2. Fourier Transform

The Fourier transform is a generalization of the complex Fourier series in the limit as $L \to \infty$.  

$$f(x) = \sum_{n=-\infty}^{\infty} A_n e^{i(2\pi nx/L)} \quad (1)$$

$$A_n = \frac{1}{L} \int_{-L/2}^{L/2} f(x) e^{-i(2\pi nx/L)} \, dx. \quad (2)$$

Replace the discrete $A_n$ with the continuous $F(k) \, dk$ while letting $n/L \to k$. Then change the sum to an integral, and the equations become

$$f(x) = \int_{-\infty}^{\infty} F(k) e^{2\pi ikx} \, dk \quad (3)$$

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi ikx} \, dx. \quad (4)$$

Here,

$$F(k) = \mathcal{F}[f(x)] = \int_{-\infty}^{\infty} f(x) e^{-2\pi ikx} \, dx \quad (5)$$

is called the forward ($-i$) Fourier transform, and

$$f(x) = \mathcal{F}^{-1}[F(k)] = \int_{-\infty}^{\infty} F(k) e^{2\pi ikx} \, dk \quad (6)$$

is called the inverse ($+i$) Fourier transform.

In physics and engineering the transform is often written in terms of angular frequency $\omega \equiv 2\pi \nu$ instead of the oscillation frequency $\nu$. This destroys the symmetry, resulting in the transform pair

$$H(\omega) = \mathcal{F}[h(t)] = \int_{-\infty}^{\infty} h(t) e^{-i\omega t} \, dt \quad (7)$$

$$h(t) = \mathcal{F}^{-1}[H(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{i\omega t} \, d\omega. \quad (8)$$

To restore the symmetry of the transforms, the convention
\[ g(y) = \mathcal{F}[f(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-i\gamma t} \, dt \quad (9) \]

\[ f(t) = \mathcal{F}^{-1}[g(y)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y)e^{i\gamma y} \, dy \quad (10) \]

is sometimes used.

Below we will use the definition of Eqs. 5 and 6. All equations have their equivalent in the time domain; this follows simply by replacing \( x \) by \( t \) and \( k \) by \( f \) (frequency in Hz).

A function \( f(x) \) has a forward and inverse Fourier transform such that

\[
\begin{align*}
  f(x) &= \left\{ \begin{array}{ll}
    \int_{-\infty}^{\infty} e^{2\pi ikx} \left[ f(x)e^{-2\pi ikx} \, dx \right] \, dk & \text{for } f(x) \text{ continuous at } x \\
    \frac{1}{2}[f(x_+) + f(x_-)] & \text{for } f(x) \text{ discontinuous at } x,
  \end{array} \right. \\
\end{align*}
\]

provided that

1. \( \int_{-\infty}^{\infty} |f(x)| \, dx \) exists.
2. There are a finite number of discontinuities.
3. The function has bounded variation.

The Fourier transform is linear, since if \( f(x) \) and \( g(x) \) have Fourier transforms \( F(k) \) and \( G(k) \), then

\[
\begin{align*}
  \int [af(x) + bg(x)]e^{-2\pi ikx} \, dx &= a\int_{-\infty}^{\infty} f(x)e^{-2\pi ikx} \, dx + b\int_{-\infty}^{\infty} g(x)e^{-2\pi ikx} \, dx \\
  &= aF(k) + bG(k). \\
\end{align*}
\]

Therefore,

\[ \mathcal{F}[af(x) + bg(x)] = a\mathcal{F}[f(x)] + b\mathcal{F}[g(x)] = aF(k) + bG(k). \quad (13) \]

The Fourier transform is also symmetric since \( F(k) = \mathcal{F}[f(x)] \) implies \( F(-k) = \mathcal{F}[f(-x)] \).

The Fourier transform of a derivative \( f'(x) \) of a function \( f(x) \) is simply related to the transform of the function \( f(x) \) itself. Consider

\[ \mathcal{F}[f'(x)] = \int_{-\infty}^{\infty} f'(x)e^{-2\pi ikx} \, dx. \quad (14) \]

Now use integration by parts
\[ \int v \, du = [uv] - \int u \, dv \quad (15) \]

with

\[ du = f'(x) \, dx \quad v = e^{-2\pi ikx} \quad (16) \]
\[ u = f(x) \quad dv = -2\pi ik e^{-2\pi ikx} \, dx, \quad (17) \]

then

\[ \mathcal{F}[f'(x)] = [f(x)e^{-2\pi ikx}]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)(-2\pi ik e^{-2\pi ikx}) \, dx. \quad (18) \]

The first term consists of an oscillating function times \( f(x) \). But if the function is bounded so that

\[ \lim_{x \to \pm \infty} f(x) = 0 \quad (19) \]

(as any physically significant signal must be), then the term vanishes, leaving

\[ \mathcal{F}[f'(x)] = 2\pi ik \int_{-\infty}^{\infty} f(x)e^{-2\pi ikx} \, dx = 2\pi ik \mathcal{F}[f(x)]. \quad (20) \]

This process can be iterated for the \( n \)th derivative to yield

\[ \mathcal{F}[f^{(n)}(x)] = (2\pi ik)^n \mathcal{F}[f(x)]. \quad (21) \]

If \( f(x) \) has the Fourier transform \( F(k) \), then the Fourier transform has the shift property

\[ \int_{-\infty}^{\infty} f(x-x_0)e^{-2\pi ikx} \, dx = \int_{-\infty}^{\infty} f(x-x_0)e^{-2\pi i(x-x_0)k} e^{-2\pi i(kx_0)} \, d(x-x_0) \]
\[ = e^{-2\pi ikx_0} F(k), \quad (22) \]

so \( f(x-x_0) \) has the Fourier transform

\[ \mathcal{F}[f(x-x_0)] = e^{-2\pi ikx_0} F(k). \quad (23) \]

If \( f(x) \) has a Fourier transform \( F(k) \), then the Fourier transform obeys a similarity theorem.

\[ \int_{-\infty}^{\infty} f(ax) e^{-2\pi ikx} \, dx = \frac{1}{|a|} \int_{-\infty}^{\infty} f(ax) e^{-2\pi i(ax)(k/a)} \, d(ax) = \frac{1}{|a|} F \left( \frac{k}{a} \right), \quad (24) \]
so \( f(ax) \) has the Fourier transform \( |a|^{-1} \mathcal{F}\left( \frac{k}{a} \right) \).

The following table summarizes some common Fourier transform pairs.

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>( F(k) = \mathcal{F}[f] )</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>( \delta(k) )</td>
</tr>
<tr>
<td>( \cos(2\pi k_0 x) )</td>
<td>( \frac{1}{2} [\delta(k - k_0) + \delta(k + k_0)] )</td>
</tr>
<tr>
<td>( \delta(x - x_0) )</td>
<td>( e^{-2\pi ikx_0} )</td>
</tr>
<tr>
<td>( e^{-2\pi k_0</td>
<td>x</td>
</tr>
<tr>
<td>( H(x) )</td>
<td>( \frac{1}{2} \left[ \delta(k) - \frac{i}{\pi k} \right] )</td>
</tr>
<tr>
<td>( \sin(2\pi k_0 x) )</td>
<td>( \frac{1}{2} i [\delta(k + k_0) - \delta(k - k_0)] )</td>
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</tbody>
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