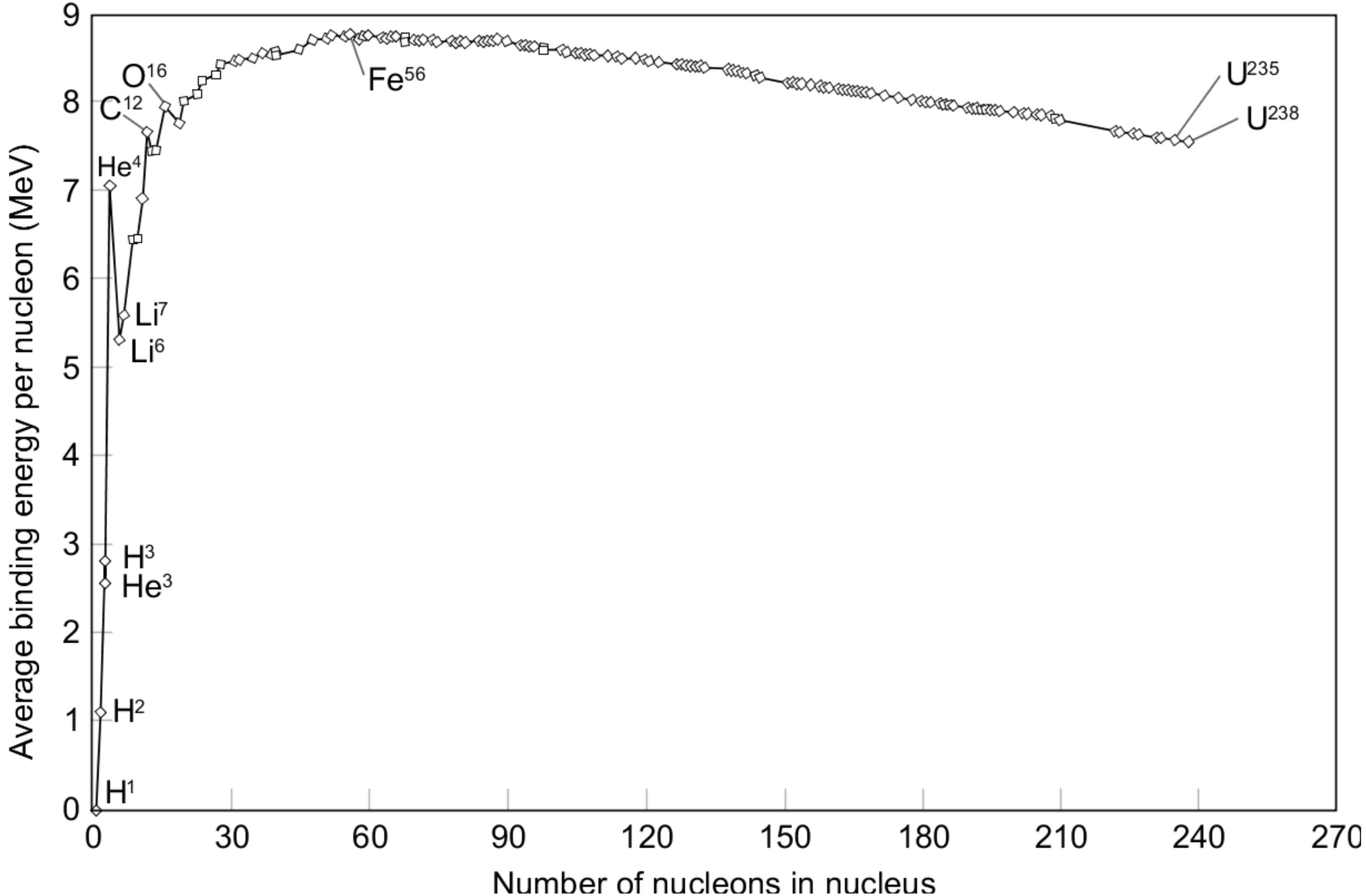
Big Bang Nucleosynthesis

Predicting the abundance of primordial elements is one of the great successes of the Big Bang theory. Other alternative theories would struggle to recover the observed abundances whereas within the Big-Bang model they are a natural predictions of the theory. For example, in the Steady State theory all elements heavier than hydrogen will have to be produced by stars which in light of the ~25% helium abundance (in mass) creates a huge problem for this theory – this fact was recognized by one of the fathers of the Steady State theory, Fred Hoyle.

Here we'll discuss how the Universe synthesized the light elements from the primordial mix of protons and neutrons. This is called the Big Bang Nucleosynthesis (BBN). In a broad sense, the discussion in this section is similar to that of the recombination and decoupling. Very early on the light elements (deuterium, tritium, helium, lithium, etc.) are in statistical equilibrium with each other. However, as the Universe expands and cools down, they freeze out at a certain redshift keeping a fixed abundance of these elements relative to hydrogen that we see around us today (in the intergalactic medium). There are some complications, relative to the CMB discussion, that have to be considered here due to the nature of the nuclear interactions involved in the process.

The most stable light element is Helium (with a binding energy of 28.5 MeV) which means in equilibrium we expect that most nuclei will end up in Helium. However, the calculation of this process involves taking into account all light elements coupled together which leads to a very complicated set of coupled equations. Here we'll simplify the process and try to account for the main results that are observed.

Big Bang Nucleosynthesis



Big Bang Nucleosynthesis

Neutron-proton freeze out:

The first step in the process of forming light elements is the formation of neutron. We first calculate the freeze-out temperature for this interaction. The formation of neutrons depends on a number of interactions the most important of which are:

$$n \rightleftharpoons p + e^- + \bar{\nu}_e, \qquad n + \nu_e \rightleftharpoons p + e^-, \quad \text{and} \quad n + e^+ \rightleftharpoons p + \bar{\nu}_e$$

These interaction's freeze-out time could be calculated using the same formula used in the neutrino freeze-out discussion (these are all weak interactions). Here however one needs to be more accurate which gives 0.8 MeV as the freeze-out temperature.

Neutron-proton ratio at freeze-out:

Here we will again use Saha's equation. In all of these interactions n and p are non-relativistic and the ratio between them is,

$$\frac{n_n}{n_p} = \exp\left[-\frac{\Delta mc^2}{k_B T_F}\right] \simeq \exp[-1.29/0.8] \simeq 0.2$$

Where 1.29MeV is the difference between the neutron and proton masses. This small number of neutrons is the main reason that BBN is an incomplete process, simply, there is not enough neutrons to push the process further and most protons remain free. The actual ratio here turns out to be ~0.17.

Big Bang Nucleosynthesis

From this number we can already guess what would be the ratio between helium and hydrogen. Assuming that all of these neutrons will end up in helium. For each 10 protons we'll have roughly 2 neutrons. Now for helium we need 2 of the protons and the rest stay free. Hence in maximum possible ratio of helium is 4/12~1/3. The actual number comes out to be smaller because during the formation of helium some of the neutrons decays (free neutron mean lifetime is 885.7 s). A proper calculation will give a fraction of $Y_p = 0.245$.

Deuterium formation:

The main channel for forming deuterium is: $p + n \rightarrow d + \gamma$

This channel dominates over the p+p and n+n channels because both will involve weak interactions (to turn p to n) and the former should overcome Coulomb barrier whereas the later has a small rate because of the small neutron fraction of baryons. This interaction should happen around 2MeV however, like recombination, the large number of photons-per-baryon the interaction freezes out only around 0.05MeV (about 3 minutes after the Big-Bang). Even at this point the number of deuterium nuclei per unit volume will be very small. As a result, there is a very low number of interactions of dd to form helium. This is known in the literature as the **deuterium bottleneck** lasting until a temperature of 0.05 MeV.

Big Bang Nucleosynthesis

Next in the process is one of the two following interaction chains:

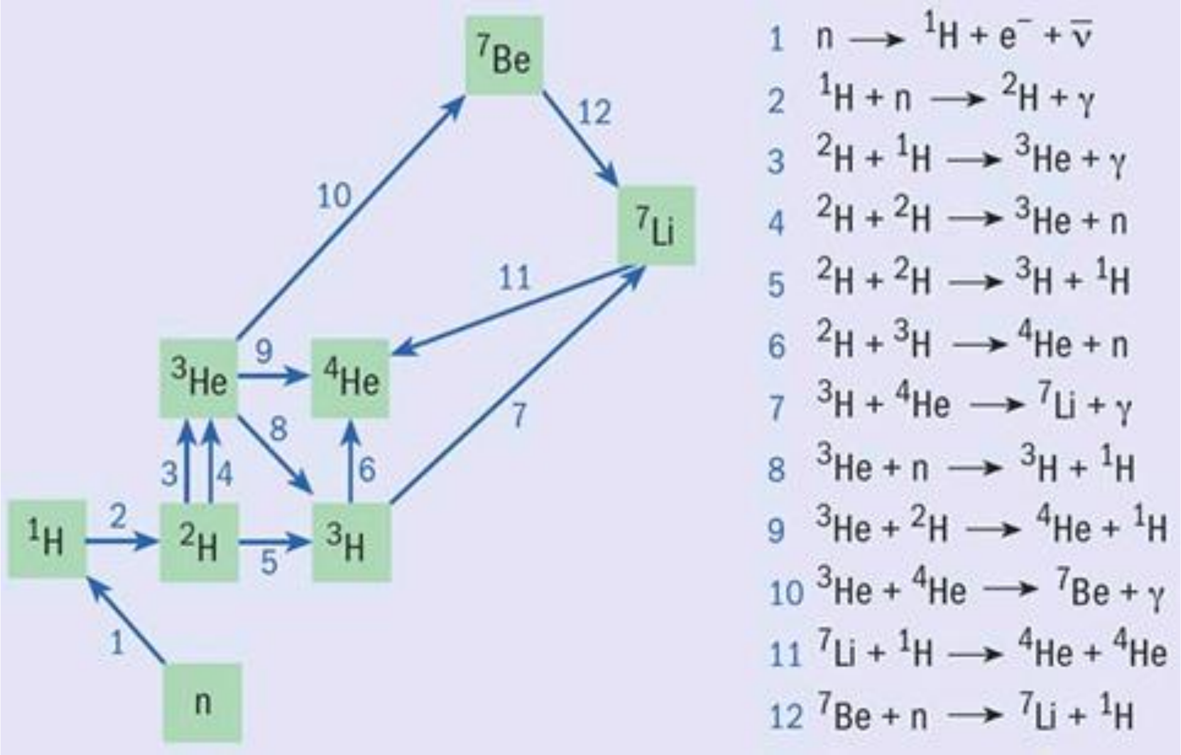
$$d + d \to {}^3\mathrm{He} + n \qquad {}^3\mathrm{He} + d \to {}^4\mathrm{He} + p$$

Or

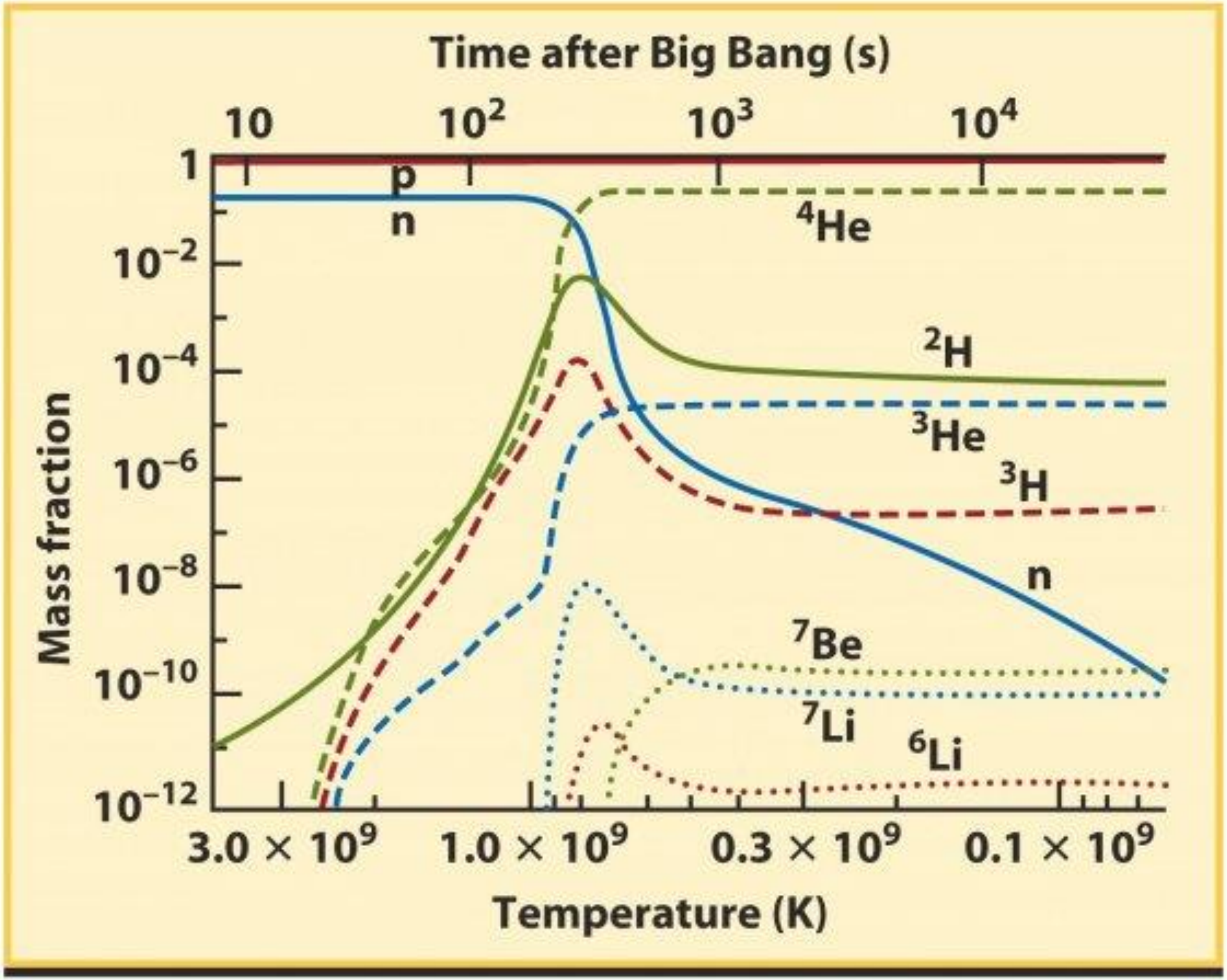
$$d + d \to t + p \qquad t + d \to {}^4\mathrm{He} + n$$

Calculating the proper numbers here yields a fraction of helium in mass Y~0.24.

In a real calculation of the BBN process, one must consider the following interactions:

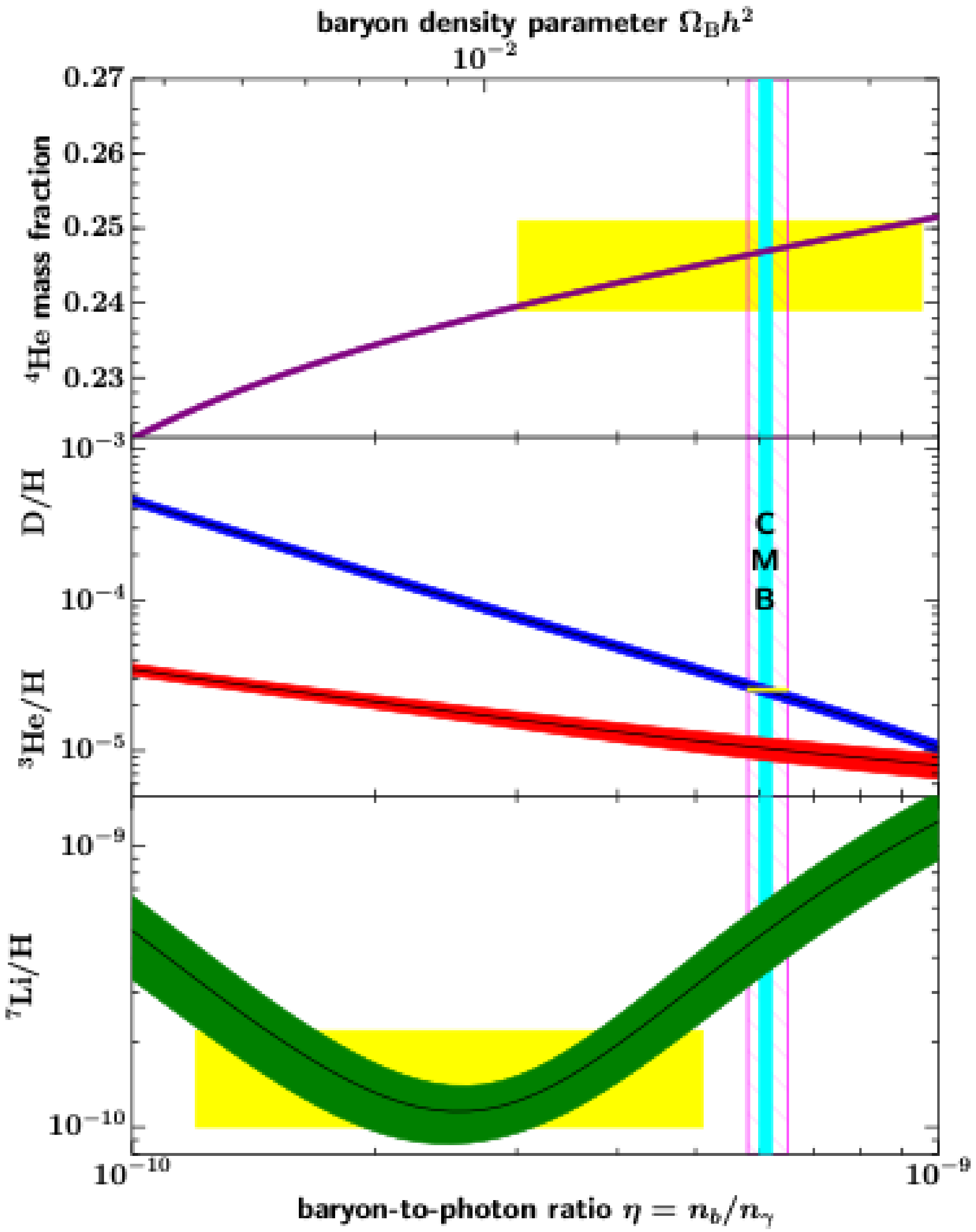


Big Bang Nucleosynthesis



Big Bang Nucleosynthesis

The mass fraction of various species as a function of the baryon density. Bands show the 95% confidence range. Boxes indicate observed light element abundances (smaller boxes: ±2σ statistical errors; larger boxes: ±2σ statistical and systematic errors). The narrow vertical band indicates the CMB measure of the cosmic baryon density, while the wider band indicates the concordance range of direct measurements of the light element abundances.

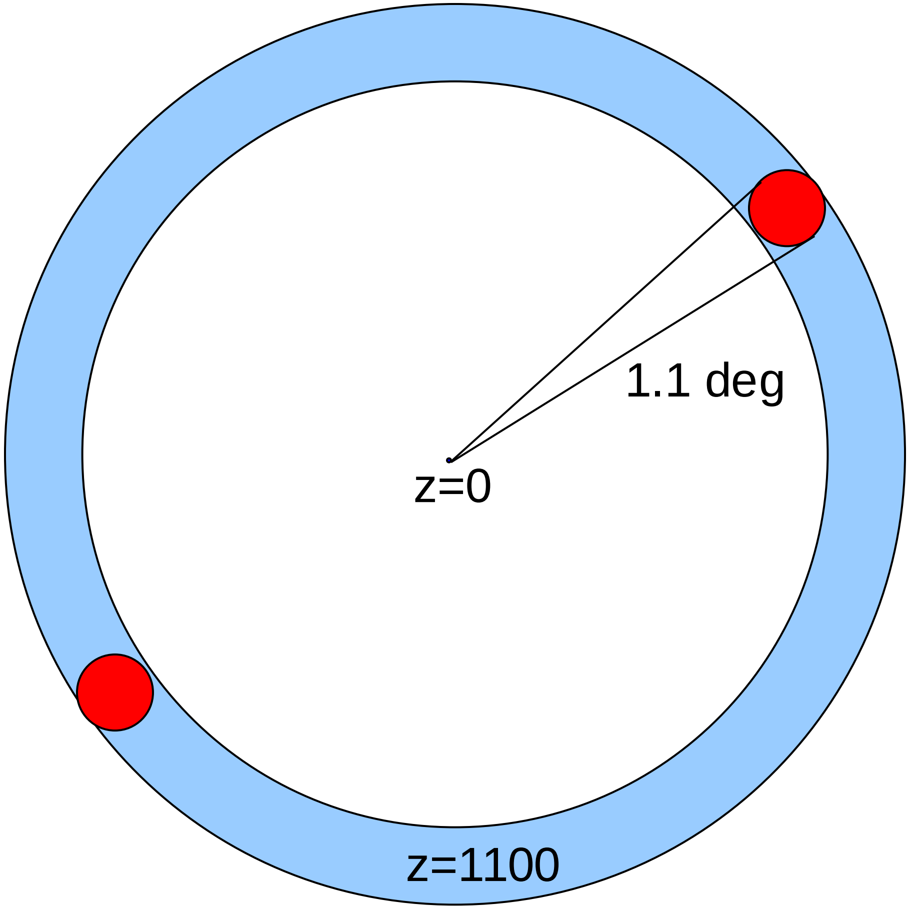


From Fields et a. 2023

Problems with the Big Bang:
1-The Horizon problem

A number of fundamental problems arise in the context of the standard big bang theory. The first is the so-called Horizon problems, which comes from the question: why causally disconnected regions on the sky have the same temperature (down to 10^{-5} accuracy)?

The horizon distance at the redshift of recombination is roughly 0.44 Mpc which corresponds to angular diameter distance as observed at z=0 of about 1.1 degrees.



How does the two red spots know how about each other and even be in thermal equilibrium even though they have never been in causal contact?

Problems with the Big Bang:
2- The Flatness problem

The Universe is currently flat with a few percent accuracy (the measurement error). We recall (from slide 42) that the curvature “density” parameter evolves like $1/(a^2H^2)$. Since currently this number is of the order of 10^{-2} at z=1100 it was $<10^{-6}$; at BBN of the order of 10^{-18}.

$$1-\Omega(t) = \frac{H_0^2(1-\Omega_0)}{H(t)^2 a(t)^2} = \frac{a^2(1-\Omega_0)}{\Omega_{r,0}+a\Omega_{m,0}}$$

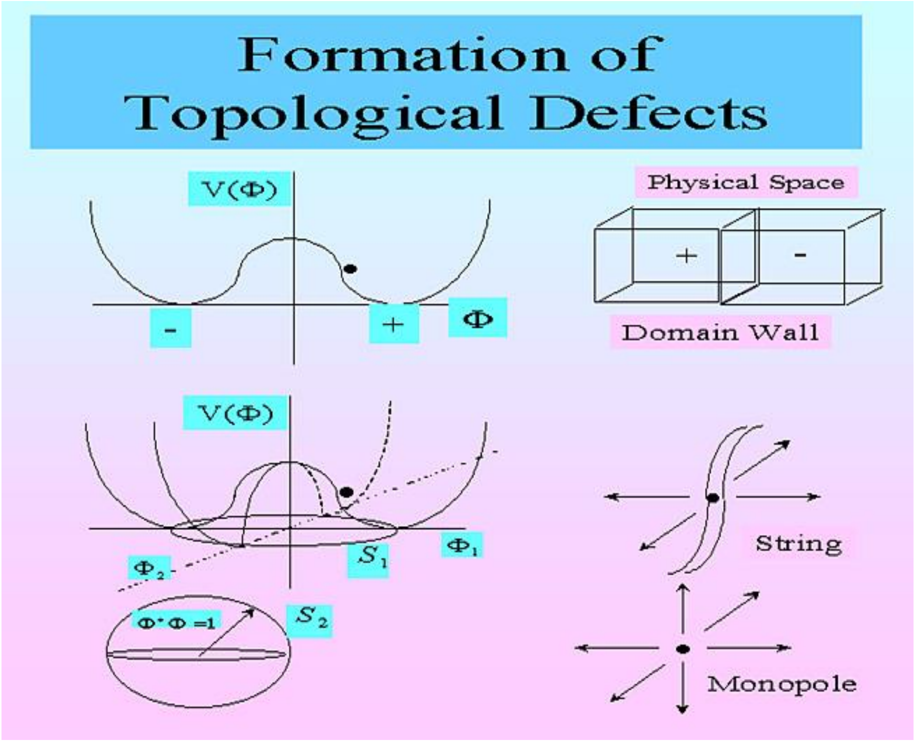
(This is a universe dominated by radiation and matter). One can see that the Planck time this was $|1-\Omega_P| \leq 2\times 10^{-62}$

This mean that the Universe started as either flat or very close to flat. The question then is why among the infinite number of possibilities for the Universe's geometry it has this very special and unique value?

To remind you this value is unique because if the spatial curvature is positive, the it will quickly dominate over the matter, and the expansion will stop and turn around, and the universe will collapse. On the other hand, if the spatial curvature is negative, the universe quickly becomes empty and cold.

Problems with the Big Bang:
3- The monopole (relics) problem

According to the grand unification theories the Universe is expected togo through a number of phase transitions as the different facets of its ingredient decouple from the others (GUT to electroweak and strong forces, ...). Such phase transitions produce so called topological defects such as magnetic monopoles, cosmic strings and domain walls (corresponding to zero, one an two dimensional defects).

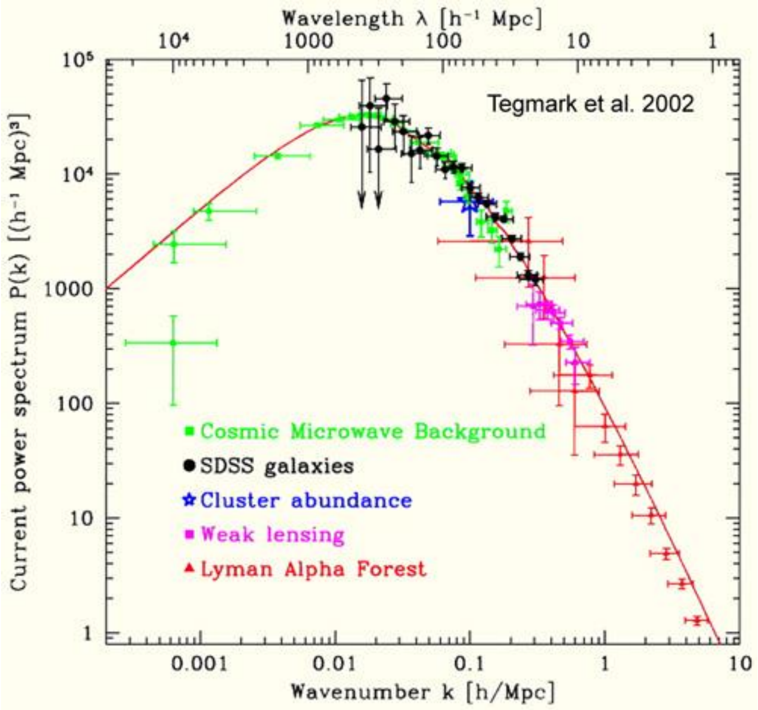


One can in principle estimate the contribution of such defects to the energy density of the Universe and it come much more than the observed value. Hence the question is where are the monopoles and the other defects that where created after the GUT era and why we don't observe their effect of the Universe or measure them in the lab?

Problems with the Big Bang:
4- Primordial fluctuation

We have so far focused on the mean properties of the Universe. However, there is another very big puzzle that has to do with the creation of the primordial density field. This puzzle has a number of elements, the first is what creates such fluctuations?

The second aspect of this puzzle is why the properties of such fluctuations are as we observe them? These fluctuations appear to be drawn from a multi-variate Gaussian field (or very close to it). Gaussian fields are characterized by their power spectrum. The second puzzle in this regards is why the primordial power spectrum seems to follow a scale independent power law behavior, i.e., P(k) ~ k?



Inflation

A solution to all of these problem was proposed independently by Alan Guth and Andre Linde called inflation. This solution considers a phase transition in which the Universe undergoes a period of accelerated expansion that takes a very small volume of the early Universe and blows it up so much and so quickly that any inhomogeneities or curvature in this volume are smoothed out, and the density of nonrelativistic particles is diluted. At the same time, any quantum fluctuations are blown up to macroscopic size, providing the seeds for large-scale structure.

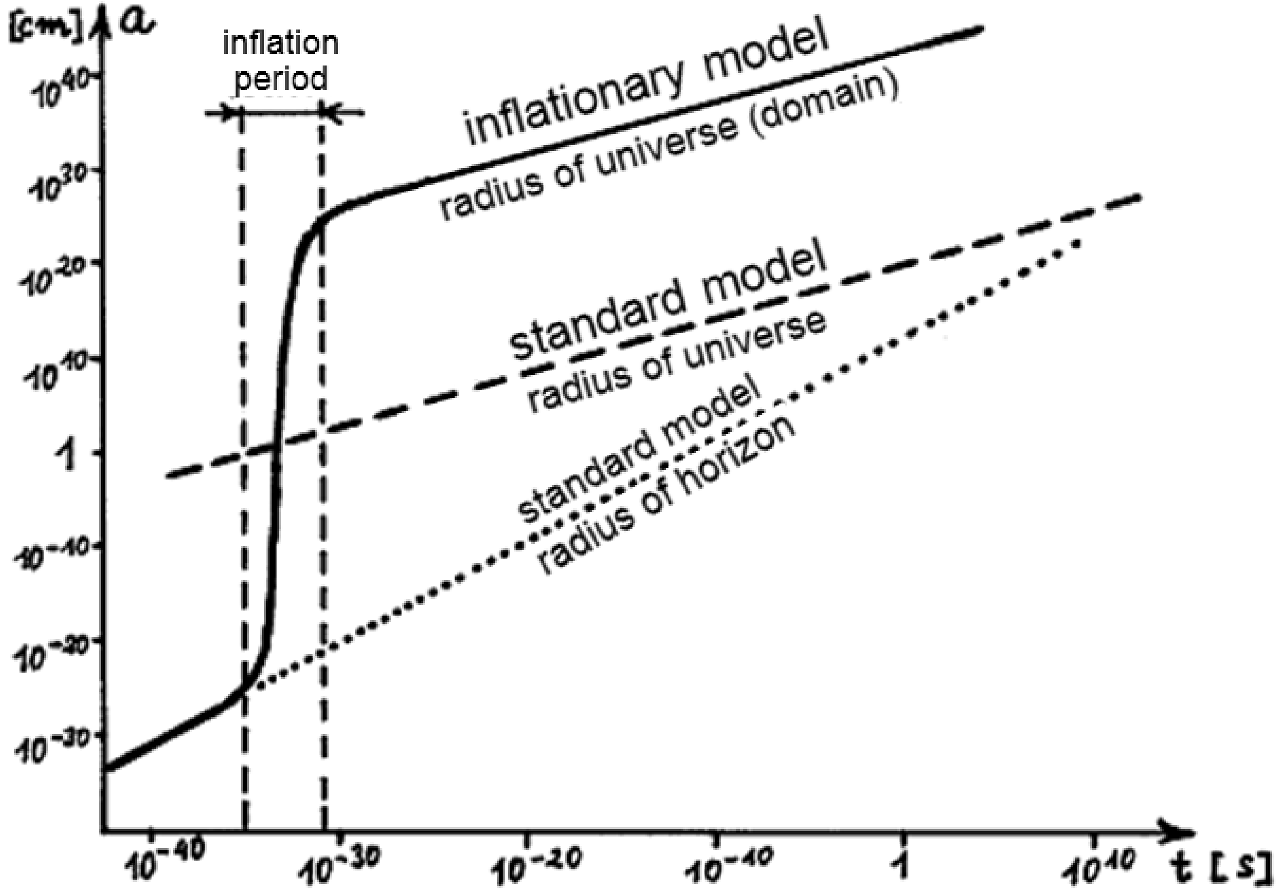
Such a solution solves all these problems in one stroke. The question hence is what can create such a behavior in the early Universe? We have already seen such a behavior when we considered a dark energy dominated Universe. In such case the expansion of the Universe is exponential and is related to the energy density if the vacuum.

Now imagine that the field that drives inflation is not the vacuum energy field but something with a much larger energy scale that occurs around (probably a little bit after) the GUT era. This would be caused by a field that describes an internal degree of freedom of the system that sets initially in a false vacuum that, as the Universe expands and cools, moves towards its real vacuum hence releasing energy that causes the Universe to expand exponentially and reheats it.

Inflation

Since the curvature density parameter scales with $(aH)^{-2}$ and the expansion of the Universe is exponential during inflation one can ask, how many e-folds inflation should give us in order to solve the flatness problem (as an example) and generically one obtains that Universe should have expanded during this phase by 60 e-folds.

This comes form the fact that the energy density around t_{Planck}: $|1-\Omega_P| \leq 2\times 10^{-62}$



Inflation

Now one can write the scale factor before, during and after inflation as: $a(t) = a_i(t/t_i)^{\frac{1}{2}}, t < t_i,$

$a_i e^{H_i(t-t_i)}, t_i < t < t_f$ and $a_i e^{H_i(t_f-t_i)}(t/t_f)^{\frac{1}{2}}, t > t_f.$

This gives the following relation between the initial and final scale factors:

$$\frac{a(t_f)}{a(t_i)} = e^{H_i(t_f-t_i)} = e^N$$

Now we recall:

$$|1 - \Omega(t)| = \frac{c^2}{R_0^2 a(t)^2 H(t)^2}$$

Now, the main idea behind inflation is to have a phase very early in the Universe (around the GUT era) in which the dominant term in the Friedmann equation behaves like the cosmological constant (Vacuum energy). This means that Hubble parameter is this phase is roughly constant and the Universe expands exponentially. Therefore,

$$|1 - \Omega(t_f)| \propto e^{-2N}$$

Starting from $|1 - \Omega(t)| \leq 0.005$ today, then using the equation on page 105, we get:

$$t_f = (N+1)t_i \sim (N+1)10^{-36}$$

This gives $a(t_f) \approx 2 \times 10^{-28}\sqrt{N+1}$. From comparing this and the $a(t_f)/a(t_i)$ one can calculate that the number of e folds from inflation is about 60.

Inflation

The Lagrangian of a scalar field is $$L_\phi = \frac{1}{2}\frac{1}{\hbar c^3}\dot{\phi}^2 - V(\phi;T)$$

Its energy and pressure are $$\varepsilon_\phi = \frac{1}{2}\frac{1}{\hbar c^3}\dot{\phi}^2 + V(\phi;T)$$ and $$p_\phi = \frac{1}{2}\frac{1}{\hbar c^3}\dot{\phi}^2 - V(\phi;T)$$

If the kinetic term is small, then $$\varepsilon_\phi \approx -p_\phi \approx V(\phi;T)$$

We now use the fluid equation to obtain $$3H\dot{\phi} = -\hbar c^3 \frac{dV}{d\phi}$$

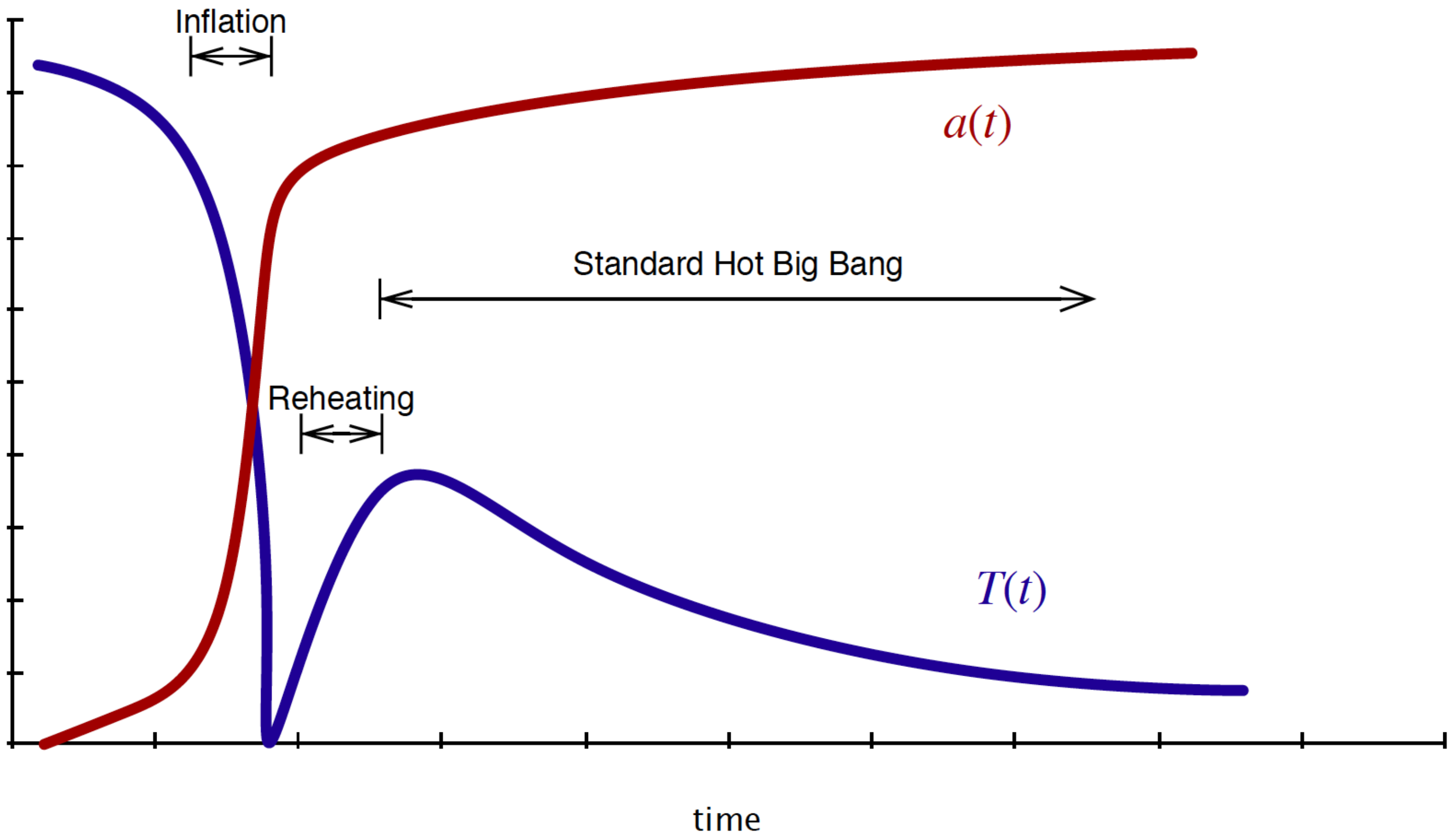
Now $$\dot{\phi}^2 \ll \hbar c^3 V$$

Therefore, $$\left(\frac{dV}{d\phi}\right)^2 \ll \frac{9H^2V}{\hbar c^3}$$

Finally, we can use Friedman equation for a flat universe to substitute for H and write:

$$\left(\frac{dV}{d\phi}\right)^2 \ll \frac{24\pi GV}{\hbar c^5}$$

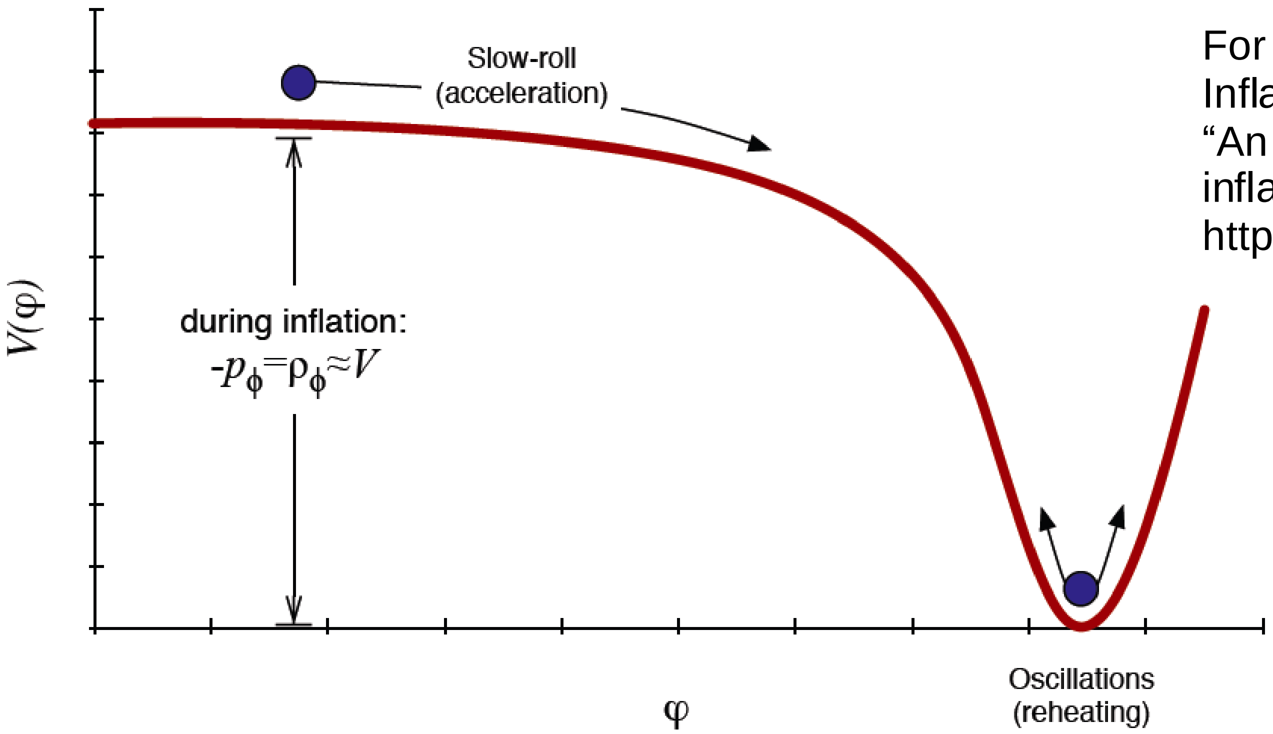
Inflation



Inflation

This is a first order phase transition such as melting of ice to liquid water which absorbs latent heat as well as changes the phase of the water from icy to liquid.
We'll not go too much in detail into the various inflationary scenario but will rather focus on one example, the so-called old-inflation scenario which is driven by a scalar field. Since we can not go into details, we'll mention that the equation that describes this scalar field is:

$$\ddot{\varphi} + 3H\dot{\varphi} + V'(\varphi) = 0$$



For a nice introduction about Inflation see the article by A. Liddle: “An introduction to cosmological inflation”. It could be obtained from: http://arxiv.org/abs/astro-ph/9901124

Inflation

The slow roll regime:
In this regime the $\ddot{\varphi}$ term is negligible, and the equation of motion is reduced to,

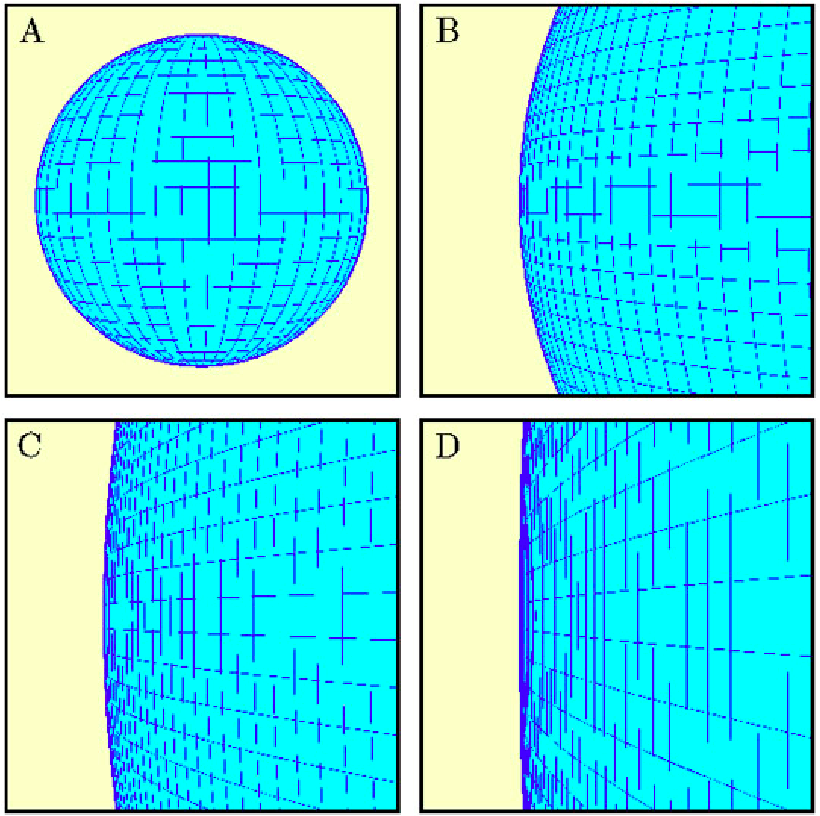
$$3H\dot{\varphi} = -V'(\varphi)$$

That is the "friction" due to the expansion is balanced by the acceleration due to the slope of the potential. Clearly, here the lower the slope is the slower the change in field is hence the more H is closer to being constant. Hence in this phase the Universe will expand exponentially. It is this phase that is responsible for the inflationary expansion of the Universe.

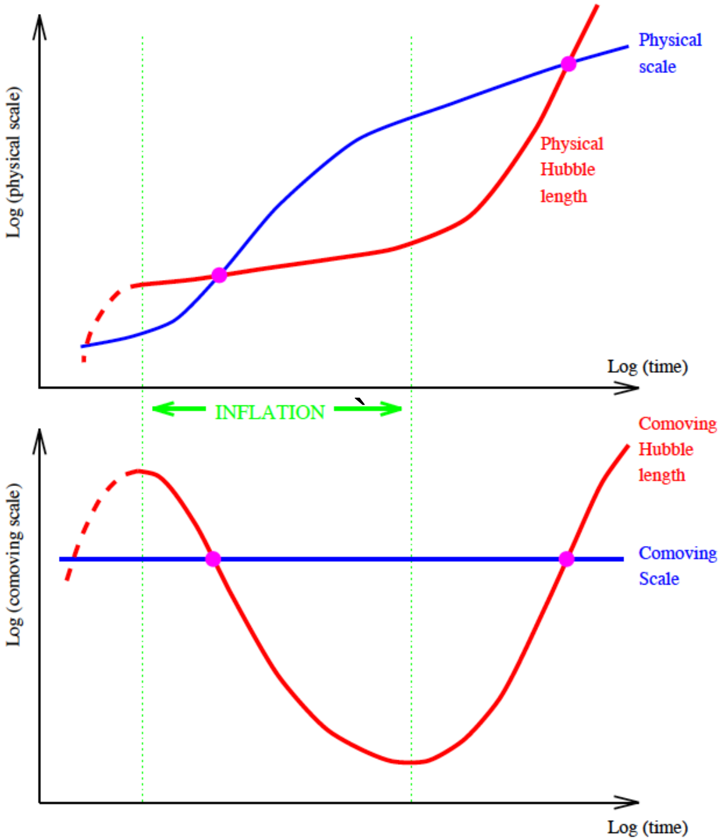
The Coherent Oscillations regime:

During this regime one can neglect the friction term (the terms involving Hubble parameter) and the field will stat oscillating around its real minimum (vacuum state). Actually, what happens here is that the field start decaying by pair-creating and hence dumping its energy in the form of heat and real particles. The oscillatory phase hence ends in what is normally called the reheating phase which marks the end of inflation.

Inflation solves the various problems!



Flatness



Horizon

The enormous expansion also solves the relics problem since their density becomes so small to render their effect virtually undetectable.

The quantum nature of the inflations field (also called inflaton) creates quantum fluctuations that explain the density fluctuations we observe. It can be shown that such fluctuations are almost scale independent, exactly like it is observed.

Baryogenesis

One can show that if there is no assymetry of baryons that we should expect,

$$\eta \equiv \frac{n_B}{n_\gamma} \simeq 10^{-20}$$

However, as we know this number is $\eta = (6.12 \pm 0.04) \times 10^{-10}$

This discrepancy forces us to consider that baryon assymetry is a fundamnetal feature of nature. Currently, there is agrrement on the nature of the process nor even when it occurs, through it is generally assumed to happen around temperature of 1-100 GeV.

During the 1960s the Andrei Sakharov articulated three conditions that need to happen for Baryogenesis to occur:

1. Nature violates Baryon number conservation
2. Baryogenesis requires C and CP violation
3. The interactions that lead to baryogenesis are out of equilibrium

Baryogenesis

The first condition is quite obvious as without it there will be no baryon antibaryon assymmetry. Remember, baryons are compose of three quarks, whereas antibaryons are made of three aniquarks. Each quark and antiquark have baryon number of 1/3 and -1/3, respectively. Therefore, baryon and antibaryons have baryon number of 1 and -1, respectively. One the other hand, Mesons, which are made of quark and antiquark, have baryon number of 0. This condition has not been observed in particle physics experiments and is not part of the standrd model of elementary particles.

The second condition needs some explanation. Consider thefollowing three operators.

The charge conjugation operator, C, operates on particles to change them to their antiparticles.
The parity reversal operator, P, reverses the sign of the coordinates.
The time reversal operator, T, reverse the direction of time.

The baryon number violation (Sakharov first condition) is not enough to guarantee baryon asymmetry, since if there is a charge conjugation symmetry, and simultaneous C and P symmetry, then these symmetries will enforce baryon antibaryon symmetry.

C and CP violations have been observed in nature back in the 1950s and 1960s and are allowed in the standard model of particle physics.

Baryogenesis

The condition number 3 regarding the out-of-equilibrium interactions is obvious considering what we mentioned regarding the expected number of baryons in the case of no baryonic asymmetry.

This condition is naturally satisfied because particle interactions freezeout when $\Gamma \lesssim H$

As I have mentioned, this issue is still open and currently there is consesus on which models will satisfy the three Sakharov conditions. Such models go beyond the standard model of elementary particles.

Example:

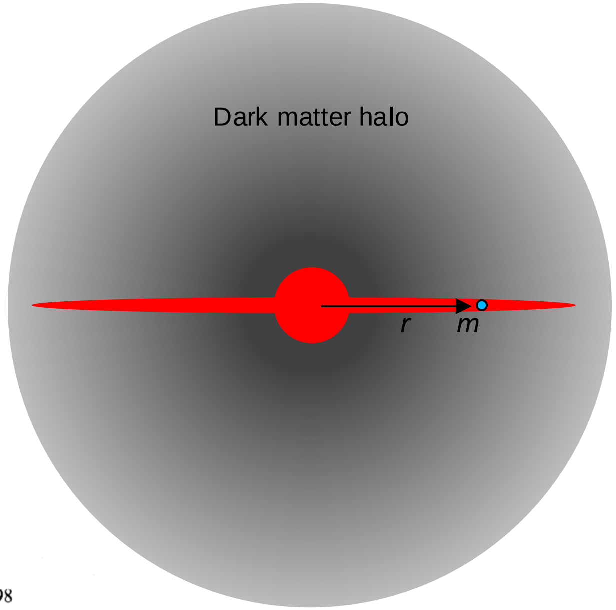
Assume a particle X decays into two particles p_1 and p_2 with a branching ratio *r*, namely a fraction *r* decays to p_1 and *(1-r)* decays to p_2. Similarly the antiparicle, $\bar{X}$, barnches into $\bar{p_1}$ and $\bar{p_2}$ with branching ratios $\bar{r}$ and $(1-\bar{r})$. Therefore, the change in the total baryonic number for this reaction is:

$$\Delta B = rB_1 + (1-r)B_2 - \bar{r}B_1 - (1-\bar{r})B_2 = (r-\bar{r})(B_1 - B_2)$$

Here the the left hand side is not zero only of bryon numer violation occure (B_1 and B_2 are different), and if C and CP symmetries are violated ($r \neq \bar{r}$).

Dark matter

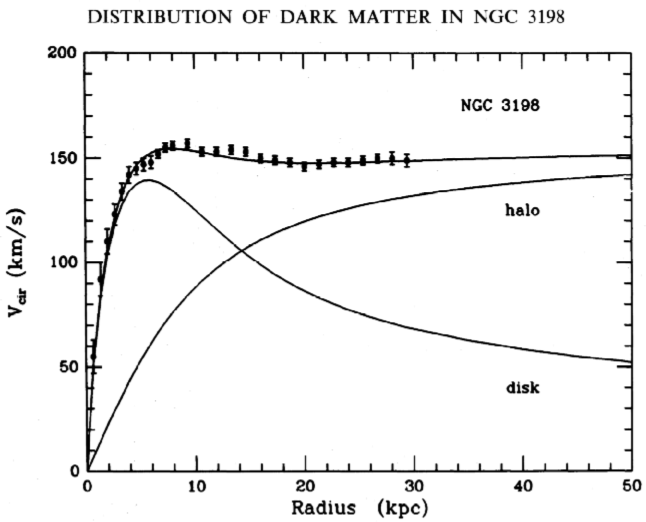
Dark matter in Galaxies: $$\frac{mv^2}{r} = \frac{GmM(<r)}{r^2}$$



$$M(<r) = 4\pi \int_0^r \rho(r) r^2 dr = \frac{v^2 r}{G}$$

$$\rho(r) = \rho_0 \left(\frac{r}{r_0}\right)^\alpha$$

$$\alpha \sim -2$$



Dark matter in clusters: x-ray images, lensing, virial theorem (motions of galaxies)

Dark matter in Clusters

The Virial theorem:

In a system with many particles the acceleration of a particle *i* is:

$$\ddot{\vec{x_i}} = G \sum_{i \neq j} m_j \frac{\vec{x_j} - \vec{x_i}}{|\vec{x_j} - \vec{x_i}|^3}$$

The potential and kinetic energy are $W = -\frac{G}{2} \sum_{i \neq j} \frac{m_i m_j}{|\vec{x_j} - \vec{x_i}|} \,\&\, K = \frac{1}{2} \sum_i m_i |\dot{\vec{x}}|^2$

It is useful to consider the moment of inertia: $I = \sum_i m_i |\vec{x}_i|^2$

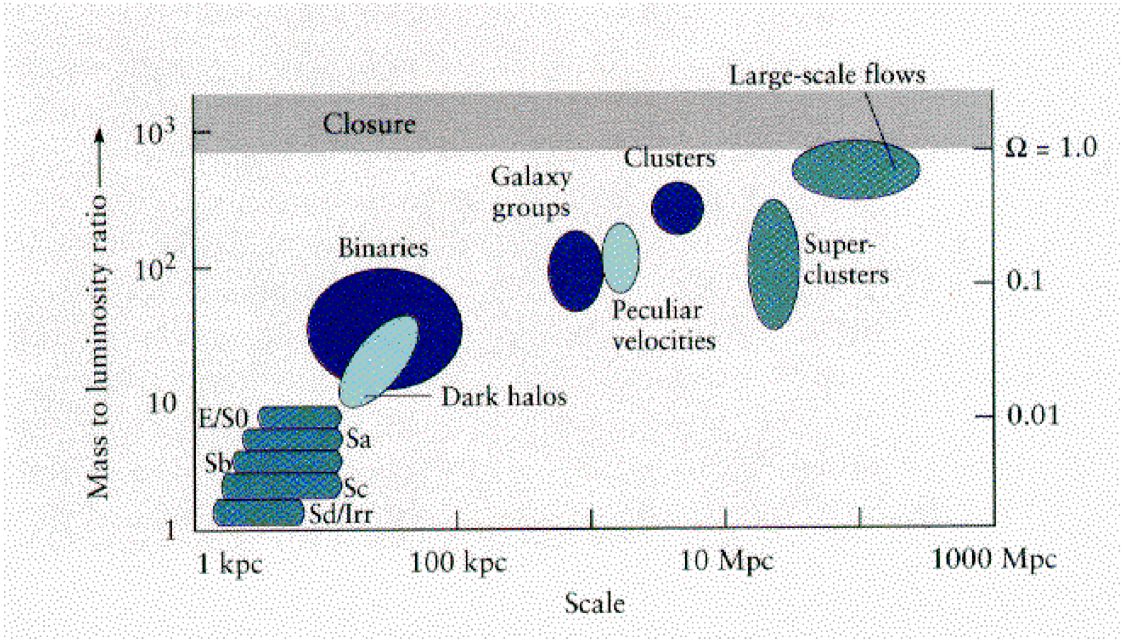
Taking the second derivative one can show that: $\ddot{I} = 2W + 4K$

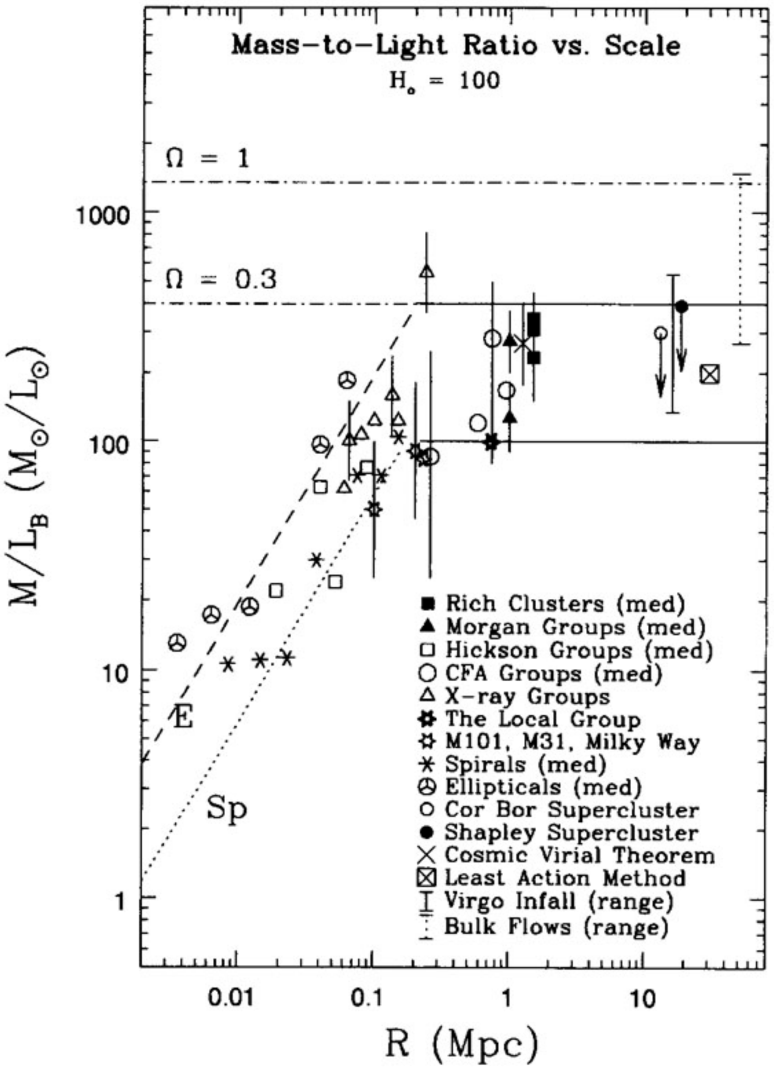
Which gives for a system with constant moment of inertia: $K = -\frac{W}{2}$

Coma cluster:

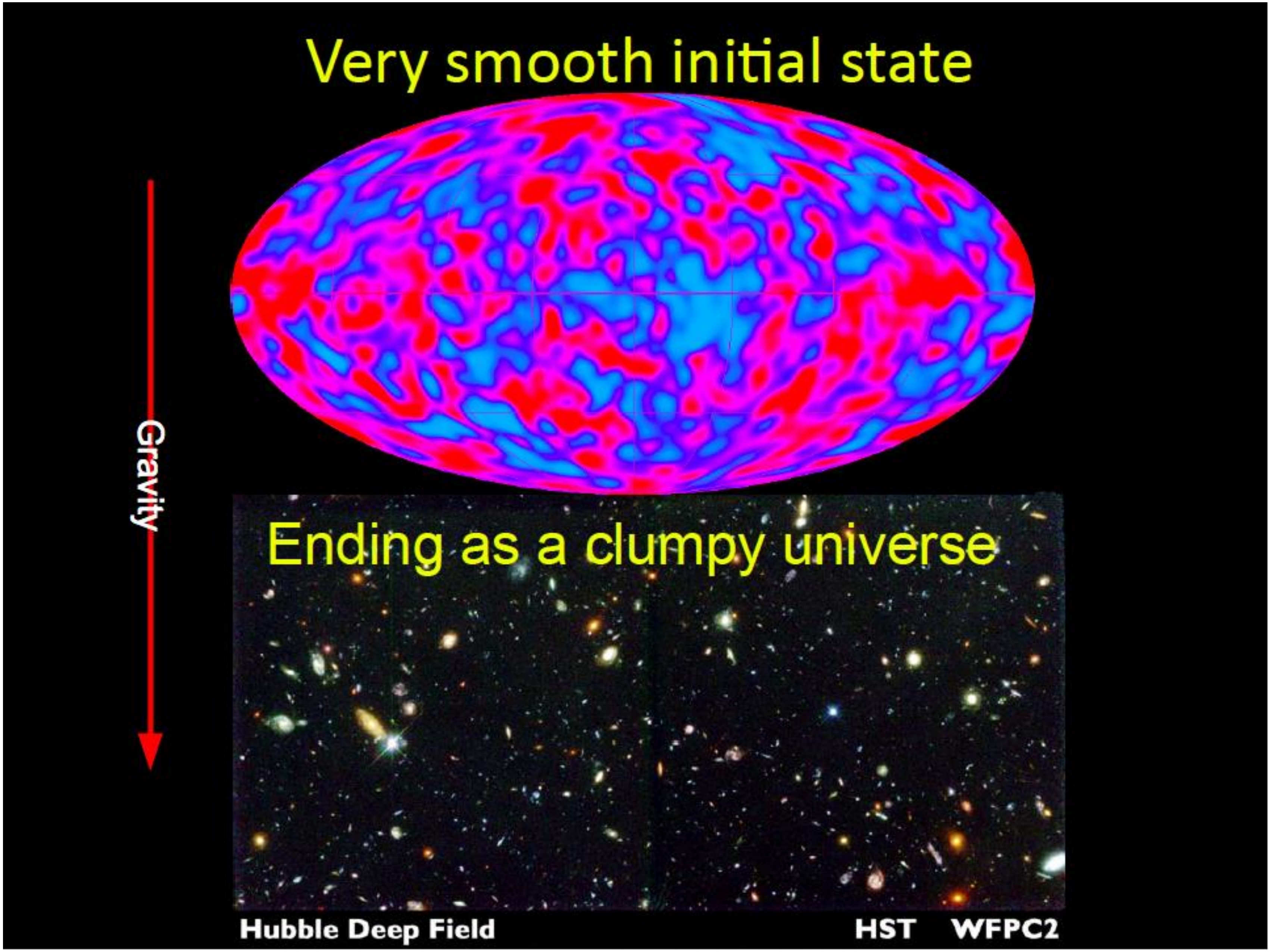
this gives a mass 400 times the mass implied by the cluster light

Dark Matter at various scales





Structure formation



Hydrodynamics and Gravitational Instability

- Basic equation in hydrodynamics: Continuity, Euler
- Jeans (Gravitational) Instability
- Fluid equations in an expanding Universe
- Linear Regime of Gravitational Instability
- Spherical Collapse model

The basic ideal fluid equations

I.1 The Fluid approximation:

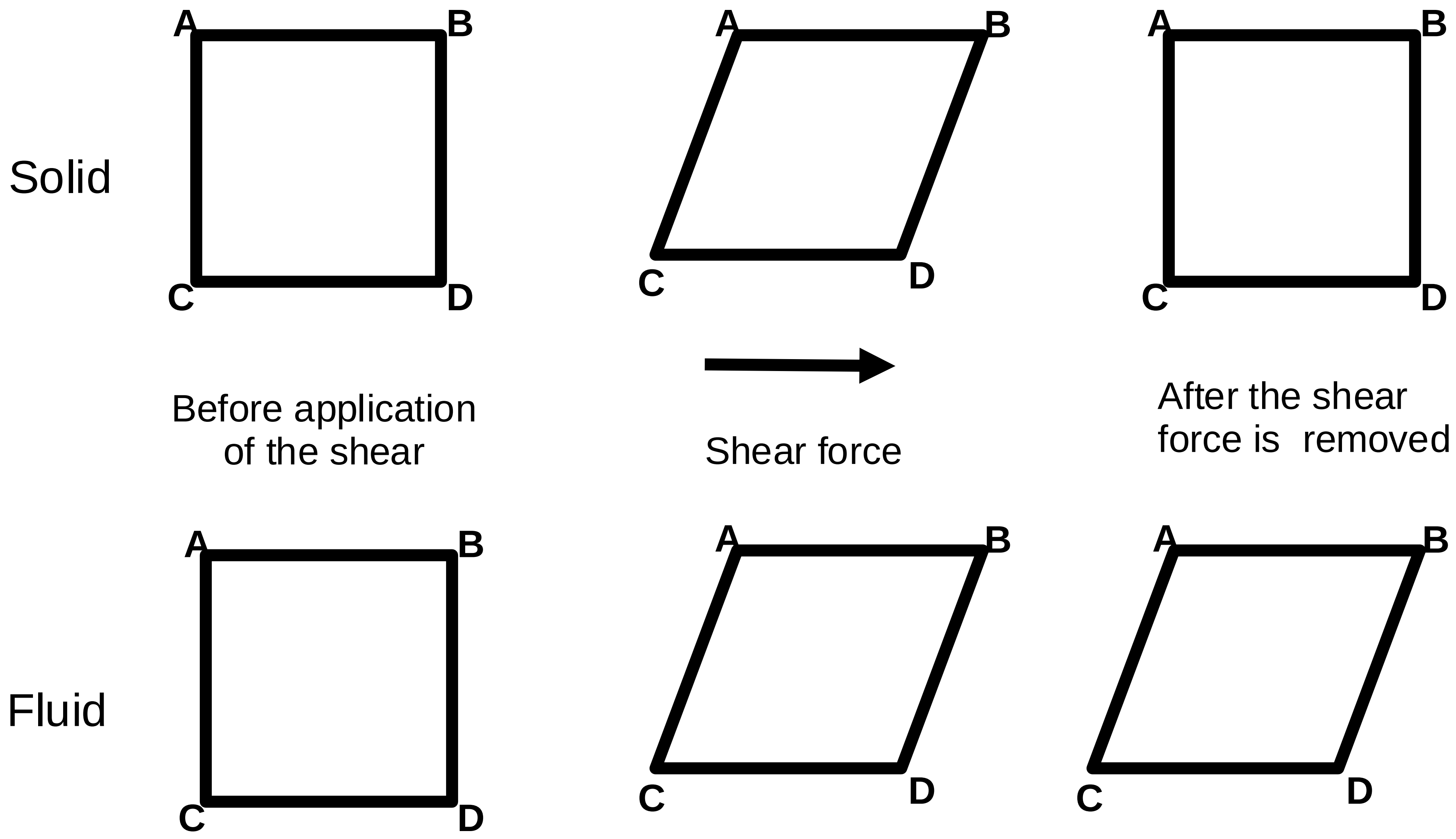
The fluid is an idealized concept in which the matter is described as a continuous medium with certain macroscopic properties that vary as **continuous** function of position (e.g., density, pressure, velocity, entropy). That is, one assumes that the scales over which these quantities are defined is much larger than the mean free path of the individual particles that constitute the fluid.

$$l_{mfp} \sim \frac{1}{s\,n}$$

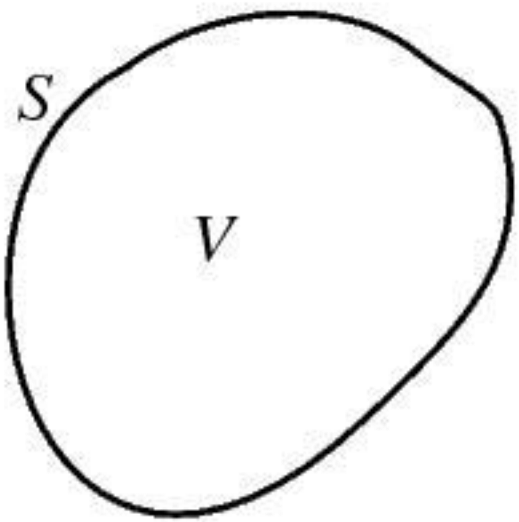
Where n is the number density of particles in the fluid and s is a typical interaction cross section.

Furthermore, for gases the kinetic energy of particles satisfies E_k>>DE, where DE is the energy required to unbind a pair of particles in the medium.

Solid vs. Fluid

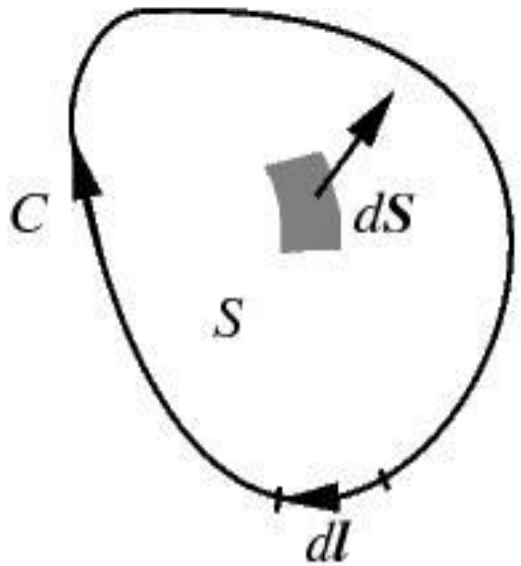


Mathematical Reminder



$$\int_S \boldsymbol{F} \cdot d\boldsymbol{S} = \int_V \nabla \cdot \boldsymbol{F} dV.$$

Gauss's Law



$$\int_C \boldsymbol{F} \cdot d\boldsymbol{l} = \int_S \nabla \times \boldsymbol{F} \cdot d\boldsymbol{S}.$$

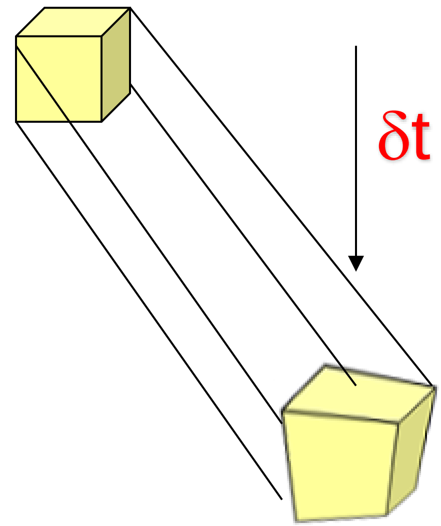
Stoke's Theorem

Convective (Material, Lagrangian) Derivative

Consider the change in a given field, say the density $\rho(\vec{r}, t)$ within a volume element moving with the fluid. After time δt the density within the volume element is $\rho(\vec{r} + \vec{v}\delta t, t + \delta t)$
Therefore, the change that the density experience is:

$$\frac{d\rho}{dt} = \frac{\rho(\vec{r} + \vec{v}\delta t; t + \delta t) - \rho(\vec{r}; t)}{\delta t}$$

$$= \frac{\partial \rho}{\partial t} + \vec{v} \cdot \nabla \rho$$



This derivative is normally called the convictive (Material or Lagrangian) derivative.
Notice that if you fix the volume element in space then the equation becomes the normal partial derivative:

$$\frac{\rho(\vec{r}; t + \delta t) - \rho(\vec{r}; t)}{\delta t} = \frac{\partial \rho}{\partial t}$$

Lagrangian vs. Eulerian Description of Fluids: The first involves a coordinate system that moves with the Fluid while the latter involves a coordinate system fixed in space.

The Continuity equation (mass conservation)

Consider a volume *V* which is fixed in space and enclose by a surface $\vec{S} = S\vec{n}$ where $\vec{n}$ is the outward pointing normal vector. The total mass of the fluid in *V* is $\int_V \rho\, dV$ where ρ(***r***,t) is the density of the fluid. The rate of change in the mass within *V* is equal to the mass flux into *V* across it surface $\vec{S}$.

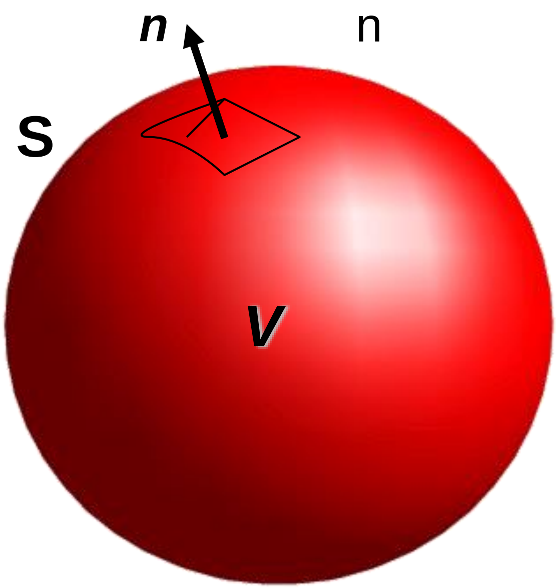
$$\frac{\partial}{\partial t}\int_V \rho \mathrm{d}V = -\oint_S (\rho\vec{v})\cdot\vec{n}\mathrm{d}S$$

Using the divergence theorem (Green's formula) one obtains

$$\frac{\partial}{\partial t}\int_V \rho dV = -\int_V \nabla\cdot(\rho\vec{v})dV$$

Since this holds for every volume, this relation is equivalent to

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0 \qquad (\mathrm{I.1})$$



One can also define the mass flux density as $\vec{j} = \rho\vec{v}$ which shows that the last equation is actually a continuity equation:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0 \qquad (\mathrm{I.2})$$

Euler's equation (momentum conservation):

The Euler (momentum) equation is obtained exactly in the same way the continuity equation is obtained with the following exceptions:

1- The volume we consider is moving with the fluid, i.e., the rate of change is determined by the convictive derivative.

2- The total change in the momentum of volume *V* is given by the total force working on the particles. This force has many component. The first is the integral of the pressure (force per unit area) over the surface ***S*** (at this stage we'll ignore other stress tensor terms that can either be caused by viscosity, electromagnetic stress tensor, etc.): $-\oint_S p\vec{n}\mathrm{d}S$

Furthermore, an external force will have to be added as $\int_V \rho\vec{f}\mathrm{d}V$. Where $\vec{f}$ is the force per unit mass, also know as ***body force***.

Therefore, the momentum change rate within a volume *V* satisfies the following integral equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\vec{v}dV = -\oint_S p\vec{n}\mathrm{d}S + \int_V \rho\vec{f}\mathrm{d}V \qquad \text{(I.3)}$$

The left-hand term of equation (I.3) is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\vec{v}dV = \int_V \rho\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}\mathrm{d}V = \int_V \rho\left(\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}\right)\mathrm{d}V$$

Euler's equation (momentum conservation)

Applying the divergence theorem to the first right hand term of equation (I.3) yields,

$$\int_V \rho \left(\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} \right) \mathrm{d}V = - \int_V \vec{\nabla} p \mathrm{d}V + \int_V \rho \vec{f} \mathrm{d}V \qquad \text{(I.4)}$$

Since this is valid for any arbitrary volume, the following differential equation always holds for an inviscid medium.

$$\frac{\partial \vec{v}}{\partial t} + \left(\vec{v} \cdot \vec{\nabla} \right) \vec{v} = - \frac{\vec{\nabla} p}{\rho} + \vec{f} \qquad \text{(I.5)}$$

In this discussion we ignored energy dissipation processes which may occur as a result of internal friction within the medium and heat exchange between its parts (conduction). This type of fluids are called **ideal** fluids.

Gravity:

For gravity, the force per unit mass is given by $-\nabla\phi$, where $\nabla^2 \phi = 4\pi G \rho$

Hydrodynamical Instabilities

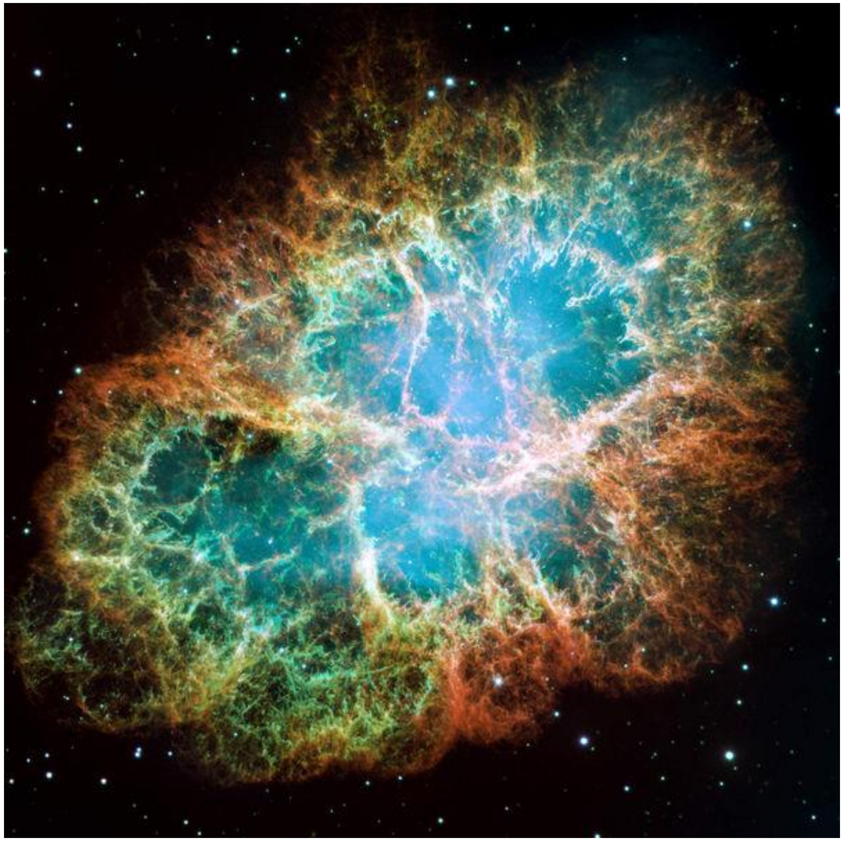
A useful way to view the reaction of a system to perturbations is to write the perturbation fields (normally possible to do) in the following form:

$$\vec{v}_1 \propto e^{\gamma t - i\omega t} f(x,y,z)$$

Obviously, the type of reaction the system has to these perturbations. i.e., stable, oscillating or unstable, depends on whether g is negative, zero or positive. respectively.

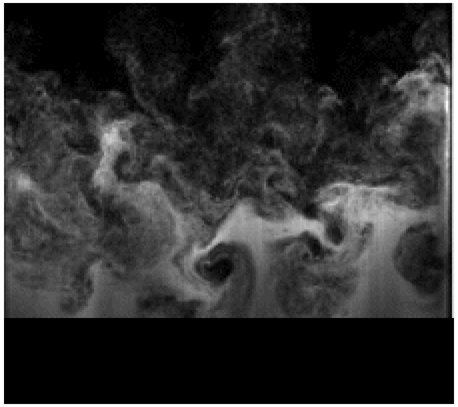
Here we'll deal with several instabilities that are common in astrophysical systems.

Rayleigh-Taylor and Kelvin-Helmholtz Instabilities



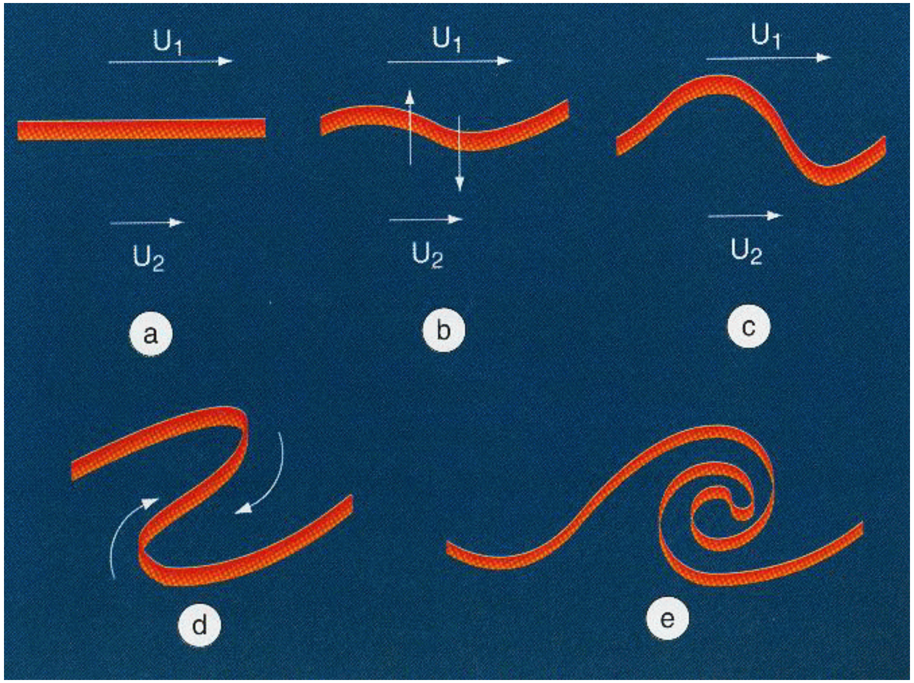
The Crab nebula

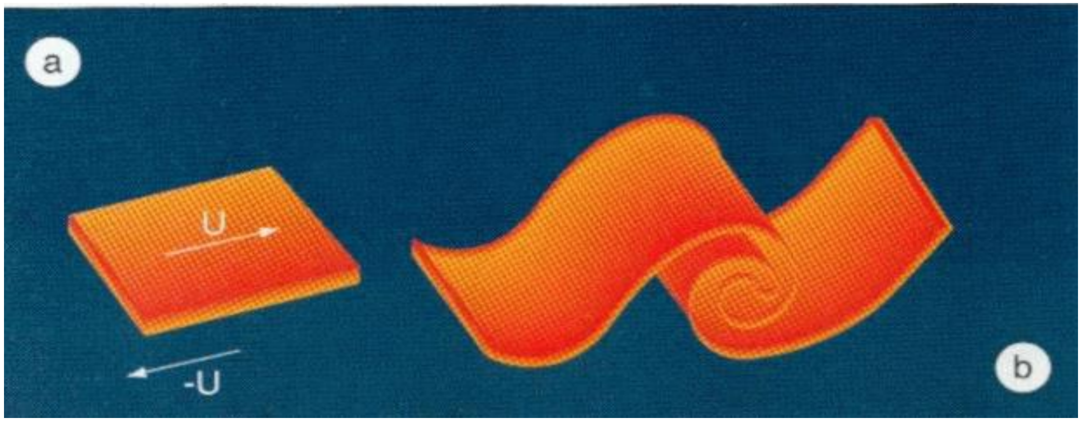
Rayleigh Taylor Instability



Rayleigh-Taylor and Kelvin-Helmholtz Instabilities (cont.)

Kelvin Helmholtz Instability





Jeans (Gravitational) Instability

In hydrodynamics, small perturbations that grow exponentially rendering the system unstable are called hydrodynamical instabilities. Here we'll examine the so called Jeans instability, which is the main "force" behind structure formation in th Universe. A useful way to view the reaction of a system to perturbations is to write the perturbation fields (normally possible to do) in the following form:

$$\vec{v}_1 \propto e^{\gamma t - i\omega t} f(x, y, z)$$

Obviously, the type of reaction the system has to these perturbations. i.e., stable, oscillating or unstable, depends on whether g is negative, zero or positive. Respectively.

We'll first show this instability in a normal system and then derive it and discuss its behavior in an expanding Universe

Jeans Instability

Assume an infinite homogeneous and self gravitating gas cloud with unperturbed $\rho_0,\ p_0,\ \mathrm{and}\ \phi_0$ which are position independent. *A side comment, such a setup is unphysical in Newtonian mechanics, still, Jeans ignored this and went ahead with his perturbative approach, this is known as the Jeans swindle.*

The first order Euler equation gives:

$$\rho_0 \frac{\partial \vec{v}'}{\partial t} = -\vec{\nabla} p' - \rho_0 \vec{\nabla} \phi'$$

Taking the divergence of both sides yields:

$$\rho_0 \frac{\partial \left(\vec{\nabla} \cdot \vec{v}' \right)}{\partial t} = -\nabla^2 p' - 4\pi G \rho_0 \rho'$$

Suppose now that the gas is ideal and isothermal then the last equation could be written as:

$$\frac{\partial^2 \rho'}{\partial t^2} = c_T^2 \nabla^2 \rho' + 4\pi G \rho_0 \rho'$$

where we used the continuity equation as well. c_T is the isothermal sound speed.

Jeans Instability

Trying a solution of the form to obtain

$$\omega^2 = c_T^2 k^2 - 4\pi G \rho_0$$

The system is clearly unstable if the dynamical time scale is smaller than the hydrodynamical time scale. Put differently, the instability criterion is:

$$k^{-1} > \sqrt{\frac{c_T^2}{4\pi G \rho_0}} \equiv \frac{\lambda_J}{2\pi}$$

where l_J is called the Jeans wavelength.
Now if the cloud is roughly spherical one can define Jeans radius (R_J= l_J/2). From this one can define a Jean mass $M_J = \frac{4}{3}\pi R_J^3 \rho_0 = \frac{\pi}{6}\rho_0 \lambda_J^3$, which gives the problem a simple interpretation. If the mass associated with the perturbation exceeds Jeans mass then the system can't react to it in time and the it becomes unstable.

Structure Formation: Linear regime

Consider the standard Newtonian equations for the evolution of the density, *ρ*, and
velocity, **u**, of a fluid under the influence of gravitational field with potential F:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0$$

$$\rho\frac{d\vec{u}}{dt} = -\vec{\nabla}p - \rho\vec{\nabla}\Phi.$$

This equation must be supplemented by Poisson’s equation to relate the gravitational field to the density of the fluid, and by an equation of state to specify the pressure *p*.

It is easier to change the coordinates to comoving positions to take the mean behavior of the Universe out, through the transformation $\vec{r} = \vec{x}/a(t)$ and to peculiar velocity $\vec{v} = a\, d\vec{r}/dt = \vec{u} - (da/dt)\vec{r}.$

Structure Formation: Linear regime

It is also easier to express the density in terms of the dimensionless over-density.

$$\delta = \rho/\bar{\rho} - 1$$

One then obtains the following set of equations:

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\vec{\nabla}_r \cdot ((1+\delta)\,\vec{v}) = 0$$

$$\frac{\partial \vec{v}}{\partial t} + \frac{1}{a}\left(\vec{v} \cdot \vec{\nabla}_r\right)\vec{v} + \frac{\dot{a}}{a}\vec{v} = -\frac{1}{a}\vec{\nabla}_r \phi$$

$$\nabla_r^2 \phi = 4\pi G a^2 \bar{\rho} \delta$$

Linearizing the fluid equation and substituting both the continuity and Poisson's equations into the divergence of Euler's equation one obtains,

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} = 4\pi G a^2 \bar{\rho} \delta$$

Structure Formation: Linear regime

The last equation could be recast in the following way:

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - \frac{3}{2}\Omega_m \left(\frac{\dot{a}}{a}\right)^2 \delta = 0$$

Which has the general solution:

$$\delta(\vec{r},\tau) = \delta_+(\vec{r})D_+(\tau) + \delta_-(\vec{r})D_-(\tau)$$

This generic solution has two modes, a growing mode and a decaying mode. The decaying mode gets suppressed very quickly and, except in certain cases, could be ignored. The growing on the other hand is the one that has the seeds of the nonlinear structures that inevitably evolve to form galaxies and galaxy clusters. The main feature that one should notice is the self similar manner in which the linear growing mode evolves with time.

Growning and decaying components

The last equation can not be solved without specifying values of the matter density and other cosmological paramters.

Radiation dominated Universe:
For example in case of highly radiation case (e.g., during the early Universe), one can ignore the Ω_m contribution to obtain the equation:

$$\ddot{\delta} + 2H\dot{\delta} = \ddot{\delta} + \frac{\dot{\delta}}{t} = 0$$

This equation can be easily solved

$$\delta(t) = B_1 + B_2 \ln t$$

Lambda dominateds:
in the case of a Lambda dominated universe the equation becomes:

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} = \ddot{\delta} + 2H_\Lambda\dot{\delta} \approx 0$$

With the solution $\delta(t) = C_1 + C_2 e^{-2H_\Lambda t}$

In a flat and matter dominated case, the solution is:

$$\delta(t) = D_1 t^{\frac{2}{3}} + D_2 t^{-1}$$

The Power spectrum

Given the statistical nature of the fluctuation is makes sense to analyze them in term of Fourier transforms. Since the evolution of the over-density is linear,

$$\ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}\Omega_m H^2 \delta = 0$$

then the Fourier transform density follows the same equation and hence has the same time dependent structure:

$$\ddot{\delta}_{\vec{k}} + 2H\dot{\delta}_{\vec{k}} - \frac{3}{2}\Omega_m H^2 \delta_{\vec{k}} = 0$$

Frourier transform of the over-density is defined as: $\delta_{\vec{k}} = \frac{1}{V}\int \delta(\vec{r}) e^{i\vec{k}\cdot\vec{r}} d^3 r$

and its inverse as: $\delta(\vec{r}) = \frac{V}{(2\pi)^3}\int \delta_{\vec{k}} e^{-i\vec{k}\cdot\vec{r}} d^3 k$

Remember, the overdensity in Fourier space is complex, $\delta_{\vec{k}} = |\delta_{\vec{k}}| e^{i\phi_{\vec{k}}}$

The power spectrum is defined as $P(k) = \langle |\delta_{\vec{k}}|^2 \rangle$

We will consider the case of a power law power spectrum: $P(k) = Ak^n$

Remember the convolution theory $\mathcal{FT}\left(f(\vec{r}) * g(\vec{r})\right) = f_{\vec{k}} \cdot g_{\vec{k}}$

The Power spectrum

We now calculate the mass at a given radius: $M(r) = \frac{4\pi}{3}\bar{\rho}(1+\delta)r^3$

where the mean mass at *r=R* is, $\langle M(r)\rangle|_{r=R} = \frac{4\pi}{3}\bar{\rho}R^3$

The variance of the mass at that radius is

$$\langle\left(\frac{M-\langle M\rangle}{\langle M\rangle}\right)^2\rangle|_{r=R} = \langle\left(\frac{\delta M}{M}\right)^2\rangle|_{r=R} = \frac{V}{2\pi^2}\int P(k)\left(\frac{3j_1(kR)}{kR}\right)^2 k^2 dk = \frac{9V}{2\pi^2R^2}\int P(k)j_1(kR)^2 dk$$

with $j_1(x) = (\sin x - x\cos x)/x^2$ is a FT of the top hat 3d funtion at radius *R*
Therefore, the mas variance for a power law power spectrum at radius R is,

$$\langle\left(\frac{M-\langle M\rangle}{\langle M\rangle}\right)^2\rangle|_{r=R} = \frac{9V}{2\pi^2}R^{-3-n}\int_0^\infty x^n j_1(x)^2 dx$$

We now wris the variance as $\langle\left(\frac{M-\langle M\rangle}{\langle M\rangle}\right)^2\rangle|_{r=R} = \left(\frac{\delta M}{M}\right)^2|_{r=R}$

Therefore, $\left(\frac{\delta M}{M}\right)|_R \propto R^{-(3+n)/2} \propto M^{-(3+n)/6}$

For n=0 the variance fits a Poisson distribution of particles (or white noise).

One can show that the peculiar potential is $\delta\Phi \sim \frac{G\delta M}{R} \propto \frac{G\delta M}{M^{1/3}} \propto M^{(1-n)/6}$

For n=1, the peculiar potential is scale invariant, which is called Harrison-Zeldovich spectrum

Non-linear Regime: Spherical Collapse

We now move to consider the simplest possible model describing the formation of an object, the so call, spherical-collapse model. Assume a spherical region with a uniform overdensity <δ> and a physical radius R in an otherwise uniform Universe. Birkhoff's Theorem (from GR) states that the contribution of the exterior material to spherically symmetric solution must be given by the Schwarzschild metric, in other words within the sphere the only thing that matters is the material inside the sphere. Hence one can write

$$\frac{d^2R}{dt^2} = -\frac{GM}{R^2} = -\frac{4\pi G}{3}\bar{\rho}(1+\bar{\delta})R$$

The first integral of this equation gives $\frac{1}{2}\left(\frac{dR}{dt}\right)^2 - \frac{GM}{R} = E$

Which can be solved: $R/R_m = \frac{1}{2}(1-\cos\eta); \;\; t/t_m = (\eta - \sin\eta)/\pi$

Non-linear regime: Spherical collapse

The collapse of the sphere to R = 0 occurs at $t = 2t_m$, and at this time the extrapolated linear overdensity is

$$\delta_{collapse} = \bar{\delta}(2t_m) = \frac{3}{20}(12\pi)^{2/3} = 1.686$$