Time evolution of gaps in tidal streams

An analytic approach

Figure 1: A stellar stream orbiting NGC5907, a spiral galaxy at $\sim 16$ Mpc. The image shows a nice example of a stellar stream that seems to be orthogonal to the disk of the galaxy. Image by Martínez-Delgado et al. (2008).

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Abstract

We model the time evolution of gaps in tidal streams caused by collisions with dark matter subhalos, whose existence is predicted by the cold dark matter cosmological model. Detecting and analyzing these subhalos is therefore key to verifying this model. We develop a prescription to describe the evolution of a gap as the divergence of two nearby orbits in action-angle variables: one orbit on each side of the gap. We test this model in a spherical and in an axisymmetric potential, finding very good matches to the N-body experiments. Overall, we find that gaps grow linear in time $t$. We conclude that the size of the gap not only depends on the geometry of the impact, and on the size of the subhalo, but also on the orbital phase of the collision. Encounters with different subhalo parameters will produce the same gap size at different times.

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1 Introduction

One of today’s challenges in galactic astronomy and cosmology is to find dark matter (DM) and determine its properties. Dark matter is the most popular answer to why gravity does not always seem to behave according to expectations. Many galaxies act like they possess more mass than is visible from the stars and the gas that they have. For example in spiral galaxies, the circular velocity curve behaves different from what is expected from the stellar density and gas (Rubin et al., 1980). The rotation curve can only be explained by adding extra (invisible) mass, thus the name: dark matter. It is believed that dark matter does not interact electromagnetically, which would explain why it is not visible. The only force that interacts with dark matter, as far as we know, is gravity. In the current model of the origin of our universe, the ΛCDM paradigm\(^3\) (Cold Dark Matter), dark matter plays a large role. The CDM gravitationally collapses and forms the backbone of the largest structures in the universe, also known as the cosmic web. Currently, we think the universe contains about 22.6% of dark matter, 4.6% is baryonic matter, and the remaining 72.8% is dark energy (Λ) (Komatsu et al., 2011). One of the key predictions of the ΛCDM model, directly related to the clustering nature of CDM, are myriads of satellites of \(10^7-10^8 M_\odot\) orbiting Milky Way like galaxies (Klypin et al., 1999; Moore et al., 1999). Fig.2 shows a simulation of a CDM simulation of a Milky Way sized halo, with many satellites orbiting the centre. Observationally, we only see a few dozen of these satellites. This mismatch is known as the missing satellite problem (Moore et al., 1999). A solution to this problem is that the satellites are purely made by dark matter, making it hard to directly detect them. Dark satellites, or dark matter subhalos, could be only detectable through their gravitational interaction with other objects. Thus to probe the dark universe we need to look for effects caused by gravity, and although the ΛCDM model is widely accepted, it is still important to test some of its predictions. This is especially important on small scales, where for example the missing satellite problem is apparent.

Gravitational lensing has proven to be a good option to probe large systems with dark matter. As an effect of general relativity, the otherwise straight path of a ray of light can be bend by massive objects. We see this effect directly on the sky, the apparent projection of such a galaxy on the sky is deformed. Sometimes the galaxy is projected multiple times. By detecting a gravitationally lensed image, and by reconstructing this image with a model, one can map the mass distribution of the lens. Unfortunately, dark satellites are not that heavy and thus lensing effects by these kind of objects are weaker, and are more difficult to detect than strongly lensed images. Apart from having to be lucky to find a lensed galaxy, it is the question whether one can break degeneracies that arise from reconstructing the image.

As another option, it has been speculated that tidal streams of stars could be used as probes for dark matter subhalos (Ibata et al., 2002; Johnston et al., 2002). More extensively explained below, stellar streams originate from tidally disrupted clusters or galaxies. While orbiting a much heavier galaxy, these systems are stretched and bend by the gravity of the host. The exact way this happens depends on the mass distribution and potential of the host. When a dark matter subhalo crosses/or interacts with such a stream it can locally create distortions and gaps in the otherwise smooth, elongated distribution of stars. Until now, there exists no simple model that explains the growth of gaps in tidal streams as a function of time. If we want to probe the population of dark matter subhalos in the proximity of the Milky Way by analysing gaps in streams, we need to understand how gaps behave in general. That is, we need to understand how these are formed exactly, and once they are formed, how they evolve.

In my thesis I focus on the following questions:

1. How can we model the evolution of gaps in tidal streams, using action-angle variables?
2. How do gaps in streams behave in a spherical potential, according to this model?
3. How do gaps in streams behave in an axisymmetric potential, according to this model?

\(^3\)Λ here comes from the expansion of the universe.
4. What are the dependencies of the gap size on the properties of the encounter (subhalo mass, size, relative velocity, and impact angle)?

This thesis is organized as follows: first the relevant theory is explained in §2, starting from some basic dynamics, followed by dynamics of streams and dynamics of subhalos. Next the model is introduced in §3, after which the N-body methods are described in §4, the results are shown in §5, and the discussion and conclusions are presented in §6.

2 Theory

In this section we summarize the necessary theory. The contents should be sufficient to understand the dynamics of streams and gaps, i.e. the topics treated in this thesis. For the interested reader we would recommend Binney & Tremaine (2008) (BT08) and Goldstein, Safko, & Poole (2014) (G14) as reading material. At the end of this thesis, we included an appendix on Hamiltonian dynamics, mostly based on G14. Fig. 1 shows an example of a stellar stream orbiting around a nearby galaxy by Martínez-Delgado et al. (2008).

2.1 Calculating an orbit

To understand the dynamics behind streams we first need some understanding of orbits of stellar objects in a galactic potential. There are multiple ways to compute an orbit of an object in a potential, an often used numerical method is the leapfrog integrator (§3.4 BT08). In this integration algorithm the change in position $x$ and velocity $v$ are evaluated at different times. First half of the drift in position is calculated, then halfway through the kick in velocity is calculated, which then is used to calculate the second part of drift in position:

$$
x_i = x_{i-1} + \Delta t v_{i-\frac{1}{2}}
$$

$$
v_{i+\frac{1}{2}} = v_{i-\frac{1}{2}} + \Delta t a_i
$$

To calculate the acceleration $a_i$ that the particle feels at time $t_i$, we need to specify the host potential, see §2.3. In the leapfrog algorithm the orbit is calculated step by step, the timestep $\Delta t$ determines the resolution of the calculated orbit. An orbit is defined by how the coordinates $x(t)$, $y(t)$, $z(t)$ depend on time. In most cases it is not possible, or very difficult, to find analytic expressions for the coordinates, hence the need for an integrator such as the leapfrog.

2.2 Actions & Angles

Action-angle variables are coordinates that have proven to be very useful in describing systems of a periodic nature, such as orbits in galactic potentials. A detailed description of action-angle variables is beyond the scope of this thesis, they are extensively discussed in G14 and §3.5 BT08. What follows is a brief summary, based on the G14 Ch.10 (see also appendix A).

As stated in the previous section, not all coordinates evolve in simple ways with time. We would like to transform a set of canonical coordinates $(q, p)$ to a set of coordinates which have this property, such a set are: action-angle variables. These variables have the property that the first set, the actions $J$, are time independent, i.e. they are constants of motion. The conjugate coordinates, the angles $\theta$, evolve linearly in time. The actions describe the evolution of particles in configuration space, whereas the angles track the position of the particles along these actions. The actions, for a set of coordinates $(q, p)$, are defined as

$$
J_i = \frac{1}{2\pi} \oint p_i \, dq_i,
$$

and the angles are defined as

$$
\theta_i = \frac{\partial W}{\partial J_i},
$$

where $W$ is Hamilton’s characteristic function and provides the transformation between the original coordinate system $(q, p)$ and the action-angle variables $(J, \theta)$ since it is a function of both $W(q, J)$. As said, the evolution of the angles is linear in time:

$$
\theta_i(t) = \theta_i(t_0) + \Omega_i(J)t,
$$

where $\Omega_i(J)$ are the frequencies. The difficulty with these variables is that they can only be derived if the Hamilton-Jacobi equation, and thus the characteristic function $W$ (appendix A) is separable. In the interest of this thesis: this puts a restriction on the kind of host potentials. An example of how action-angle variables are calculated is given in appendix A.
2.3 Host potentials

To calculate orbits we need to specify the orbit of the host galaxy. In this thesis, we start with a simple spherical potential, and later we go to an axisymmetric potential with two components.

2.3.1 NFW

The spherical potential that we use is an NFW potential (Navarro, Frenk, & White, 1997), which has the form

$$\Phi_{\text{NFW}}(r) = -\phi_0 \frac{\ln (1 + r/r_s)}{r/r_s},$$

where $r_s$ is the scale radius, and

$$\phi_0 = \frac{GM_{200}}{r_s (\ln (1 + c) - c/(1 + c))},$$

where $M_{200}$ is the mass within a sphere of radius $r_{200}$, which in turn is defined as the radius at which the average density inside the sphere equals 200 times the critical density of the universe, $c$ is the ‘concentration’ $c = \frac{r_{200}}{r_s}$. We can calculate the acceleration of a particle due to the central potential at distance $r'$ in the radial direction

$$a(r') = -\frac{\partial \Phi}{\partial r}.$$  

(7)

For the NFW potential this is

$$a_{\text{NFW}}(r) = -\frac{\phi_0 r_s}{r} \left( \frac{\ln (1 + r/r_s)}{r} - \frac{1}{r_s + r} \right).$$

(8)

This acceleration is in the radial direction, to compute an orbit we transform this acceleration to Cartesian coordinates. Via a Jacobian transformation, and using the fact that there in the spherical case there is only one relevant coordinate: $r$, we find the transformation

$$a(x_i) = \frac{\partial r}{\partial x_i} a(r) = a(r) \frac{x_i}{r} \text{ where } x_i = x, y, z.$$  

(9)

For an axisymmetric potential the transformation is less trivial, there are two relevant coordinates and the force has multiple components.

In our experiments we use an NFW potential with $M_{200} = 3 \cdot 10^{11} M_\odot$, and $r_s = 15.61$ kpc. The velocity curve for these specific parameters is shown in Fig.3. The Milky Way typically has a velocity of $\sim 200$ km/s, however the contribution of the halo component is in the order of $\sim 120$ km/s at the position of the Sun. The potential that we model here has the right halo contribution, but the overall circular velocity is too low in comparison to the rotation curve of the Milky Way.

![Figure 3: Velocity curves of the potentials described in this section. The curves for the Stäckel are calculated in the plane of the disk. Although selected that way, it seems that the velocity curves do not match that of the Milky Way very well. At 8 kpc, the velocity should be $\sim 220$ km/s.](image-url)
2.3.2 Stäckel

As stated above, in the case of galactic dynamics, there is a serious restriction in the kind of possible potentials which can be explored in action-angle variables. The main difficulty is that the Hamilton-Jacobi equation that is generated by the potential has to be separable. Stäckel\(^4\) showed that the only set of coordinates in which the Hamilton-Jacobi equation is separable are confocal ellipsoidal coordinates. Here we use spheroidal coordinates \((\phi, \lambda, \nu)\), following the notation from Dejonghe & de Zeeuw (1988). These coordinates are derived from cylindrical coordinates, accordingly \(\phi\) is the azimuthal angle. The other two, \(\lambda\) and \(\nu\) are the roots for \(\tau\) of

\[
\frac{x^2}{\tau + \alpha} + \frac{z^2}{\tau + \gamma} = 1,
\]

where \(x^2 = x^2 + y^2\), and \(\alpha\) and \(\gamma\) determine the exact shape of the ellipsoid. Instead of \(\alpha\) and \(\gamma\), often \(a\) and \(c\) are used: \(-a^2 = \alpha\), \(-c^2 = \gamma\). When \(a = c\), the potential is spherical, when \(a > c\) it is oblate, and when \(a < c\) it is prolate.

Further on we only use Stäckel potentials that are axisymmetric, in a triaxial case one has to separate the \(\frac{\omega^2}{\tau + \alpha}\) term to break the symmetry around the \(z\)-axis.

In the axisymmetric case, Stäckel potentials have the form

\[
\Phi(\lambda, \nu) = \frac{(\nu + \gamma)G(\nu) - (\lambda + \gamma)G(\lambda)}{\lambda - \nu},
\]

where \(G(\tau)\) is an arbitrary function that determines the shape of the potential. One simple choice for \(G(\tau)\) is the Kuzmin-Kutuzov model, where

\[
G(\tau) = \frac{GM}{c + \sqrt{\tau}},
\]

where \(M\) is the mass of the system, and \(G\) is the gravitational constant. If we substitute this in eq.(11), we find

\[
\Phi(\lambda, \nu) = -\frac{GM}{\sqrt{\lambda + \nu}}.
\]

It is possible to add more components to this potential (Batsleer & Dejonghe, 1994; Famaey & Dejonghe, 2003). For example, we can consider two components. However, these two components cannot be entirely independent, they have to be described by the same set of ellipsoidal coordinates to remain of the Stäckel form. This puts some restrictions on our choice of the two components. In general the addition of an extra component, e.g. adding a disk to a halo, is by adding a second component to the \(G(\tau)\) function

\[
G(\tau) = \frac{GM_h}{\sqrt{\tau} + c_h} + \frac{GM_d}{\sqrt{\tau - q} + c_d}.
\]

Here \(q\) is an arbitrary parameter set by the constraint, namely that to keep the sum of the two components of the Stäckel form: \(\lambda_h - \nu_h = \lambda_d - \nu_d\). This constraint can be rewritten, using the \(q\) parameter:

\[
\lambda_d = \lambda_h - q, \quad \nu_d = \nu_h - q, \quad q \geq 0.
\]

The parameters \(a\) and \(c\) are connected in a similar way

\[
a_d = \sqrt{a_h^2 - q}, \quad c_d = \sqrt{c_h^2 - q}.
\]

Let us now define the ratio \(a_h/c_h = \epsilon_h\) as a new parameter. After some simple algebra we find

\[
q = \epsilon_h^2 \frac{\epsilon_h^2 - \epsilon_d^2}{1 - \epsilon_d^2}.
\]

With all these tools we can now construct a two-component potential with a halo and a disk. The potential is by construction separable, so we can compute the action-angle variables \((\theta_\phi, \theta_\lambda, \theta_\nu)\) and \((J_\phi, J_\lambda, J_\nu)\).

\(^{4}\)BT08 p.228

4
Typical values for a two component Stäckel potential (Stäckel potential 2) that produces a velocity curve like that of the Milky Way (Famaey & Dejonghe, 2003) are

\[
\begin{align*}
M_{\text{tot}} &= 3.596 \cdot 10^{11} \, M_\odot \\
k &= 0.101 \\
a &= 2.0 \, \text{kpc} \\
e_h &= 1.02 \\
e_d &= 80.0
\end{align*}
\]

Here \( k \) is the fraction of mass in the disk \( M_d = M_{\text{tot}} \cdot k \), the mass of the halo is the remaining mass \( M_h = M_{\text{tot}} \cdot (1 - k) \).

To test the model, we first try a Stäckel potential (Stäckel potential 1) with only one component, but with a much more oblate halo. For this potential we change two parameters

\[
\begin{align*}
k &= 0.0 \\
e_h &= 1.5
\end{align*}
\]

The velocity curves for these potentials are shown in Fig.3. It seems that the scale \( a = 2.0 \, \text{kpc} \) of the potential is a bit too low, a velocity curve with \( a = 7.0 \, \text{kpc} \) looks more like that of the Milky Way. Equipotential plots of both Stäckel potentials are shown in Fig.4, and Fig.5.

Figure 4: Equipotential contours of Stäckel potential 1: a one-component (oblate) potential. The properties of the potential are: \( M = 3.596 \cdot 10^{11} \, M_\odot, r_a = 2.0 \, \text{kpc}, \epsilon_h = 1.5 \).
2.4 Physics of streams

Tidal streams are nothing more than tidally disrupted initially bound groups of stars (e.g. globular clusters, dwarf galaxies, or other objects) that are in the process of being accreted by a much heavier host. As an example, let us consider a globular cluster. Typically, globular clusters contain \(10^5 - 10^6\) stars, often within \(~\)tens pc (Sparke & Gallagher, 2000). For a satellite that enters the potential of a more massive host, we can define the Jacobi radius \(r_J\) (or Hill, or Roche radius) §8.3 BT08. If a bound particle (e.g. a star) is within a distance \(r_J\) then it will stay bound to the satellite, but when it is outside this distance it will be tidally stripped by the host galaxy. The Jacobi radius is defined as

\[
r_J = \left( \frac{m_{\text{sat}}}{3M_{\text{host}}} \right) \frac{1}{2} R_0,
\]

where \(R_0\) is the physical distance between the satellite and the host, \(m_{\text{sat}}\) is the mass of the satellite, and \(M_{\text{host}}\) is the mass of the host. With this simple argument, if we set \(m_{\text{sat}} = 10^6 M_\odot\), \(M_{\text{host}} = 10^{12} M_\odot\), and \(R_0 = 10\) kpc, we find \(r_J \approx 70\) pc. This is larger than the the typical size of globular clusters in the Milky Way: the stars are no longer strongly bound to the cluster, the gravitational pull of the Milky Way takes over.

The orbit of a star in some potential is determined by its initial position \(x_0\) and velocity \(v_0\). Or, expressed in other quantities, by the energy \(E(x, v)\) and angular momentum \(L(x, v)\) (or by the actions \(J\)) of the star (Johnston, 1995). All the stars in a globular cluster share the same systemic velocity, but have a random peculiar velocity:

\[
v_i = \bar{v}_{\text{systemic}} + \delta v_{i,\text{peculiar}},
\]

where \(\bar{v}_{\text{systemic}}\) is the average velocity of the cluster, and \(v_i\) the velocity of star \(i\). The same is true for the position \(x\) of the stars. Initially, the stars are on approximately the same orbit around the host galaxy. However, because the stars all have a slightly different position and velocity, their orbits will diverge after some time, and after they become unbound. At this point, when the stars are much further apart than the Jacobi radius, the internal gravitational attraction of the cluster can be neglected.

Essentially, one can see a stellar cluster as an ensemble of stars clustered around the centre of mass. After some time the distribution will get elongated and stretched in some form (Helmi & White, 1999). Depending on the orbit, the
streams can look very differently, see Fig.6. When put on a circular orbit, a stream will mostly grow in one direction (panels D and F). If the stream is put on a radial orbit, the spreading will occur also in other directions, creating shell-like structures (panels A and B). As there are many possible orbits, gradually ranging from circular to radial, there are many apparent shapes for streams, ranging from narrow structures to shell structures (panels C and E).

![Fig.6: Tidally disrupted objects (Hendel & Johnston, 2015). The panels A-F show a variety of apparent shapes that streams can have, ranging from narrow to dispersed.](image)

2.5 Streams in Actions and Angles

Computing the orbits of all the stars in the stream can be computationally expensive. Instead of describing a stream as an ensemble of stars that are on nearby orbits, one can also look at their dispersion as a way of describing its internal properties (Gomez & Helmi (2007) building up on work of Helmi & White (1999)). All the stars in the stream travel in the same host potential, it can be assumed that the internal gravitational interaction of the stream can be neglected if the mass ratio is large. If we are able to write the distribution function of the progenitor of the stream in action-angle variables, we would have a framework in which we can very simply evolve a stream analytically. This section is mainly based on the work of Helmi & White (1999) and that of Gomez & Helmi (2007). Let us assume the progenitor follows a Gaussian distribution in configuration and velocity space. In matrix form this can be expressed as

\[
f(x, v, t^0) = f_0 \exp \left(- \frac{1}{2} \Delta_\omega^0 \sigma_\omega^0 \Delta_\omega^0 \right).
\]  

(20)

In this notation, \( \Delta_\omega^0 \) is a 6D-vector containing the configuration and velocity elements, \( \Delta_\omega^0,i = \xi_i - \xi_i^0 \) where \( \xi_i = x_i \) for \( i = 1, 2, 3 \) and \( \xi_i = v_i \) for \( i = 4, 5, 6 \), and \( \Delta_\omega^0 \) is the Hermitian conjugate. Furthermore, the matrix \( \sigma_\omega^0 \) is diagonal and has the elements \( \sigma_{ii}^0 = 1/\sigma_\xi^2 \). Note that this distribution is in Cartesian coordinates \( \omega = (x, v) \), and it is at \( t = t_0 \), it is still a constant distribution. To let it evolve in time, we first need to transform the distribution to action-angle variables \( w = (\theta, J) \). This transformation, which is a local transformation evaluated at \( \xi_i^0 \) because the system initially is small in all directions, is given by the matrix \( M_0^{-1} \), at \( t = t_0 \). The transformation of the separate parts of eq.(20) is as follows:

\[
\Delta_\omega^0 = M_0^{-1} \Delta_\omega^0,
\]

\[
\sigma_\omega^0 = M_0^{-1} \sigma_\omega^0 M_0^{-1}.
\]

(21a)

(21b)

where the elements of the transformation matrix are \( M_{i,j}^{-1} = \frac{\partial \varpi_j}{\partial \varpi_i} \). Substituting these terms in eq.(20) gives

\[
f(\theta, J, t^0) = f_0 \exp \left(- \frac{1}{2} \Delta_{\theta}^0 \sigma_\theta^0 \Delta_{\theta}^0 \right).
\]  

(22)

Note that the new distribution function has a similar analytic form, but now in action-angle variables. Now recall that in action-angle variables the time evolution (eq.(4)) is trivial: \( \theta_i = \theta_i(t_0) + \Omega_i(J)t \). The other variable \( J_0 \), is constant, so for this one we can drop the \( t_0 \) subscript. Combining this, the transformation to time dependent action-angle
variables is given by

$$\Delta_w(t) = \Omega' \Delta^0_w,$$  \hspace{1cm} (23a)

$$\sigma_w(t) = \Omega'^\dagger \sigma^0_w \Omega',$$ \hspace{1cm} (23b)

where

$$\Omega' = \begin{bmatrix} I_3 & \partial \Omega/\partial J_t \\ 0 & I_3 \end{bmatrix}. \hspace{1cm} (24)$$

Here \( \partial \Omega/\partial J \) is a 3x3 matrix with elements \( \partial \Omega_i/\partial J_i \). This, finally, gives us a distribution in action-angle variables that we can evolve in time:

$$f(\theta, J, t) = f_0 \exp\left(-\frac{1}{2} \Delta^\dagger_w(t) \sigma_w(t) \Delta_w(t)\right). \hspace{1cm} (25)$$

Note that this final distribution has the same analytic form as the two constant distributions eq.(20) and eq.(22), the final one will vary in form as a function of time.

With these analytic tools one can investigate the evolution of a distribution of orbits as a function of time. After evolving the distribution for some time \( t \), we can transform it back to Cartesian coordinates by applying the inverse transformations. The elements of the inverse transformations have to be evaluated locally at the new coordinates of the centre of the distribution \( x_c \).

Using a distribution function means that we do not need the information of every single particle, we can easily compute (time) average properties of the system. For example, once transformed back to Cartesian coordinates, we can compute the distribution of the velocities at some time \( t \). The complete distribution function \( f(x, v, t) \) is centered at some point \( x_c \). To obtain the velocity distribution function \( f_v \), we simply set \( x = x_c \), obtaining

$$f_v(v, t) = f(x_c, v, t). \hspace{1cm} (26)$$

When the elements in the dispersion matrix in this function \( f_v(v, t) \) are diagonalized, this distribution function has an ellipsoidal shape. Another thing we can do, is computing a central density. To compute this, we have to integrate over the velocities

$$\rho(x_c) = \int f(x_c, v, t) \, d^3v. \hspace{1cm} (27)$$

Furthermore, we can also predict how the stream will grow. In appendix C it is shown that the dispersions of the velocities decrease in time, meaning that the velocity distribution becomes more concentrated. Now, since the number of stars in the stream is constant, we know that the integral of the distribution function over all possible positions and velocities is a constant:

$$\int f(x, v, t) \, d^3x \, d^3v = \text{constant}. \hspace{1cm} (28)$$

Because the velocity part concentrates over time, the positional part has to grow. The rate with which this happens as a function of time depends on the kind of potential. In HW99 it is shown that the evolution of a stream, e.g. its density, goes with \( 1/t^n \) where \( n \) is the number of degrees of freedom. This implies that the volume occupied by streams in a triaxial potential grow with power \( n = 3 \), in a spherical potential with power \( n = 2 \), unless the orbit is circular, in which case the stream grows with \( n = 1 \).
2.6 Subhalo interaction

Now that we have some understanding of how streams are created and evolve, we introduce subhalos. To describe the interaction between a subhalo and a tidal stream one can use the impulse approximation BT08 §8.2. The interaction takes place on a relatively short timescale ($\sim$Myr) compared to the dynamical timescale of a stream ($\sim$Gyr). Conceptually, when a subhalo crosses a stream it changes the orbits of the stars that are close to the encounter. Stars that are in the front of the encounter get pulled back, and stars that are behind the encounter get pulled forward. This causes a compression of the density along the stream. After a short time, the compression is turned to an expansion, finally resulting in a gap, see Fig.7. Here I follow the notation of Erkal & Belokurov (2015a) also in Helmi & Koppelman (2016). We assume here for simplicity the stream is oriented along the $y$-axis at the time of the subhalo flyby, and is moving in the positive $y$-direction in the $xy$-plane, as shown in Fig.8. The subhalo moves in an arbitrary direction, with velocity $\vec{w} = (w_x, w_y, w_z)$, where $w = \sqrt{w_x^2 + w_y^2 + w_z^2}$. In this frame, $b$ is the point of closest approach, and $\alpha$ is the angle of the subhalo’s velocity in the $x-z$ plane: $\tan \alpha = -\frac{w_x}{w_z}$.

Figure 7: Schematic evolution of a gap extracted from Erkal & Belokurov (2015a).

To galaxy center

Figure 8: Axis convention, figure extracted from Erkal & Belokurov (2015a).
Let us redefine the velocity of the subhalo as \((-w_\perp \sin \alpha, w_\parallel, w_\perp \cos \alpha)\), where \(w_\perp = \sqrt{w_\perp^2 + w_\parallel^2}\), and \(w_\parallel = v_y - w_y\) the relative velocity in the frame of the stream. In this frame, using the impulse approximation, one can compute the accelerations of particles in the stream:

\[
\Delta v_i = \int_{-\infty}^{\infty} a_i(x, y, z) \, dt,
\]

where \(a_i\) is the acceleration field due to the subhalo on a particle located at \((x, y, z)\). For simplicity only the Plummer sphere is considered for the subhalo potential here, and the stream is assumed to be a one-dimensional structure along the \(y\)-axis. In this case, one finds:

\[
\Delta v_x = \frac{2GM_s (bw_\perp \sin \alpha + yw_\parallel \cos \alpha)}{w \left( (b^2 + r^2_s)w_\perp^2 + w_\perp^2 y^2 \right)}, \tag{30a}
\]

\[
\Delta v_y = \frac{-2GM_s w_\perp^2 y}{w \left( (b^2 + r^2_s)w_\perp^2 + w_\perp^2 y^2 \right)}, \tag{30b}
\]

\[
\Delta v_z = \frac{2GM_s (bw_\perp \cos \alpha - yw_\parallel \sin \alpha)}{w \left( (b^2 + r^2_s)w_\perp^2 + w_\perp^2 y^2 \right)}. \tag{30c}
\]

These three expressions can be calculated numerically for other profiles than the Plummer sphere, such as a (truncated) NFW potential. Depending on the exact profile, the curves for \(\Delta v\) look a bit different. A NFW profile typically affects a larger part of the stream, for a similar mass, but the amplitude of the kick is lower.

Figure 9: The change in velocity along the stream for \(\Delta v_y\). The different lines are for different subhalo masses and scale radii. The red lines are for a Plummer sphere from eq.(30b), the blue lines are for a NFW profile and are numerically integrated.
In Fig.9 we see typical profiles of the change in velocity $\Delta v_y$ in the $y$-direction along the stream for both a Plummer sphere and a NFW profile. Increasing lines of the same colour indicate increasing subhalo size (mass and scale radius). After some simple algebra we find the maxima of this curve:

\[
\Delta v_{x,\text{max}} = \pm \frac{GM_s w_|| \cos \alpha}{w^2 \sqrt{b^2 + r_s^2}} \\
\Delta v_{y,\text{max}} = \pm \frac{GM_s w_\bot}{w^2 \sqrt{b^2 + r_s^2}} \\
\Delta v_{z,\text{max}} = \pm \frac{GM_s w_|| \sin \alpha}{w^2 \sqrt{b^2 + r_s^2}}
\]

and the position of the peaks $y_{\text{max}}$ along the stream,

\[
y_{\text{max}} = \pm \frac{w \sqrt{b^2 + r_s^2}}{w_\bot}.
\]

These expressions show that the maximum amplitude of the velocity change depends on the subhalo mass and on the speed of the encounter, while the location of the maxima is only weakly dependent on the subhalos properties through $r_s$.

3 Modelling Gaps

One possibility to model a gap in a stream is by following the time evolution of the dispersion of the orbits that are affected by the encounter. In this framework the gap grows just like a stream, after all the gap is some perturbation of a part of the stream, just with slightly different initial conditions. The orbits that are affected by the encounter receive a kick in velocities. Since the kick is not uniformly applied to particles along the stream, the local velocity dispersion will be different from before. As the stream grows in length, it will decrease in density, since the number of stars in the stream remains constant. At the location of the encounter the stream will grow a bit faster, and thus create a local minimum in the density: a gap.

We chose to simplify the framework from a model that predicts dispersions to one that predicts the separation of orbits. This is because it is not straightforward to compare the dispersion of orbits in the gap to its size in 1D (what is measured in the N-body). In the end, when measuring a gap in a stream, one only needs to know the separation of two orbits or particles: one in front of the gap and one just behind it.

3.1 Separation of orbits $\Delta$

For the evolution of a gap we use the evolution of two nearby orbits: one on each side of the gap. What follows here has been reported in Helmi & Koppelman (2016). We can use action-angle variables to track the evolution of a gap by the separation of two nearby orbits.

The initial separation in position $\Delta X_0$ and in velocity $\Delta V_0$ can be transformed to action-angle coordinates with the Jacobian transformation matrix $M_0$, just like eq.(21a) in §2.5

\[
\begin{bmatrix}
\Delta \Theta_0 \\
\Delta J_0
\end{bmatrix} = M_0 \begin{bmatrix}
\Delta X_0 \\
\Delta V_0
\end{bmatrix},
\]

where $M_0 = \partial(\Theta, J_0)/\partial(X, V)$, in the notation of §2.5 $\begin{bmatrix}
\Delta X_0 \\
\Delta V_0
\end{bmatrix} = \Delta_0$, and $\begin{bmatrix}
\Delta \Theta_0 \\
\Delta J_0
\end{bmatrix} = \Delta^0_\infty$. Because the time evolution of these new variables is simple, we can transform the static vectors to time-dependent version by

\[
\begin{bmatrix}
\Delta \Theta \\
\Delta J
\end{bmatrix} = \Omega' \begin{bmatrix}
\Delta \Theta_0 \\
\Delta J_0
\end{bmatrix},
\]

with

\[
\Omega' = \begin{bmatrix}
I_3 & \partial \Omega / \partial Jt \\
0 & I_3
\end{bmatrix}.
\]
This is the equivalent of the matrix product in eq.(23a). Recall that $J$ is time independent, i.e. $J = J_0$. Now we are in a position to evolve the separation vector in action-angle variables as a function of time. Then, at some time $t$ we can transform the vector back to Cartesian coordinates by

$$\begin{bmatrix}
\Delta X \\
\Delta V
\end{bmatrix} = M_t^{-1} \begin{bmatrix}
\Delta \Theta \\
\Delta J
\end{bmatrix},$$  \hspace{1cm} (36)

where $M_t^{-1}$ is a matrix that transforms the vector locally, at time $t$. If we combine eq’s (33),(34), and (36), we find:

$$\begin{bmatrix}
\Delta X \\
\Delta V
\end{bmatrix} = M_t^{-1} \Omega' M_0 \begin{bmatrix}
\Delta X_0 \\
\Delta V_0
\end{bmatrix}. $$ \hspace{1cm} (37)

This finally puts us in a position to evolve gaps in streams, starting from their initial properties at the time of the encounter. A scheme of the transformations is shown in Fig. 10.

Figure 10: Scheme of the transformations from Cartesian to Action-Angle and back.

### 3.2 Matrix transformations & time evolution

Let us take a better look at eq.(37) from §3.1:

$$\begin{bmatrix}
\Delta X \\
\Delta V
\end{bmatrix} = M_t^{-1} \Omega' M_0 \begin{bmatrix}
\Delta X_0 \\
\Delta V_0
\end{bmatrix}. $$

The vector $\begin{bmatrix}
\Delta X \\
\Delta V
\end{bmatrix}$ is a vector with the elements $(\Delta x, \Delta y, \Delta z, \Delta v_x, \Delta v_y, \Delta v_z)$. The power of the model is in the product of the matrices $M_t^{-1} \Omega' M_0$. Recall that $M_0$ is the transformation from Cartesian coordinates to action-angle variables, $\Omega'$ is time evolution in action-angle variables, and $M_t^{-1}$ is the (local) transformation back to Cartesian coordinates. If the model holds, we can see what it predicts for the evolution of gaps in streams (Helmi & Koppelman, 2016). The matrix $\Omega'$ has a submatrix $\partial \Omega / \partial J$ that dominates at large times since the other submatrices are either the identity matrix, or zero. From that we can conclude that the time evolution of $X$ goes as

$$\Delta X \sim t M_{t,1}^{-1} [\partial \Omega / \partial J] \Delta J_0, \hspace{1cm} (38)$$

where $M_{t,1}^{-1}$ is the upper left submatrix of $M_t^{-1}$, it transforms from physical to angle coordinates. The spatial separation is given by $|\Delta X|$, or $(\Delta X \Delta X)^{1/2}$, which then is

$$|\Delta X| \sim (t M_{t,1}^{-1} [\partial \Omega / \partial J] \Delta J_0)^{1/2} (t M_{t,1}^{-1} [\partial \Omega / \partial J] \Delta J_0)^{1/2}$$

$$\sim t (\Delta J_0^T [\partial \Omega / \partial J] [M_{t,1}^{-1}]^T M_{t,1}^{-1} [\partial \Omega / \partial J] \Delta J_0)^{1/2}$$

Which reduces to

$$|\Delta X| \sim t (\Delta J_0^T C_{x,\Omega} \Delta J_0)^{1/2}, \hspace{1cm} (39)$$
where \(C_x, \Omega = [\partial \Omega / \partial J] (M^{-1}_{x} \phi) [\partial \Omega / \partial J] \) is a symmetric matrix. The separation of orbits \(|\Delta X|\) depends both on the properties of the orbit and on the properties of the encounter, i.e. the impact geometry and subhalo mass and size. In turn, matrix \(C_x, \Omega\) only depends on the properties of the orbit. Let us look into matrix \(\Delta J_0\), which can be separated into a part that is only depending on the orbit, and a part that depends on the subhalo encounter. In eq.(33) we see that both \(\Delta X_0\) and \(\Delta V_0\) determine \(\Delta J_0\). Considering that the dominating term is that of the change of velocity in the \(y\) direction (eqs.(33)b), we can split \(\Delta J_0\) in two parts \(\Delta J_0 = 2\Delta v_y^{\text{max}} f_{\text{orb,0}}\), where \(f_{\text{orb,0}}\) is some vector function depending only on the location of the impact. To specify \(f_{\text{orb,0}}\) we need to know in what kind of potential these orbits reside, let us consider a spherical potential. In the plane of motion, where we are left with only two dimensions, this function is \(f_{\text{orb,0}} = [x_0, (v_y,0 - x_0\Omega_y)] / \Omega_y\). Here \(x_0\) is the \(x\)-position at of the encounter \(t = 0\), \(v_y,0\) the \(y\)-velocity, and \(\Omega_y\) & \(\Omega_z\) are the action-angle frequencies in the plane. Recall

\[
\Delta v_y^{\text{max}} = \pm \frac{GM_0 w\perp}{w^2 \sqrt{b^2 + r_s^2}}
\]

Combining all of this one finds

\[
|\Delta X| \sim t \frac{2GM_0 w\perp}{w^2 \sqrt{b^2 + r_s^2}} \left( f_{\text{orb,0}}^\dagger C_x, \Omega f_{\text{orb,0}} \right)^{1/2}
\]

### 3.3 Initial conditions

Suppose we have an orbit for the centre of a gap, one question remains: what are good initial conditions, i.e. what is the correct initial separation in \(\Delta X_0\) and \(\Delta V_0\) for the gap? From the impulse approximation we got \(\Delta v\) and \(\Delta y\), which are two times eq.(31) and eq.(32) respectively. The equations give the amplitudes of the curve that describes the change in velocity along the stream, the actual separation in velocity \(\Delta V_0\) is then two times the amplitude. These conditions are only valid when we can neglect the internal velocity gradient of the stream. This is only true for cold, thin streams, typically of several Gyr old and for which the orbital gradient is zero. For younger streams, we need to find the internal velocity separation: the orbits of stars A and B that are used to measure the size of the gap have some internal separation in velocities, even when the stream has experienced no encounter. To test whether one can neglect these gradients or not, one can compare an experiment of a stream without an encounter with an experiment of the exact same stream, but with an encounter. First, we find the particles, in the experiment with the encounter, that have the largest change in velocity: the stars A and B. Then, we look for the same particles in the stream without the encounter, but at the time of the encounter, and calculate their initial separation in velocities. If these internal velocity differences are of the order, or even larger than the \(\Delta v\) from the encounter, one has to take these into account when calculating the evolution of the gap size. The aim of the model is to be able to predict the size of gaps, completely independent of N-body simulations. So, we are after finding some way to determine the internal velocity gradient, without using an N-body.

#### 3.3.1 Spherical potential

The stream that we model in the spherical potential is thin and we can describe it as a one-dimensional structure. For such a stream it is easier to find the initial conditions because we only have to take one dimension into account. We find the initial conditions by evolving the separation of the two stars, a few Gyr before the encounter up until this takes place. When the separation is evolved forward for \(\sim 2\) Gyr up to the time of the encounter, we find the gradient \(\partial v / \partial x\) at the time of the encounter. Note that now the initial conditions for \(\Delta X_0\) and \(\Delta V_0\) are arbitrary as we are only interested in the ratio of the two.

Assuming the velocity gradient is linear, which is only true on small scales, we can find the internal separation in velocity:

\[
\Delta v_i^{\text{int}} = \frac{\partial v_i}{\partial x} \Delta x + \frac{\partial v_i}{\partial y} \Delta y + \frac{\partial v_i}{\partial z} \Delta z,
\]

where \(i = (x, y, z)\). If the stream is narrow, \(\Delta x\) and \(\Delta z\) can be neglected, and the internal velocity separations are:

\[
\begin{align*}
\Delta v_x^{\text{int}} &= \frac{\partial v_x}{\partial y} 2y_{\text{max}} \\
\Delta v_y^{\text{int}} &= \frac{\partial v_y}{\partial y} 2y_{\text{max}} \\
\Delta v_z^{\text{int}} &= \frac{\partial v_z}{\partial y} 2y_{\text{max}}
\end{align*}
\]
3.3.2 Stäckel potential

Streams in Stäckel potentials are unfortunately not always thin enough to describe them as one-dimensional structures. Now we can no longer neglect $\Delta x$ and $\Delta z$ and have to use the full equation (41). This adds a complication, we do not know what $\Delta x$ and $\Delta z$ are. In theory we could use the size of the stream, however in practice the velocity gradients are so large that we need a preciser way of finding these values.

As stated in §2.3, we use two Stäckel potentials. For potential 1, which only has one component, we were able to estimate the velocity gradient via a slightly different method. For this, we use the velocity gradient of the orbit of the gap. This is in general less accurate than the method described in the section above. Particles in a stream are not all on the same orbit, and thus the velocity gradient of the orbit of one particle is different than the velocity gradient of the orbits of multiple particles.

To obtain the initial conditions with this method, we integrate the orbit of the gap starting before the encounter and zooming in on the position where the encounter occurs. At this position, we locally approximate the velocity gradient to be linear. Because this is done for one orbit, and not for a stream, we can neglect the $\Delta x$ and $\Delta z$ and we end up with similar equations as eqs.(42).

For Stäckel potential 2, the orbital gradients are very large (especially in the vertical direction, because the orbit is so close to the plane of the disk) and with the time resolution we have used it is not possible to accurately measure the gradient. Therefore, in this case we use the gradient directly measured from the relevant particles in the N-body.

4 N-body

To validate the model, we performed N-body simulations with GADGET-2.

4.1 Setting up GADGET-2

To perform the N-body simulations we modified GADGET-2 (Springel, 2005). GADGET-2 is a particle code: one can input particles and GADGET-2 computes the gravitational attraction of all the particles and evolves them in time. The calculation of the force on each of the particles is also called an N-body simulation, as it takes care of N bodies/particles. For this N-body calculation Gadget uses a leap-frog algorithm, see §2.1. The code is very extensive, it can take care of different types of particles (gas, stars, DM) and has a good SPH (Smoothed Particle Hydrodynamics) solver. Despite being very extensive, GADGET-2 does not come with the implementation of a galactic potential or ‘flying’ subhalos. One can choose to model these by inserting particles distributed by the density function of the potential of choice. We choose to implement rigid versions of the potentials, which are less computational heavy. A modified version of GADGET-2, with the implementation of a spherical galactic host potential, was provided by Tjitkse Starkenburg. We modified the code such that it now can host a moving subhalo, and with an option for a host potential in Stäckel coordinates. With this version one can explore the behaviour of streams-subhalo encounters both in spherical and axisymmetric potentials.

The rigid potential is modelled by calculating and adding the force due to the potential for each particle. In section §2.3 the acceleration is derived for a spherical NFW potential. GADGET-2 uses the acceleration to calculate the orbit of the particle, so we can compute and add this term for each particle and let the code compute the orbit in the potential.

4.2 Specific set-up - spherical

The host potential of the N-body simulations is a spherical NFW potential, with mass $M_{\text{host}} = 3 \cdot 10^{11} \text{M}_\odot$, and scale radius $r_{\text{host}} = 15.6 \text{kpc}$. The stream progenitor is modeled as an ensemble of $10^4$ particles of mass $M_{\text{particle}} = 100 \text{M}_\odot$ that are distributed following a Gaussian profile in 6D configuration and velocity space. Initially, this distribution has the dispersions $\sigma_{\text{pos}} = 0.05 \text{kpc}$, and $\sigma_{\text{vel}} = 2 \text{kpc/Gyr} \approx 2 \text{km/s}$. This is a very cold and compact stellar object, which does not necessarily model a globular cluster or dwarf galaxy. We want the stream to be thin and short, if the initial distribution is larger the stream will grow much longer. The stream is put on an eccentric orbit with apocenter $\sim 71 \text{kpc}$ and pericenter $\sim 45 \text{kpc}$, as shown in Fig.11. Before introducing a subhalo, the stream is evolved for 2 Gyr. The orbit of the stream is shown in Fig.11, and a stream evolving along this orbit is shown in Fig.12.
Figure 11: Orbit of the stream, the red dot indicates the position where the subhalo impacts the stream. For this orbit, apocenter = 71 kpc, and pericenter = 45 kpc. The potential that is used here is the one described in §2.3.1.

Figure 12: Stream evolving in an NFW potential, projected on the xy-plane. The progenitor of this stream is a Gaussian, with dispersions $\sigma_{\text{pos}} = 0.05$ kpc and $\sigma_{\text{vel}} = 2$ kpc/Gyr. The potential that is used here is the one described in §2.3.1.
4.3 Specific set-up - Stäckel

4.3.1 No disk

As a first try, for simplicity we first set the mass of the disk to zero. What remains is the Stäckel potential 1 described in §2.3, Fig.4. The mass of the halo is $3.596 \cdot 10^{11} M_\odot$, and its scale $a = 2.0$ kpc. These parameters, except for the axis-ratio, are chosen to model a Milky Way like galaxy. However, the scale $a$ is set a bit too small, which causes the circular velocity to be too large at 8 kpc, the distance of the Sun, as shown in Fig.3. Our choice of the axis-ratio is $a_h/c_h = \epsilon_h = 1.50$.

In Fig.13 we see the orbit of a star in the described potential. The plane of motion of the orbit is inclined in the $z$-plane and moves up and down, i.e. it precesses. The apocenter and pericenter of this specific orbit are $r_a = 12.8$ kpc, $r_p = 45.5$ kpc, which is a more inner orbit than the one used in the spherical case. The progenitor of the stream is similar to the one used in the spherical potential: the density follows a Gaussian distribution with $\sigma_{pos} = 0.05$ kpc and $\sigma_{vel} = 2$ kpc/Gyr. In this potential, the stream is evolved for 2 Gyr before the subhalo is inserted, see Fig.14.
Figure 14: Stream evolving in a single component Stäckel potential (Stäckel potential 1 §2.3), projected on the $xy$-plane. The initial dispersions of the progenitor of the stream are $\sigma_{\text{pos}} = 0.05\,\text{kpc}$ and $\sigma_{\text{vel}} = 2\,\text{kpc/}\text{Gyr}$, distributed like a Gaussian.

4.3.2 With a disk

Next, to try the model in a two-component potential (Stäckel potential 2) we added a disk with a fraction of $k = 0.101$ of the total mass of the system. The mass of the system is $3.596 \cdot 10^{11}\,\text{M}_\odot$, and its scale $a = 2.0$. These parameters, are again chosen to model a Milky Way like galaxy. Our choice of the axis-ratio for the halo is $a_h/c_h = \epsilon_h = 1.02$, and for the disk $a_d/c_d = \epsilon_h = 80$. Unfortunately, the scale $a$ is set a bit too small, which causes the circular velocity to be too large at 8 kpc, the distance of the Sun, as shown in Fig.3.
In Fig.15 we see the orbit of a star in the described potential, the initial position and velocity that of the orbit of Fig.13. The plane of rotation of the orbit is much less inclined in the z-plane as before, the halo is much more spherical and there is now a disk. The apocenter and pericenter of this specific orbit are \( r_a = 14.5 \) kpc, \( r_p = 42.8 \) kpc, which is a more inner orbit than the one used in the spherical case. The progenitor of the stream is similar to the one used in the spherical potential: the density follows a Gaussian distribution with \( \sigma_{\text{pos}} = 0.05 \) kpc and \( \sigma_{\text{vel}} = 2 \) kpc/Gyr. In this potential, the stream is evolved for 2 Gyr before the subhalo is inserted.

### 4.4 Inserting a subhalo

After the formation of the stream, a subhalo is inserted in the simulation for a total of \( \sim 0.5 \) Gyr, after which it is removed. At \( t = t_{\text{hit}} \), a direct encounter (impact parameter \( b = 0 \)) of the subhalo and the stream takes place, causing a gap in the stream. Here, only encounters with Plummer spheres are presented. For the Plummer sphere it is easy/possible to analytically compute the impulse, in case of an NFW this is more complex because we need to compute the impulse numerically. Different simulations have been carried out, all with the same set-up, but differing in subhalo masses and scale radii: \([\log_{10} M^{\text{sub}} [M_\odot], r_s^{\text{sub}} [\text{kpc}]] = ([6.9, 0.38], [7.2, 0.59], [7.5, 0.9], [7.9, 1.35])\) (Tolstoy et al., 2009). These combinations are derived from physically motivated NFW potentials, for which a relation exists between mass and scale radius, by equating the NFW potential to the Plummer potential, both at the NFW scale radius. First we set the total mass of the Plummer to be equal to \( M_{200} \), then we can calculate the Plummer scale radius \( r_{pl} \):

\[
\Phi_{\text{NFW}}(r = r_s) = \Phi_{\text{Plummer}}(r = r_s) \tag{43a}
\]

\[
\phi_0 \log 2 = \frac{GM}{\sqrt{r^2 + r_{pl}^2}} \tag{43b}
\]

\[
r_{pl}^2 = \left( \frac{GM}{\phi_0 \log 2} \right)^2 - r_s^2 \tag{43c}
\]

Once GADGET-2 is working with a host potential, it is little work to add a subhalo. There are two major differences: the subhalo has a different potential, and it moves on some orbit in the host potential, i.e. is not centred at zero. To let the subhalo move around we centre it on a particle that is put on the desired orbit, the mass of the particle is negligible. The force that a subhalo on such an orbit exerts on particles in the simulation can be computed in a similar way as is done in §2.1 for the host potential. First we input the desired potential,

\[
\Phi_{\text{Pl}}(r) = -\frac{GM_s}{\sqrt{r^2 + r_{pl}^2}} \tag{44}
\]
next we calculate the acceleration for each of the particles (eq. (7))

\[ a_{Pl}(r) = -\frac{GM_s r}{(r^2 + r_{pl}^2)^{3/2}}, \tag{45} \]

where \( r \) is the distance to the subhalo, \( r = \sqrt{(x_{sub} - x)^2 + (y_{sub} - y)^2 + (z_{sub} - z)^2} \). In the simulations in this thesis, we only want the subhalos to encounter the stream once. To make sure the subhalo does not come close to the stream at later times we remove it after the encounter. Typically, the subhalo is only active in the simulation for \( \sim 0.4 - 0.6 \) Gyr.

5 Results

5.1 Inspection of an encounter

To see if the approximations in our model hold, we compare them with an N-body experiment. Fig.16 shows a cold stream in a spherical NFW potential §2.3.1, which has a direct collision with \( M = 10^{7.5} M_\odot \) subhalo with \( r_s = 0.90 \) kpc. The subhalo, indicated by the red dot, is directly colliding with the stream as shown in the top right panel. Because the subhalo moves along with the stream with some angle, the interaction is relatively long. This results in a very prominent gap in the stream, as shown in Fig.16.

![Figure 16: A cold stream orbiting in a spherical NFW potential at different times before, at, and after impact with dark subhalo. The time of the impact \( t_0 = 0.0 \) Gyr, the properties of the subhalo are \( M = 10^{7.5} M_\odot \) and \( r_s = 0.90 \) kpc. The red dot is the location of the subhalo. The red arrow is the velocity vector (in scale) of the subhalo and the black arrow is the velocity of the stream (also in scale).](image-url)
In section §2.6 we discussed the application of the impulse approximation to dynamically model the interaction of a stream and a subhalo. In Fig.17 we see that this approximation holds very well. The particles in the N-body (black dots) are a bit scattered and have some offset at large $y$ with respect to the curves from the impulse approximation (green lines). In the bottom right panel it is shown that the stream is not exactly a linear, one-dimensional structure at the time of the collision. The scatter can be explained because the stream has some finite size in the $x$ and $z$ direction, the offset is related to the curvature of the stream at large $y$ distances.

Figure 17: Difference in velocity along the stream (same stream as Fig.16) at different times after impact with dark subhalo. The black dots are data from an N-body experiment described here, the green curves are the expressions for the change of velocity. The red dots indicate the 30 particles that receive the largest change in velocity in the $y$-direction. The time of the impact $t_0 = 0.0$ Gyr, the properties of the subhalo are $M = 10^{7.5} M_\odot$ and $r_s = 0.90$ kpc.

5.2 Measuring the gap size

Comparing the results of the model with the results off the N-body experiments is not straightforward. The difficulty here lies in finding a way to measure the gap size from the N-body. In some cases the gap is clearly present, in most cases it is only visible for encounters with very massive subhalos, and in all cases the gap has no clear shape and boundaries. We set the subhalos on a trajectory that collides directly with the streams because this creates the largest and best visible gaps, making our task easier.
Since near the ends of the gap, the density is higher, one way of defining the gap is by looking at the density of the stream along the stream. A first attempt of measuring the gap size is by tracing the particles in the N-body that are found in the peaks of the density. So, by that definition, the gap size is defined as the distance between the peaks in the density along the stream. This approach works fine, however to measure the size this way one needs many particles to accurately measure the density along the stream. Currently we are running N-body experiments with $10^4$ particles, which is not enough for this approach.

A more accurate way of tracing the peaks is by tracing the particles that receive the largest impulse from the encounter (red dots in Fig.17). These particles have the largest changes in velocity, and will end up in the peaks of the density profile of the stream as can be seen in Fig.18. Then, we find two sets of 30 particles that have the largest difference in velocity, one set with a positive difference (set A) and one negative (set B), the red dots in figure. Finally, the gap size is defined as the distance between these two sets

$$
\Delta r_{\text{gap}} = \sqrt{(\bar{x}_A - \bar{x}_B)^2 + (\bar{y}_A - \bar{y}_B)^2 + (\bar{z}_A - \bar{z}_B)^2},
$$

where $\bar{x}_A$ is the mean value for $x$ of the particles in set A, etc.

Figure 18: Density of a stream (same stream as Fig.16) orbiting in a spherical NFW potential at different times after impact with dark subhalo. The time of the impact $t_0 = 0.0$ Gyr, the properties of the subhalo are $M = 10^{7.5} M_\odot$ and $r_s = 0.90$ kpc.
Figure 19: A gap growing in a stream in a NFW potential. The different colours indicate the models for the same stream, only encountered with different subhalos ($r_s$ and $M$ in the legend), the solid black lines show what we measure in the Nbody. The dotted lines near the solid black lines indicate a $1\sigma$ error on the measurement of the gap size. The dashed lines are the best linear fits for each gap.

5.3 N-body vs model: Spherical potential

The $\Delta r_{\text{gap}}$ can be directly compared to the separation of the two orbits $\Delta X$. In Fig. 19 we see that the predicted gap sizes reproduce the measured sizes from the N-body experiments very well. The specifics of the orbit of the stream are exactly similar to the one described in section §4.2. Furthermore, the solid black lines are the measurements from the N-body and the dotted lines correspond to $1\sigma$ scatter. Finally, to compare the sizes are fitted with a straight line, plotted as dashed lines. Recall that we use two sets of particles to measure the gap size, the solid line here is the mean of all the individual separations between the particles in the two sets. The $1\sigma$ scatter is the standard deviation of these distances. In this figure, the coloured solid lines are the predictions, one for each side of the gap. To obtain these orbits, we traced the particles that receive the largest kick in velocity. These particles are traced back in an experiment without an encounter, their velocities and positions are used to compute the two orbits. This way we make sure the orbit of the gap is in between the two modelled orbits. In the panel below is the distance to the centre of the orbit, plotted as a function of time. At apocenter, the gap is the smallest, and at pericenter it is the largest. The growth of the gaps clearly depends on the properties of the subhalo. In general, the rate of growth is linear in time, the exact slope depends on the properties of the orbit, subhalo, and on the geometry of the encounter.
5.4 Degeneracy in size

Looking at the equations from Erkal & Belokurov (2015a) §2.6, it is evident that there is some degeneracy in the impulse from an encounter with a subhalo. Both the properties of the subhalo (mass and scale radius), and the geometry of the impact ($\alpha$ and $w_\perp/w$) determine the size of the impact. This degeneracy is shown in Fig.20 and Fig.21.

![Gap growth vs Mass and Geometry](image)

**Figure 20:** Example A: a gap in a stream orbiting in a spherical NFW potential along a different trajectory than the stream used in §5.3. Increasing lines of the same colour are gaps due to encounters with larger subhalos. The different colours indicate the angle of the impact with respect to the stream velocity. Here, c-a-b are in the plane of motion, as shown with the arrows. Angle d is perpendicular to the plane of motion, coming out of the paper.

In these figures the coloured lines are for different impact geometries, increasing lines of the same colour indicate different subhalos. One can see that for example A the dominant factor that determines the size of the gap is the size of the subhalo, the top lines are all of a different colour. On the other hand, for example B, the dominant factor is the impact geometry: the red lines dominate over the others.

It has been speculated that this degeneracy possibly can be disentangled by taking the full 6D phase space information into account (Erkal & Belokurov, 2015b). Unfortunately, by looking at the results of the gap size, we can add another degeneracy onto this. That is, the size of the gap is also dependent on the orbital phase. Because of the growing and decreasing nature of gaps we can not easily trace back the initial conditions of the gap just by looking at its size. For example, if we draw a horizontal line at a size of 50 kpc through Fig.20 or Fig.19 it would cross multiple lines indicating that a gap of a given size can be formed at different times by subhalos of different size and/or orbital properties.
Figure 21: Example B: a gap in a stream orbiting in a spherical NFW potential along a different trajectory than the stream used in §5.3. Increasing lines of the same colour are gaps due to encounters with larger subhalos. The different colours indicate the angle of the impact with respect to the stream velocity. Here, c-a-b are in the plane of motion, as shown with the arrows. Angle d is perpendicular to the plane of motion, coming out of the paper.

Figure 22: Orbits for examples A and B, both projected on the xy-plane. The host potential is the NFW described in §2.3.1. The red dots indicate the position of the encounter of the streams and the subhalo.
5.5 N-body vs model: Stäckel potential 1

As stated in §3.3.2 finding the initial conditions for $\Delta X_0$ and $\Delta V_0$ is more complex for streams in Stäckel potentials. To find the initial conditions for the gap measured in the stream in Stäckel potential 1 we use the method described in §3.3.2, which reproduce initial conditions that are less good as for the spherical case.

![Graph showing gap growing in a stream in a one-component potential (Stäckel potential 1). The different colours indicate the models for the same stream, only experiencing an encounter with different subhalos ($r_s$ and $M$ in legend), the solid black lines show what we measure in the N-body. The dashed lines near the solid black lines indicate a 1σ error on the measurement of the gap size, obtained in a similar way as described in §5.3. The dashed straight lines show the best fit for a straight line.](image)

Figure 23: A gap growing in a stream in a one-component potential (Stäckel potential 1). The different colours indicate the models for the same stream, only experiencing an encounter with different subhalos ($r_s$ and $M$ in legend), the solid black lines show what we measure in the N-body. The dashed lines near the solid black lines indicate a 1σ error on the measurement of the gap size, obtained in a similar way as described in §5.3. The dashed straight lines show the best fit for a straight line.

In Fig.23 it is shown that the model reproduces the orbital cycles measured in the N-body experiments, the amplitude is not always matched perfectly. The specifics of the orbit of the stream are exactly similar to the one described in section §4.3.1. Furthermore, the solid black lines are the measurements from the N-body and the dashed lines correspond to 1σ scatter. To obtain the orbit of the gap, we traced the particles that are closest to the impact. These particles are traced back in an experiment without an encounter, their velocities and positions are used to compute the orbit.

At late times, the peaks of the gap sizes in the N-body are lower than those predicted by our model. We think this is partly due to the resolution of the N-body, and because the measurements in the N-body do not orbital take winding of the gap into account, whereas the model does. In the N-body there is a limitation to the possible size of the gap because of the method we measure the size, this does not occur in the model.

If we compare the evolution of the size of the gap to the one modelled in the spherical potential we see that the
oscillatory behaviour is much stronger. This is explained by the different types of the orbits. In the Stäckel potential, the orbit is more inside, as shown in §4, and there are many more orbital cycles. Although the strong oscillatory behaviour makes it hard to do a good fit, the rate of growth still is fitted well by a linear relation, as shown by the dashed lines in Fig.23. It seems that the gaps grow at a slower rate, in comparison to the gaps in the spherical potential, however this could be because of the different orbits.

5.6 N-body vs model: Stäckel potential 2

As stated in §3.3.2 we were not able to accurately reproduce the initial conditions for the gap in Stäckel potential 2. The chosen orbit resides close to the plane of the disk, and because of that the stream is significantly extended in the z-direction. To compute the results shown in Fig.24 we use initial conditions measured in the N-body.

Figure 24: A gap growing in a stream in a two-component potential (Stäckel potential 2). The different colours indicate the models for the same stream, only experiencing encounters with different subhalos (r, and M in legend), the solid black lines show what we measure in the N-body. The dashed lines near the solid black lines indicate a 1σ error on the measurement of the gap size, obtained in a similar way as described in §5.3. The dashed straight lines show the best fit for a straight line.

In Fig.24 we see that the predicted gap sizes reproduce the measured sizes from the N-body experiments reasonably well. The orbital cycles are matched perfectly, the size of the gap not always. The specifics of the orbit of the stream are exactly similar to the one described in section §4.3.2. Furthermore, the solid black lines are the measurements from the N-body and the dashed lines correspond to 1σ scatter. To obtain the orbit of the gap, we traced the particles that are closest to the impact. These particles are traced back in an experiment without an encounter, their velocities
and positions are used to compute the orbit. At late times, the peaks of the gap size produced in the N-body are lower than those predicted by our model. The oscillatory behaviour makes it hard to fit the evolution of the gap sizes, however a linear fit seems to fit reasonably well. If we compare Fig.24, and Fig.23 we see that the first one grows much faster. The initial conditions for the progenitor (orbit and size) are exactly the same, the only thing that changed is the potential: there is an extra component, and the axis-ratio of the halo is different.

6 Discussion & Conclusion

6.1 Comparing to other studies

Investigating gaps in streams (or streams in general) is a hot topic because of their cosmological relevance. Here we compare our results to what other works have found. The change in velocity of particles along a stream after an encounter is consistent with other results (Carlberg, 2013; Erkal & Belokurov, 2015a). Caustic features that are produced in the density along the stream after an impact, e.g. Fig.18, are seen by others as well. Erkal & Belokurov (2015a) describe that a stream grows in three phases: a compression phase, an expansion phase (growth $\sim t$), and a caustic phase (growth $\sim \sqrt{t}$). In our results, we see a compression and a growth, see Fig.19, however we do not see the caustic phase growth. As predicted by our model, the stream that we simulate grows faster than the $\sqrt{t}$ rate predicted by Erkal & Belokurov (2015a). In a more recent study (Sanders et al., 2016), this caustic phase is also not found. There are two major differences that possibly cause this difference in the growth rate. First, the streams considered in the earlier works are mostly on circular streams. As explained in this work, gaps (and streams in general) grow differently on circular orbits. The degrees of freedom are less on a circular orbit, thus the gap will grow differently. However, this does not explain the difference completely. Secondly, in their study Erkal & Belokurov (2015a) neglected the velocity gradient that is present in streams. One has to take this velocity dispersion into account when investigating the growth rate of gaps in streams. This might not explain the different rate of growth. However, in the aforementioned studies, it is found that the growth rate at very late times $t >> 1$ does not match the claimed $\sqrt{t}$ growth rate very well.

Perhaps the largest difference with respect to most of the previous studies on gaps in streams, apart from the growth rate that we find, is our definition of the size of a gap. Often, what is used is the size of the area that has an under density. That is, the gap size is then defined by the area where $\rho_{\text{perturbed}}/\rho_0 > 1$. Although we find that this approach works, we define the size of the gap by the distance between two particles that receive the largest kick in velocities from the encounter. In simulations with a low amount of particles this new definition is more accurate, and it is easier to simply trace two particles. When looking at Fig.18 we see that these particles end up in the peaks of the density along the stream, and thus are tracing size of the gap. Another benefit from our approach is that the size of the gap directly is measured in Cartesian coordinates. Other works often express the size of the gap in parallel angle ($\Delta \theta$ action-angle space), or in physical angle ($\Delta \phi$). For a better comparison of the two methods, we measured the size of the gap with the method of the densities, as shown in Fig.25. The rate of growth of the sizes measured by the two methods is very similar but does not match completely. The method of Erkal & Belokurov selects a smaller part of the stream to be the ‘gap’. Naturally, the size difference of two parts in a stream, one being larger than the other, grows in time. The rate of growth of both methods is linear in time.
6.2 Conclusion

Let us now look back at the questions asked in the introduction:

- We find that gaps are well described by the divergence of two nearby orbits. This divergence is evolved in action-angle variables, and can be computed in different potentials.

- In the spherical case, the proposed method to model sizes of gaps in tidal streams produces sizes that match well with N-body simulations. The reproduced rate of growth is different from earlier findings (Erkal & Belokurov, 2015a), however the most recent studies might point this out as well (Sanders et al., 2016).

- The model is very sensitive in a Stäckel potential, making it difficult to reproduce the size of the gaps. Although the oscillatory behaviour makes it hard to fit the gap sizes convincingly, the evolution is fitted well by a linear relation with time.

- Deriving the properties of subhalos encountering with a stream might be more difficult than is expected. Next to the degeneracy in the impact geometry and mass and size of the subhalo, also the age of the gap is unknown. For a certain size, one can find multiple fitting subhalos, at different gap-ages, see Fig.19.

- Overall, we find that, after a small compression phase, gaps grow linearly in time $\Delta X \sim t \cdot f(M, r_s, w)$. The properties of the subhalo and the relative velocity of the impact determine the slope of the growth rate.

6.3 Future Prospects

A possible application of the model is to predict distribution of gap sizes in currently known streams. At the moment, most streams are not observed with a high enough quality to detect gaps. In the near future, with Gaia, we expect that there will be many more observations of streams, and detections of gaps. However, even before that, we can calculate the power spectrum of gap sizes that should be present in streams near the Milky Way, assuming a subhalo
distribution according to the ΛCDM theory. Recently Erkal et al. (2016) have predicted a spectrum of this kind, without using a model like the one described in this thesis. Possibly, improvements can be made to their predictions. As another application, if there are accurate detections of gaps in streams, one could use the model to possibly infer the subhalo properties. However, this might be difficult in reality. To do this, one has to know the potential of the Milky Way, the orbit of the gap, and the age of the gap.

Of course, there are possible future improvements as well. One of the limitations of the model at the moment is that we can only accurately measure the initial separations of a gap in a stream that is sufficiently thin. In this thesis, we had to use an N-body to measure the initial conditions for one of the gaps. Ideally, we would like to skip the N-body and be able to analytically infer the initial conditions of such a gap.

Another improvement would be to extend the model from an axisymmetric potential to a triaxial potential. This does not necessarily improve the best fit for the Milky Way, but it does give us some extra freedom in the choice of our potential. Next to that, the type of subhalo potential could be improved. In this thesis, we have chosen to model the dark matter subhalos as Plummer spheres. A better fitting potential would be a (truncated) NFW profile. Unfortunately, there is no analytic solution to the integral used in the impulse approximation (eq.29) for a NFW, so this has to be done numerically.

Apart from the type of subhalo potential, it has been speculated that the subhalo might not be spherical (Vera-Ciro et al., 2014). Possibly, subhalos are more realistically modelled by non-spherical structures, which might complicate encounters.

7.0 Acknowledgement

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Appendices

A Hamiltonian Dynamics

This section was written to refresh the reader on Hamiltonian Dynamics, Hamilton’s characteristic function, Hamilton-Jacobi Theory, and Action-Angle Variables. For a complete overview I recommend the book Classical Mechanics by Goldstein (Goldstein et al., 2014), of which this section is a condensed summary.

A.1 Hamiltonian

Classically, Hamiltonian mechanics is a different mathematical tool to solve problems in classical mechanics. It differs from Newtonian and Lagrangian dynamics in the sense that it is more abstract and ‘mathematical’. One of the advantages is that it seeks to describe the motion of a system in phase space with $2n$ independent coordinates. Thus, the number of independent variables is larger than for Lagrangian dynamics, where the number is $n$.

The $2n$ variables are chosen to be $n$ configuration coordinates $q_i$ and $n$ conjugate momenta $p_i$. These variables are conjugate coordinates. The coordinates $q_i$ are familiar from the Lagrangian formulation, whereas the momenta are not directly used in this formulation. The momenta are defined as

$$p_i = \frac{\partial L}{\partial \dot{q}_i}. \quad (47)$$

The Hamiltonian, which is the core of this formalism, is defined as

$$H(q, p, t) = \dot{q}_i p_i - L(q, \dot{q}, t), \quad (48)$$

where $L$ is the Lagrangian. Now, the equations of motion can be directly derived from the Hamiltonian

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad (49a)$$
$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \quad (49b)$$

As a simple, though insightful example, the concepts treated here are used in the Kepler problem. The Lagrangian for this problem, in the plane of motion, reads

$$L(r, \phi, \dot{r}, \dot{\phi}) = \frac{1}{2} \left( \dot{r}^2 + r^2 \dot{\phi}^2 \right) - \Phi(r), \quad (50)$$

where $\Phi(r) = -\frac{G M m}{r}$. The Hamiltonian now reads

$$H = p_i \dot{q}_i - L = p_r \dot{r} + p_\phi \dot{\phi} - \frac{1}{2} \left( \dot{r}^2 + r^2 \dot{\phi}^2 \right) + \Phi(r)$$

$$= \frac{1}{m} \left( \frac{p_r^2}{r^2} + \frac{p_\phi^2}{r^2} \right) + \Phi(r), \quad (51)$$

note that the indices are spherical coordinates in the plane of motion, which is a bit cheating since one would in a general case not know that the $\theta$ coordinate can be neglected. The full 3D Hamiltonian reads a bit different:

$$H = \frac{1}{m} \left( \frac{p_r^2}{r^2} + \frac{p_\theta^2}{r^2 \sin^2 \theta} + \frac{p_\phi^2}{r^2} \right) + \Phi(r), \quad (52)$$

In the case like for the Kepler problem, when the Hamiltonian is time-independent, it is called a restricted Hamiltonian. A cyclic coordinate is one that does not explicitly appear in the Lagrangian, and thus also not in the Hamiltonian. Just like in Lagrangian dynamics, the conjugate momentum to a cyclic coordinate is conserved. In turn, the Hamiltonian now has one less dimension. In the example of the Kepler problem, the $\phi$ coordinate is a cyclic coordinate: only its momentum appears in the Hamiltonian.
A.2 Coordinate Transformations

In Goldstein §9.1 we read that the transformation from one set of canonical coordinates \((q, p)\) to another \((Q, P)\) satisfies

\[
p_i q_i - H = P_i Q_i - K + \frac{dF}{dt},
\]

where \(K\) is the new Hamiltonian, and \(F\) is the generating function of the transformation. Often, it is possible to pick a generating function such that eq.(53) reduces to

\[
K = H(q, p, t) + \frac{\partial F'}{\partial t}.
\]

This requires the relations

\[
p_i = \frac{\partial F'}{\partial q_i}, \quad P_i = \frac{\partial F'}{\partial Q_i},
\]

(55a)

(55b)

to hold. In Goldstein Chapter 10 we read that, for the further derivations, it is convenient to choose a generator function that is dependent on the old coordinates and new momenta: \(F' = F'(q_i, P_i, t)\). Note that we here put constraints on the selection of all the new momenta, they are no longer variables. If we ensure that the new Hamiltonian \(K\) equals zero, by setting the time derivatives of the new set of coordinates to zero, eq.(53) reduces even more

\[
H\left(q_1, \ldots, q_n; \frac{\partial F'}{\partial q_1}, \ldots, \frac{\partial F'}{\partial q_n}; t\right) + \frac{\partial F'}{\partial t} = 0.
\]

(56)

This equation is known as the Hamilton-Jacobi equation, the generating function, and solution to this equation, is often denoted by \(S\) and is called Hamilton’s principal function. In the case that the Hamiltonian in this equations is not explicitly time dependent, Hamilton’s special function reduces can be written in the form

\[
S(q, \alpha, t) = W(q, \alpha) - \alpha t,
\]

(57)

where \(\alpha\) are the new momenta \(P_i\). For the characteristic function \(W\) holds a similar relation as for the generator function

\[
p_i = \frac{\partial W}{\partial q_i},
\]

(58)

Whenever the old Hamiltonian is restricted, we can write it as a function of the characteristic function and reduce the Hamilton-Jacobi equation to

\[
H\left(q_i, \frac{\partial W}{\partial q_i}\right) = \alpha_1,
\]

(59)

where \(\alpha_1\) is a constant of integration, usually this is the energy of the system.

A.3 Actions for a Kepler potential

Now we will derive the action variables for the Kepler problem. In this problem, there is a central potential

\[
V(r) = -\frac{k}{r}.
\]

(60)

To find the actions, we first need to find Hamilton’s characteristic function \(W\), which is derived from the Hamiltonian. For this potential, we can write the Hamiltonian (eq.(52)), setting the Hamiltonian constant equal to the total energy \(E\),

\[
H = \frac{1}{2m} \left( p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2 \theta} \right) + V(r) = E.
\]

(61)

This Hamiltonian is completely separable, combining it with eq.(58) gives us

\[
\left( \frac{\partial W_r}{\partial r} \right)^2 + \frac{1}{r^2} \left[ \left( \frac{\partial W_\theta}{\partial \theta} \right)^2 + \frac{1}{\sin^2 \theta} \left( \frac{\partial W_\phi}{\partial \phi} \right)^2 \right] + 2mV(r) = 2mE.
\]

(62)
However, since $\phi$ is a cyclic coordinate, we can write

$$W_\phi = \alpha_\phi \phi,$$

(63)

where $\alpha_\phi$ is some constant of integration. Now we can contracting all terms depending on $\theta$ in one function

$$f(\theta) = \left( \frac{\partial W_\theta}{\partial \theta} \right)^2 + \frac{1}{\sin^2 \theta} \left( \frac{\partial W_\phi}{\partial \phi} \right)^2.$$

(64)

In Goldstein p.447 it is shown that this function has to be a constant $f(\theta) = \alpha_\theta^2$. We can rewrite this to obtain

$$\frac{\partial W_\theta}{\partial \theta} = \sqrt{\alpha_\theta^2 - \frac{\alpha_\phi^2}{\sin^2 (\theta)}}.$$

(65)

Substituting this, the Hamiltonian reduces to

$$\left( \frac{\partial W_r}{\partial r} \right)^2 + \frac{\alpha_\theta^2}{r^2} + 2mV(r) = 2mE,$$

(66)

from which we get the last term that we are after: $\frac{\partial W_r}{\partial r}$

$$\frac{\partial W_r}{\partial r} = \sqrt{2mE - 2mV(r) - \frac{\alpha_\theta^2}{r^2}}.$$

(67)

The actions are defined as

$$J_i = \oint p_i \, dq_i = \oint \frac{\partial W_i}{\partial q_i} \, dq_i.$$

(68)

The action $J_\phi$ is easy to compute:

$$\oint \frac{\partial W_\phi}{\partial \phi} \, d\phi = \oint \alpha_\phi \, d\phi = 2\pi \alpha_\phi.$$

(69)

The action $J_\theta$ is a bit trickier

$$\oint \frac{\partial W_\theta}{\partial \theta} \, d\theta = \oint \sqrt{\alpha_\theta^2 - \frac{\alpha_\phi^2}{\sin^2 (\theta)}} \, d\phi,$$

(70)

which I will not solved here, but it can be solved with some complex substitutions, see G14 p.467-468. The action $J_r$ is the toughest

$$\oint \frac{\partial W_r}{\partial r} \, dr = \oint \sqrt{2mE - 2mV(r) - \frac{\alpha_\theta^2}{r^2}} \, dr,$$

(71)

where $V(r) = -\frac{k}{r}$. This equation can only be solved with a contour integral in the complex plane, see G14 p.469-470.

**B Liouville’s Theorem**

This appendix is mainly based on §7.2.2 BT08 it is meant to briefly explain Liouville’s theorem, I do not wish to prove or explain the matter in a detailed way.

Liouville’s equations are a generalization of the collisionless Boltzmann equation. In these equations, a stellar system is described in 6N-dimensional space, containing the positions ans velocities of N stars. In general, when investigating systems with many stars, it is often more usefull to look at statistical properties, e.g. averages in time and/or space, rather than to look at the behaviour of single stars. So, also in this formalism, we are interested in the average behaviour of the system, thus we investigate the distribution function of the stellar system $f^{(N)}(\mathbf{w}_1, \ldots, \mathbf{w}_N, t)$, where $\mathbf{w}_i = (\mathbf{q}_i, \mathbf{p}_i)$. This distribution function describes the probability of finding a star at a certain point in the 6N-dimensional space, in other words, this function is a probability density function.

The evolution of the distribution function in this 6N space is described by Liouville’s equations, which look much like the collisionless Boltzmann equation. Liouville’s theorem, directly derived from these equations, states that the probability density in the 6N-dimensional space is conserved. Note that this is still in the collisionless regime.
C General behaviour of $\sigma_w(t)$

Let us take a look at the matrix $\sigma_w(t)$ from eq.(23b), which describes the time evolution of the distribution function that describes a stream ($\S$2.5).

$$\sigma_w(t) = \Theta^\dagger \sigma_0 \Theta$$  \hspace{1cm} (72)

If we write $\partial \Omega/\partial t = \Omega^*$

$$\begin{bmatrix} I_3 & 0 \\ \Omega^* & I_3 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I_3 & \Omega^* \\ 0 & I_3 \end{bmatrix}$$

Here we did not yet specify the elements of $\sigma_0$. $A, B, C, D$ are 3x3 block sub matrices containing the positional dispersions $A$, velocity dispersions $D$, the other two contain the cross terms.

$$= \begin{bmatrix} A & B \\ \Omega^* A + C & \Omega^* B + D \end{bmatrix} \begin{bmatrix} I_3 & \Omega^* \\ 0 & I_3 \end{bmatrix}$$

$$= \begin{bmatrix} A & A\Omega^* + B \\ A\Omega^* + C & \Omega^* A\Omega^* + C\Omega^* + \Omega^* B + D \end{bmatrix}$$

Using the fact that $\Omega^* \sim t$, we can focus on the terms that dominate at large times. The terms that dominate in the evolution of the distribution are

$$\begin{bmatrix} A & At \\ At & At^2 \end{bmatrix}$$ \hspace{1cm} (73)

From this, we can see that the sub matrix for the velocities, i.e. the bottom right submatrix in $\sigma_w(t)$ grows in time. Since the elements of the matrix $\sigma_{w,ii} = 1/\sigma_{ii}^2$, this means that the dispersions actually decrease in time.