Meshing with topological guarantees

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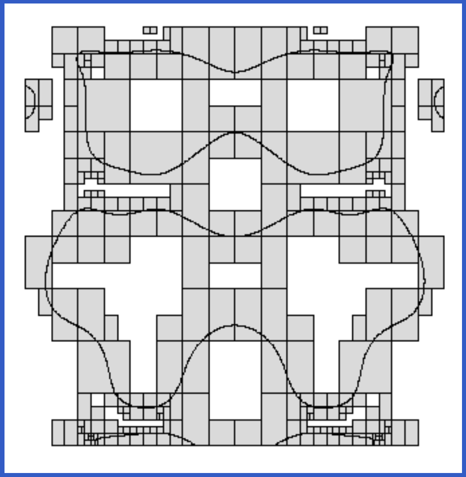
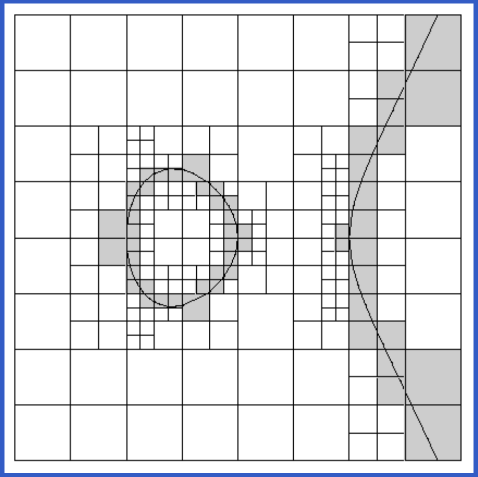
Outline

- Related work
- Implicit curves and surfaces
- Interval arithmetic
- Curve approximation
- Surface meshing
- Level sets
- Results

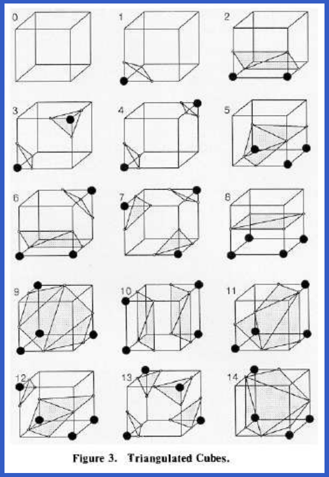
Related work

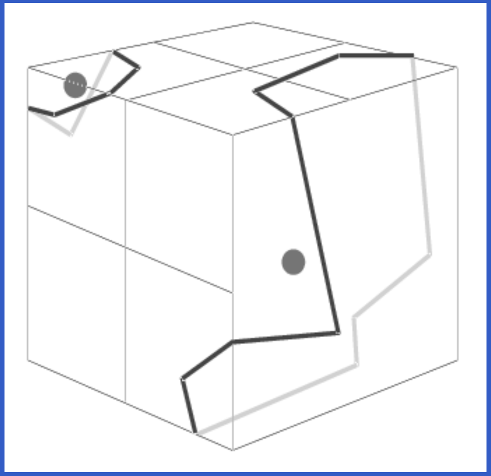




- J.M. Snyder: *Interval analysis for computer graphics* (1992)
- H. Lopes, L.B. Oliveira and L.H. de Figueiredo: *Robust adaptive polygonal approximation of implicit curves* (2002)

Related work





- W.E. Lorensen and H.E. Cline: *Marching cubes: a high resolution 3d surface construction algorithm* (1987)
- K. Ashida and N.I. Badler: *Feature preserving manifold mesh from an octree* (2004)

Related work

- J.M. Snyder (1992), Shrinkwrap (1996), Stander and Hart (1997)
- J.D. Boissonat, S. Oudot: *An effective condition for sampling surfaces with guarantees* (2004)
- S. Cheng, T. Dey, E. Ramos, T. Ray: *Sampling and meshing a surface with guaranteed topology and geometry* (2004)
- J.D. Boissonat, D. Cohen-Steiner and G. Vegter: *Isotopic Implicit Surface Meshing* (2004)

Implicit surfaces

Zero-set of smooth function $F\colon \mathbb{R}^3 \to \mathbb{R}$, where $\nabla F \neq 0$ on $F^{-1}(0)$.

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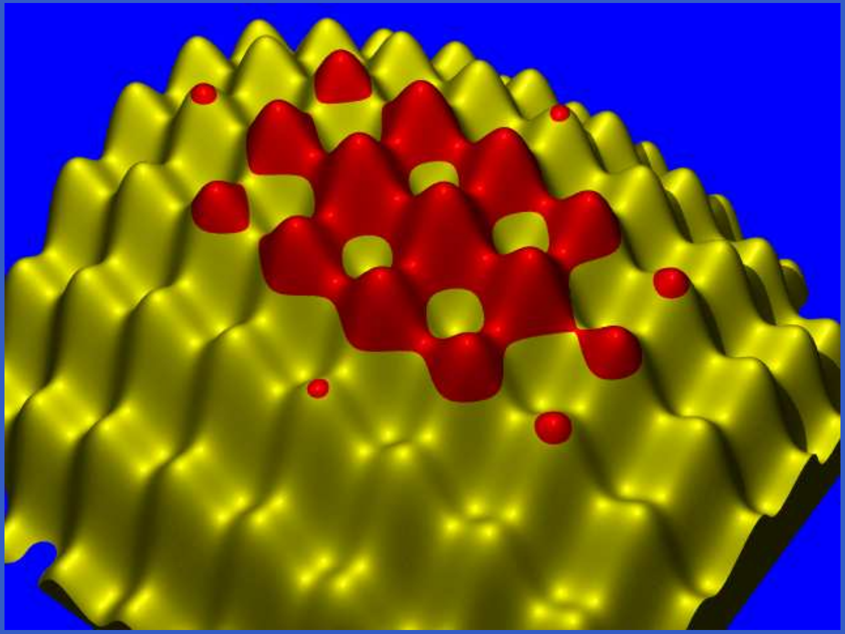
- Advantages: smooth surface, fast ray-tracing, deformable, easy blending and metamorphosis.
- Applications: biology (cell modeling), animation (metaballs), molecular modeling.

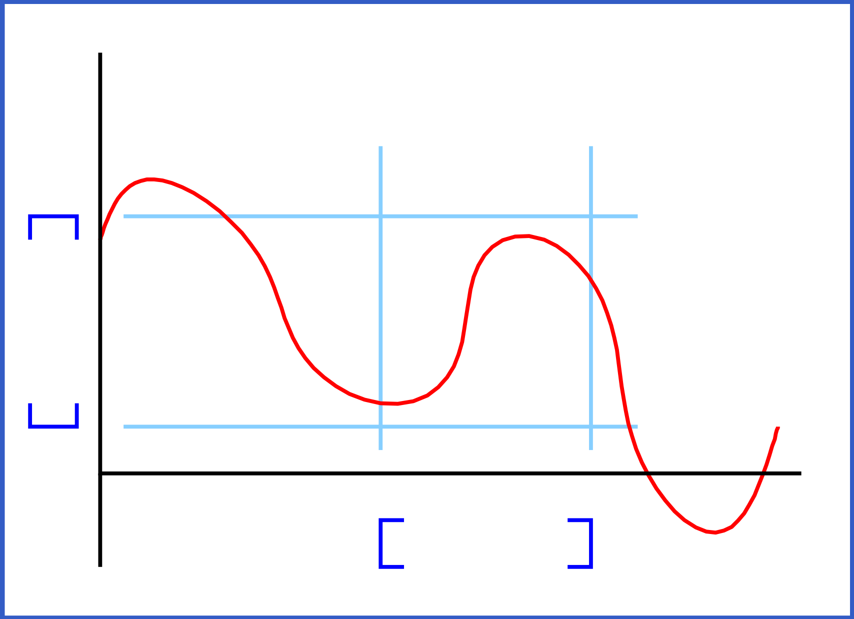
Implicit curves

Zero-set of smooth function $F\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$



Implicit curve is the ‘coastline’ of a height function.

Interval example



Interval arithmetic

- Function $f: \mathbb{R} \to \mathbb{R}$
  Inclusion function $\Box f$ with
  $\Box f([x_1, x_2]) = [y_1, y_2]$ such that
  $x \in [x_1, x_2] \Rightarrow f(x) \in [y_1, y_2]$.

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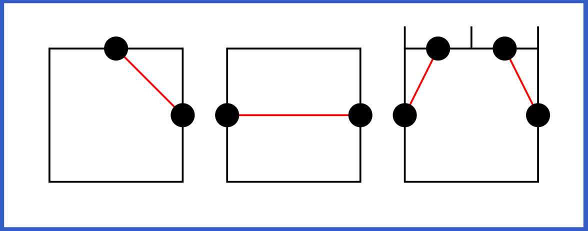
- Example: $f(x) = x^2$
  $\Box f([-1, \pi]) = [0, 9.9]$
- Inclusion function is *convergent* if
  $w(\Box f(I)) \longrightarrow 0$ for $w(I) \longrightarrow 0$

2D-Algorithm

Isotopic curve approximation

- Initialize quadtree to bounding square
- Subdivide until ‘magic’
- Balance the tree
- Create vertices on edges with sign-change
- Connect vertices locally:



Magic formula

The interval condition:

$$0 \notin \Box F(C) \quad \vee \quad 0 \notin \langle \Box \nabla F(C), \Box \nabla F(C) \rangle .$$

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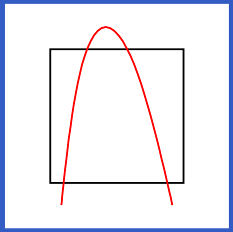
Implies

- Parametrizability
- No undetected loops



Correctness

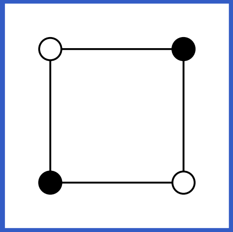
- Regular grid and constraints on function
- Removing the constraints
- Balanced quadtrees:
  refining to a complete quadtree does not change the isotopy

Regular grid

Constraints on the curve C:

- C does not pass through grid vertices
- C intersects edges at most once

Conclusion: at most two intersections for each square.

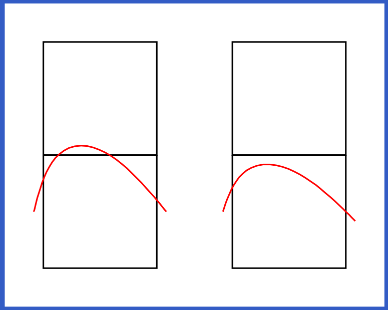


Removing constraints

- If function passes through vertices, shift it towards $F(x, y) + \epsilon = 0$.
  Symbolically: consider $F \geq 0$.

Removing constraints

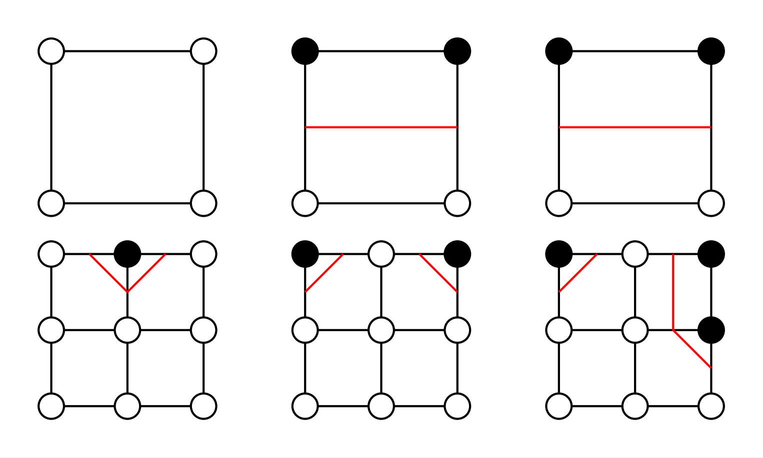
- If function passes through vertices, shift it towards $F(x, y) + \epsilon = 0$.
  Symbolically: consider $F \geq 0$.
- If we have multiple intersections, push them through the edges.



Balanced quadtrees

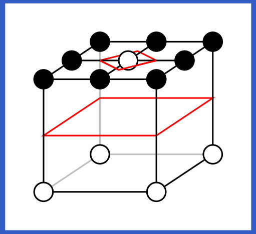
Subdivide the quadtree until it is a complete tree.

Topological changes

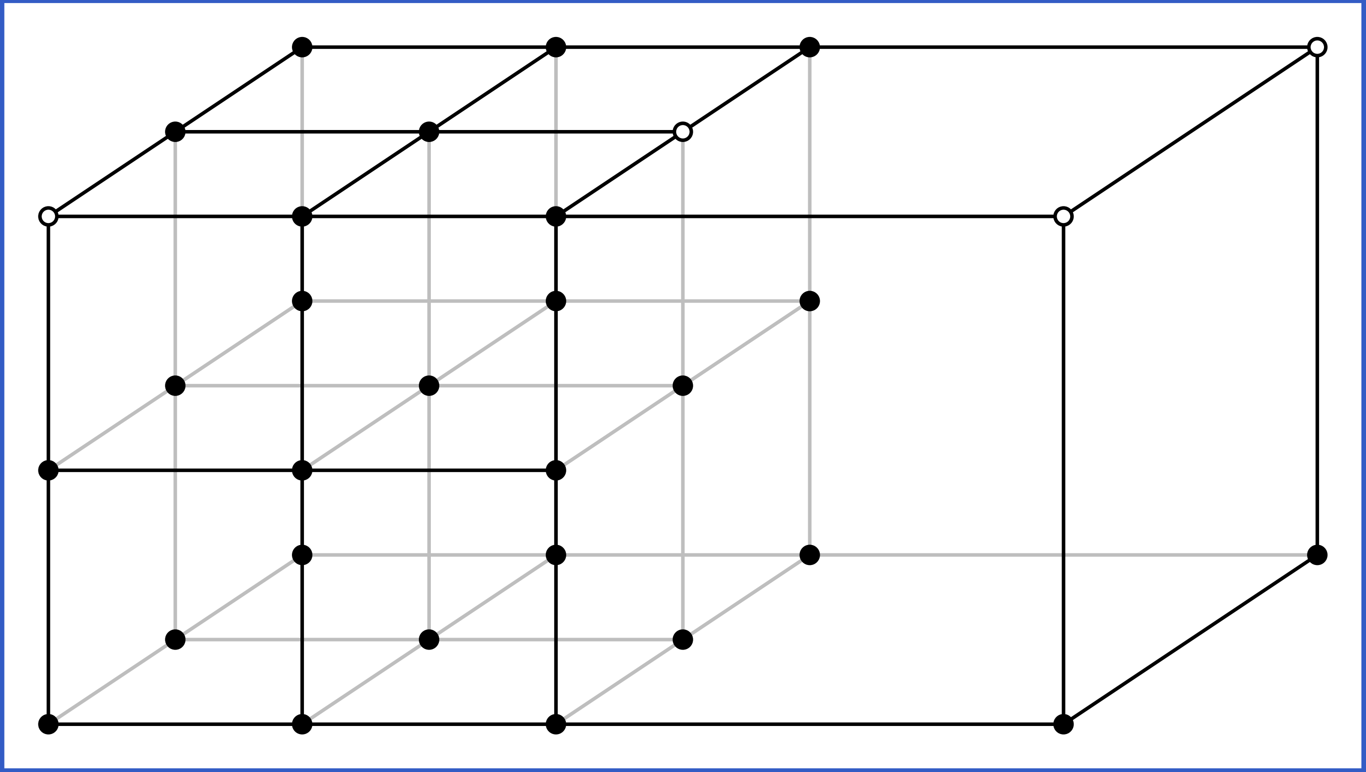


Implicit surfaces

- Initialize octree to bounding square
- Subdivide until 'magic'
- Balance the tree
- Create vertices on edges with sign-change
- Connect vertices on faces of octree
- Connect loops of edges



Meshing



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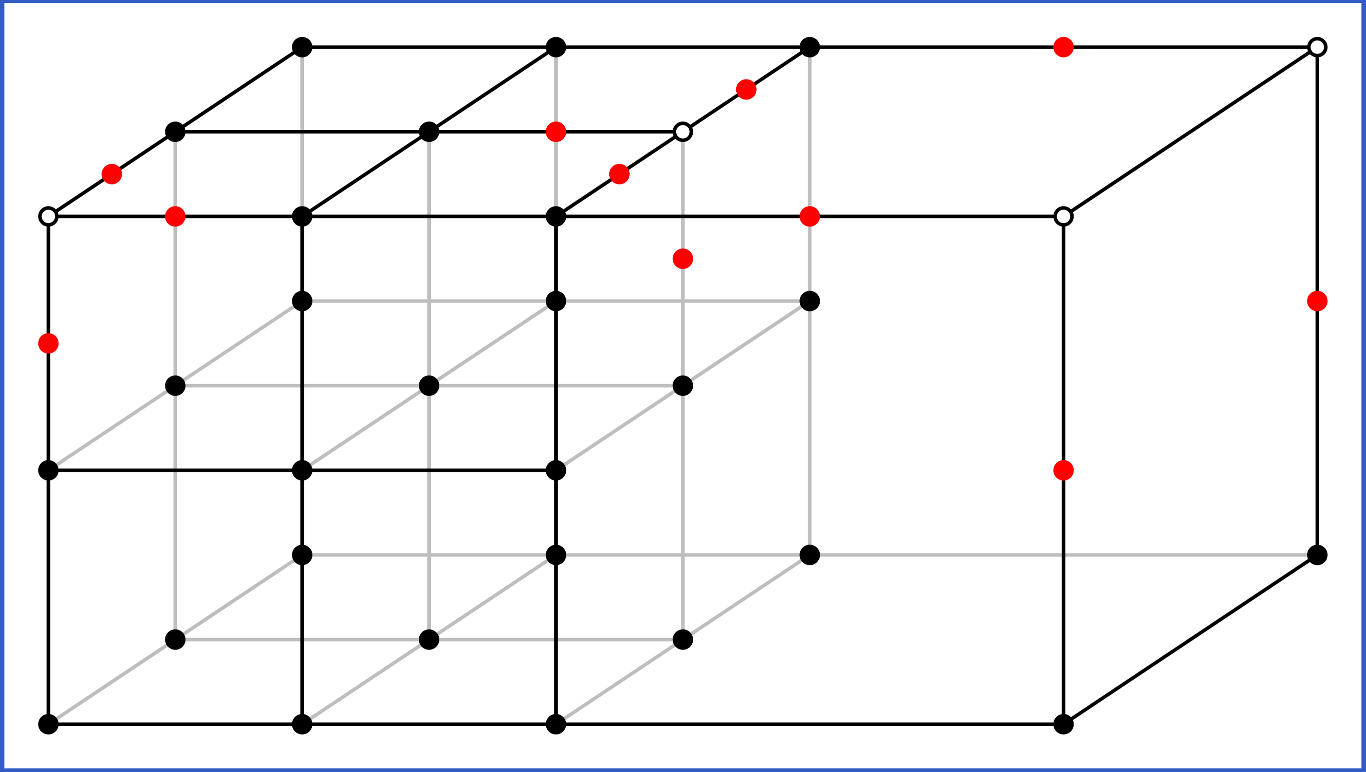

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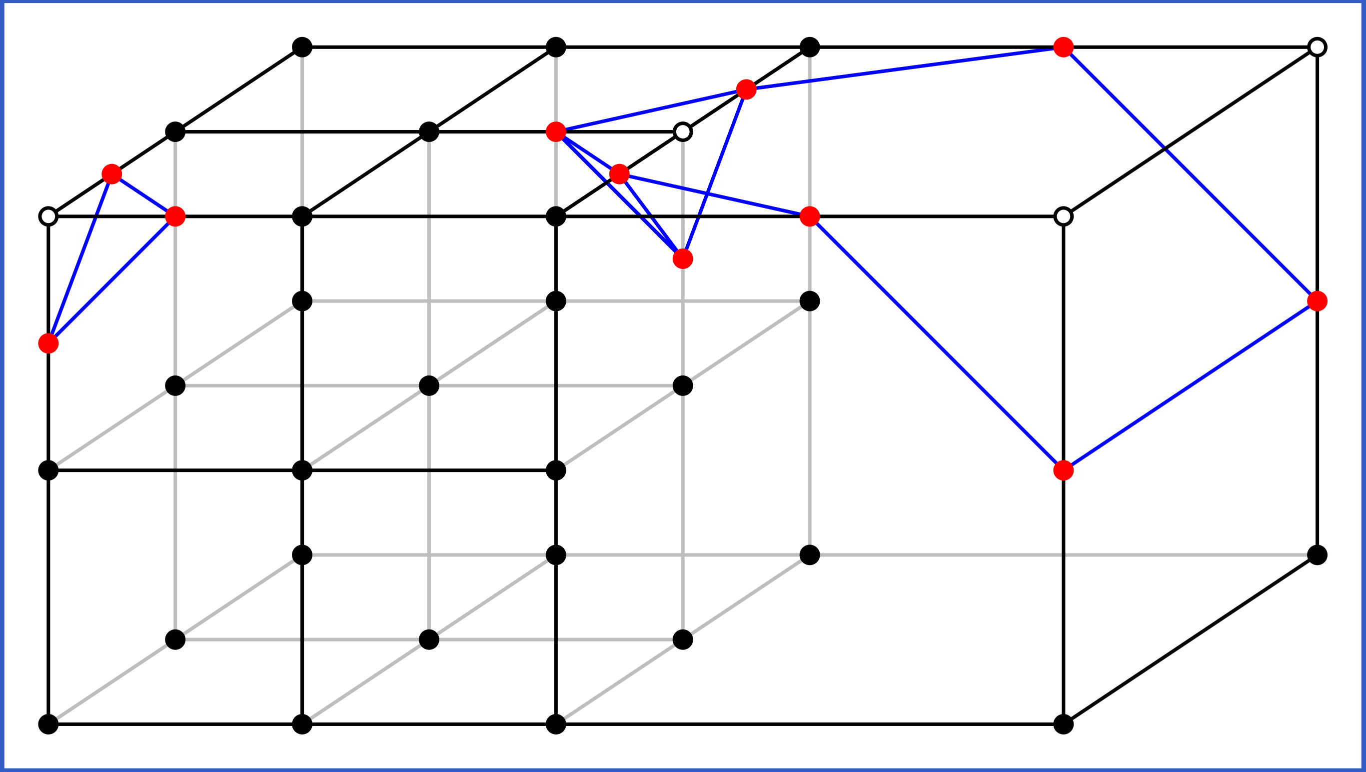

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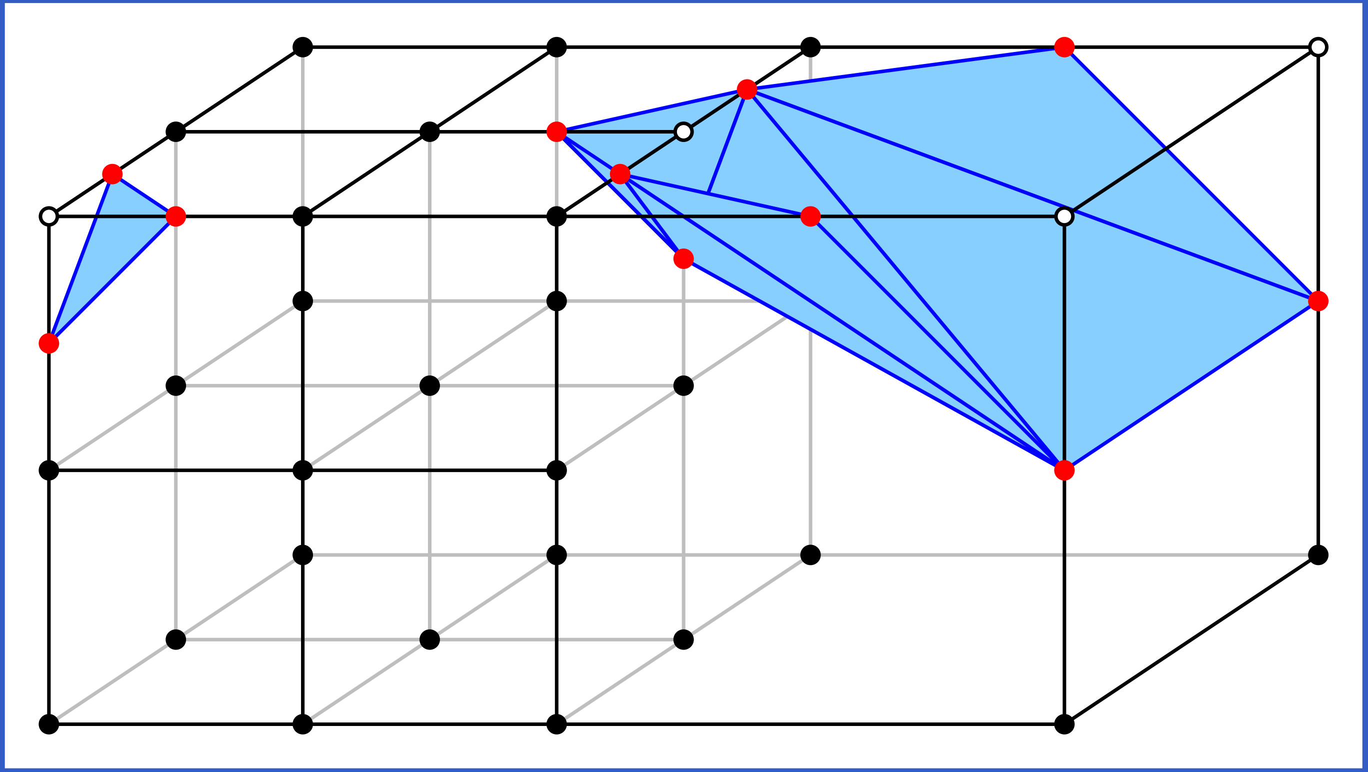
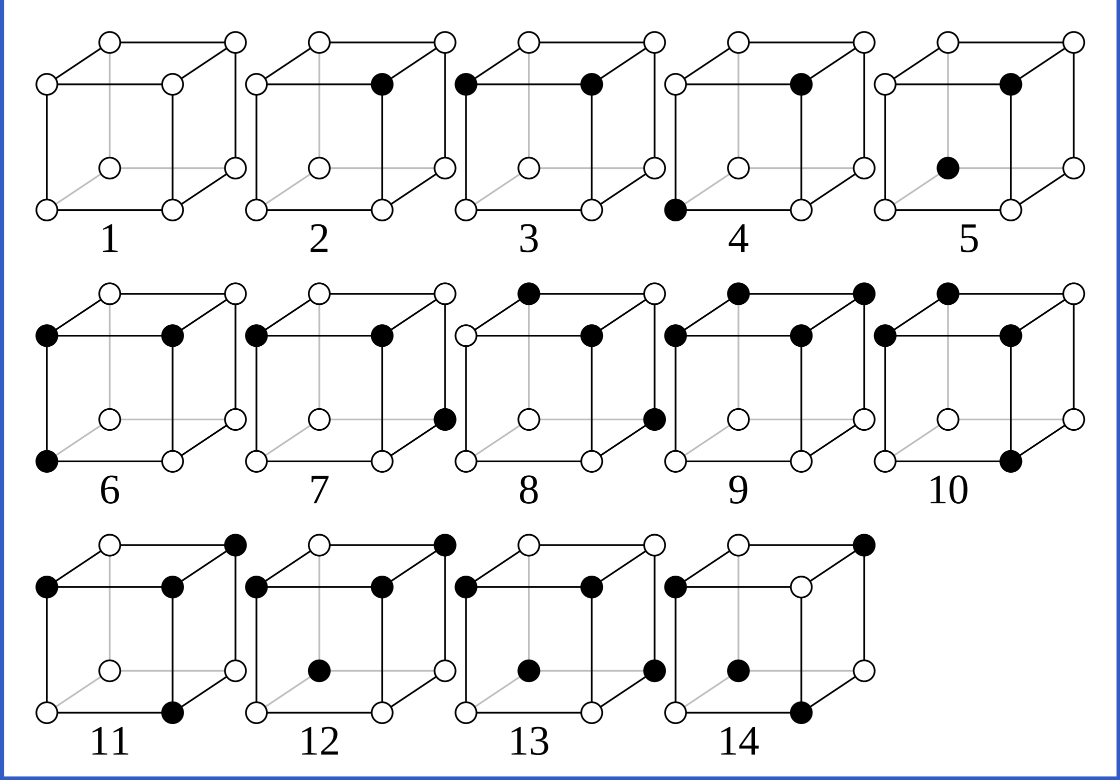
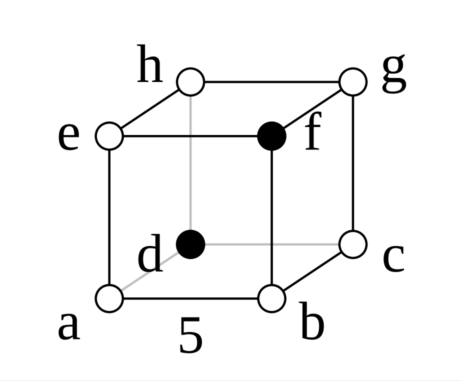


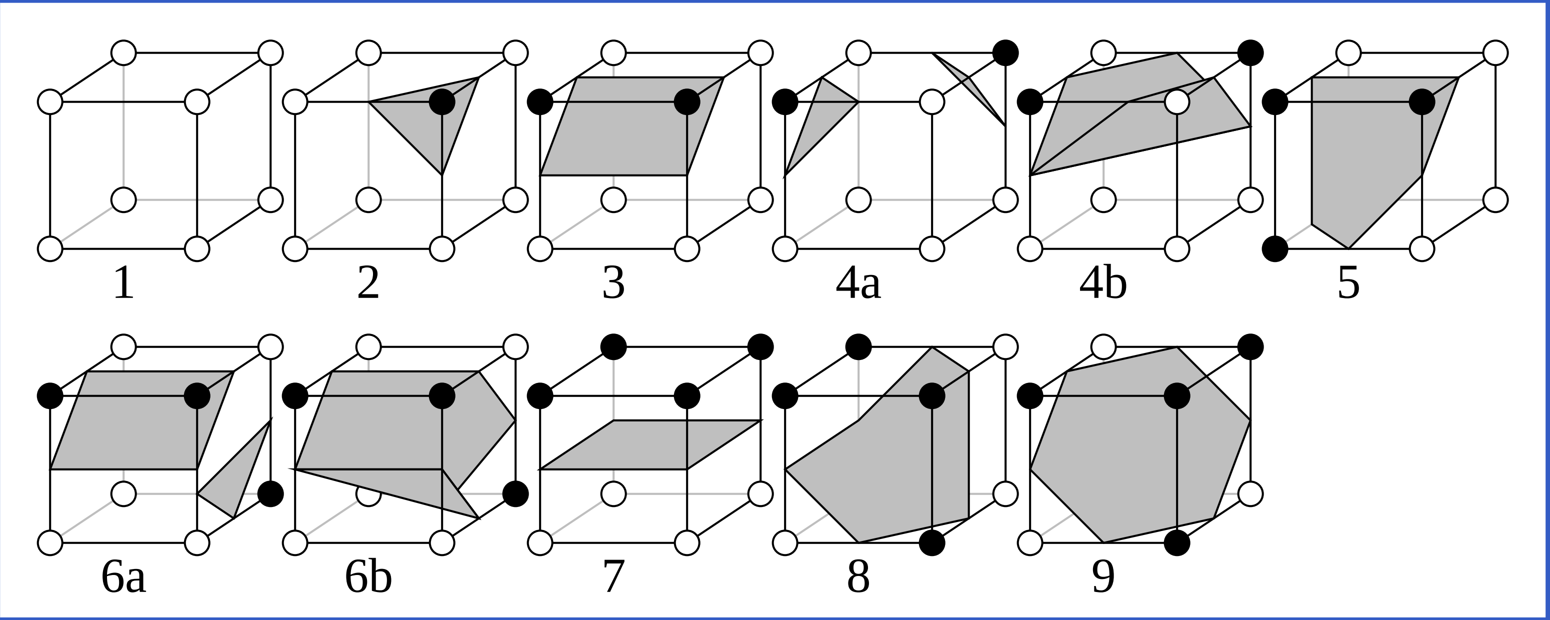
Regular grid



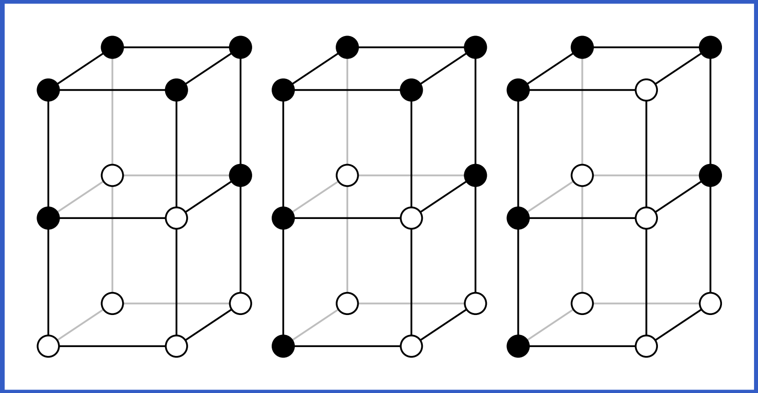
Impossible sign configurations

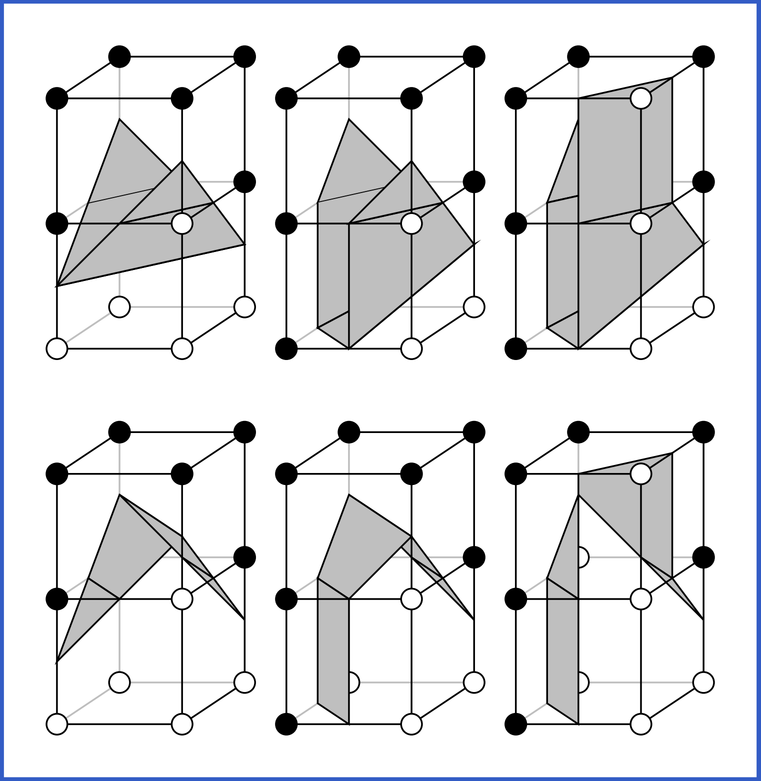


Triangulation of subcubes

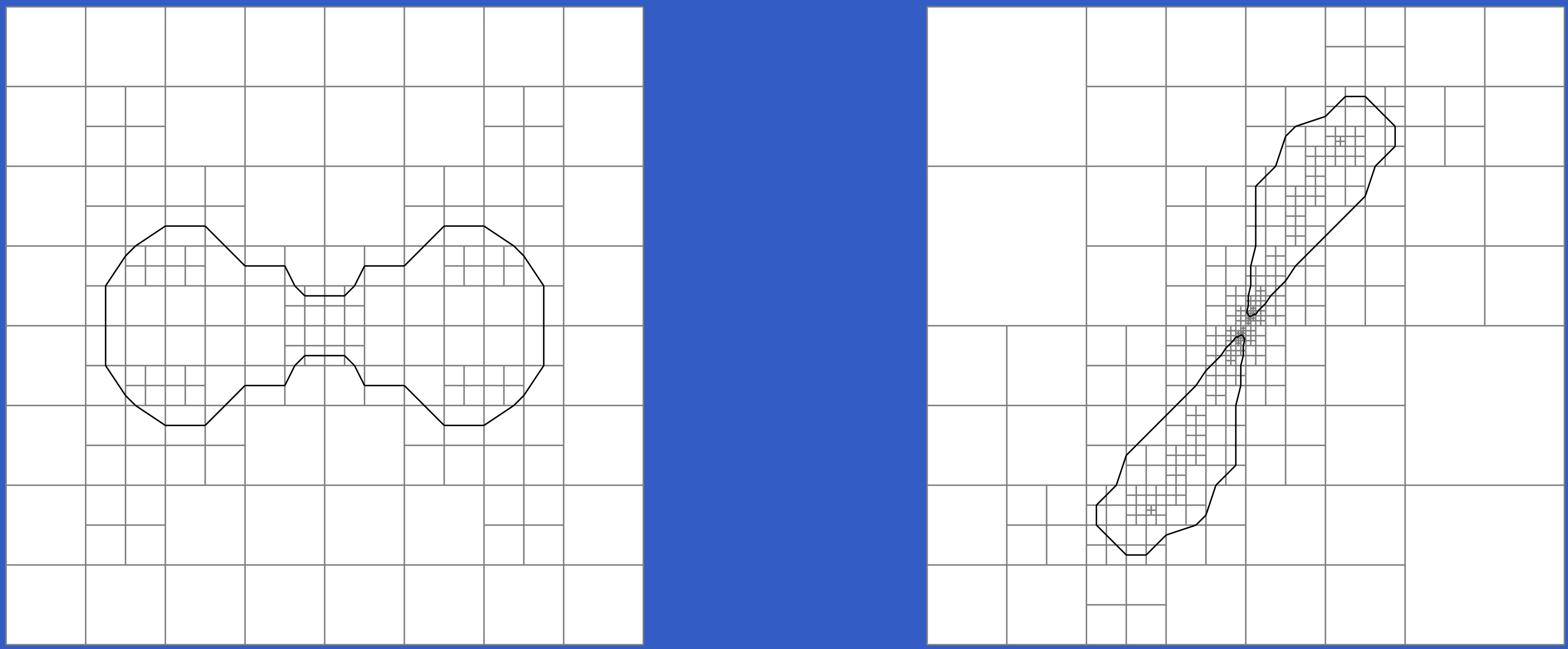


Gluing ambiguous faces



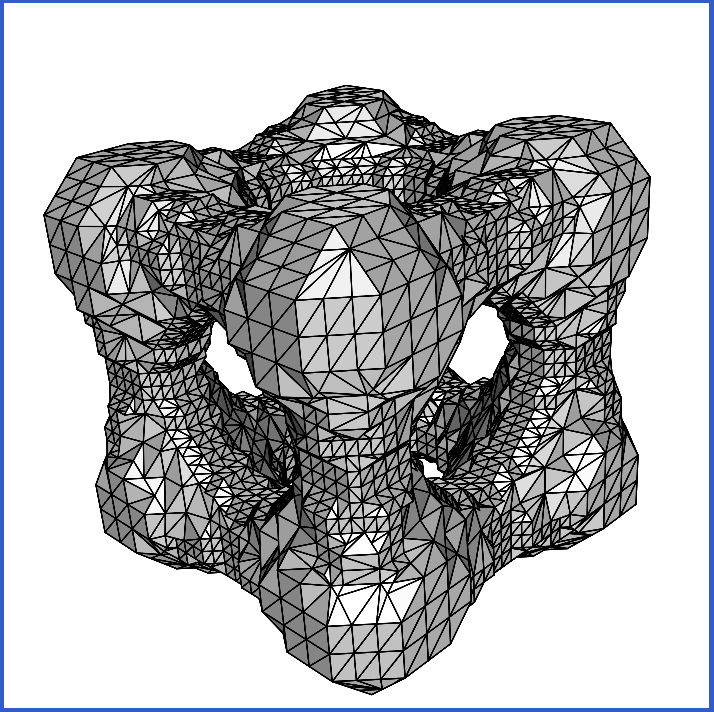


Results



Implicit curves $f(x,y) = x^2(1-x)(1+x) - y^2 + 0.01$ and $f(x,y) = x^2 - xy + y^4 + 0.0001$

Results



Tangle cube:

$$f(x, y, z) = x^4 - 5x^2 + y^4 - 5y^2 + z^4 - 5z^2 + 10$$

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- Precompute $\Box F(C)$ for all leaves of octree.

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- For maximum-level leaves for which gradient variation bound does not hold, test whether $\theta \in \Box F(C)$. If so, mark leave.

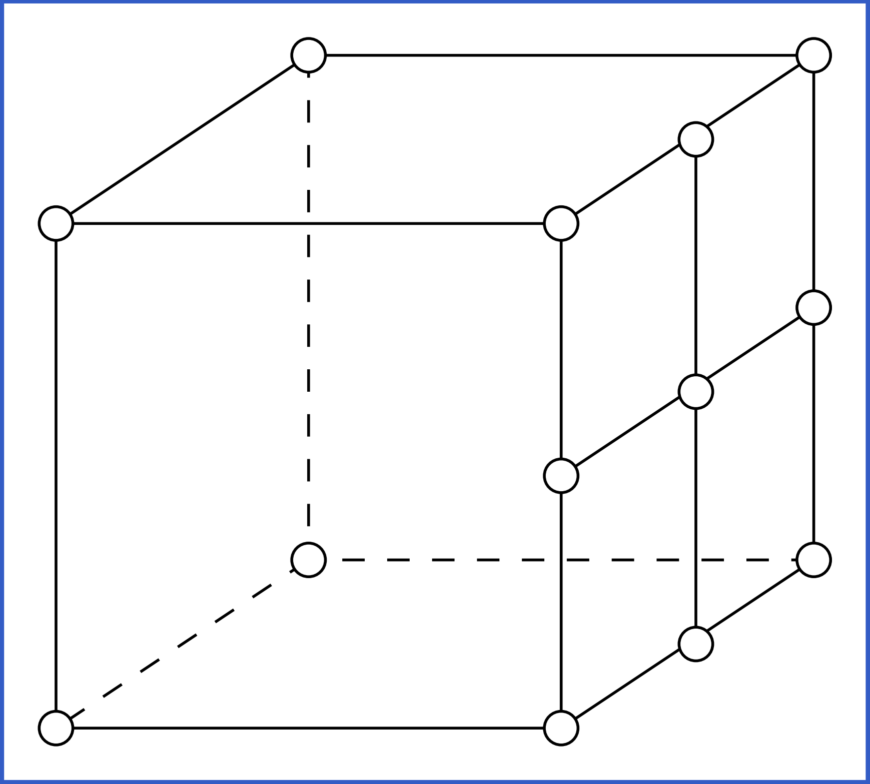
Tetrahedra

Tetrahedrization for a more convenient meshing algorithm.

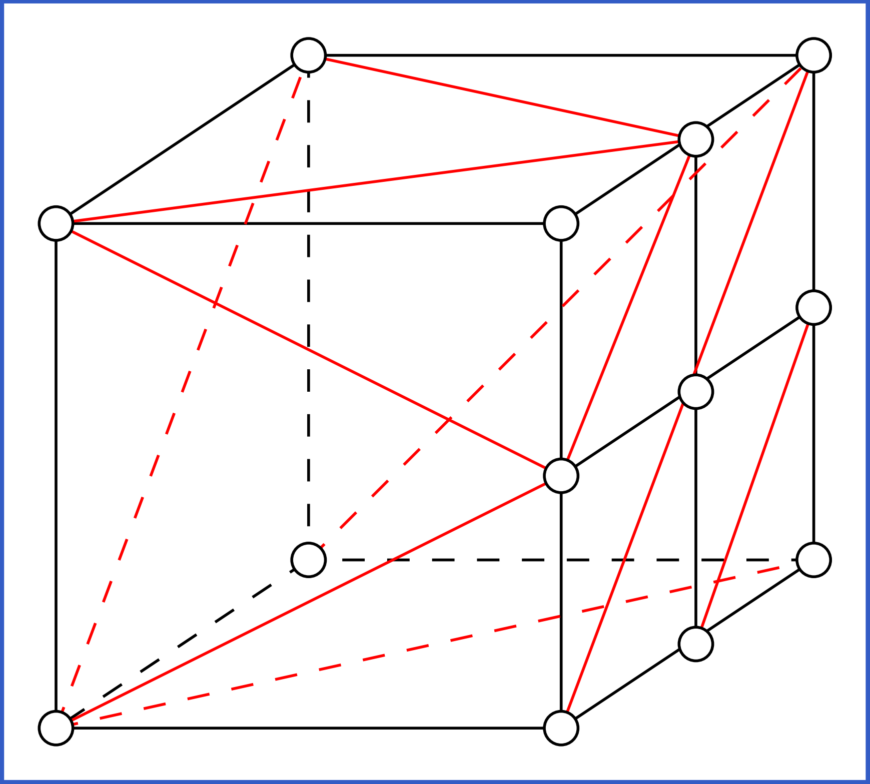
- Triangulate faces of octree such that adjacent cells match.
- Add vertex in cell centre.
- Connect this vertex to the triangles on the cell boundary.
- Mesh tetrahedra (0–2 triangles for each tetrahedron).

Resulting mesh is isotopic to the implicit surface.

Tetrahedrization



Tetrahedrization



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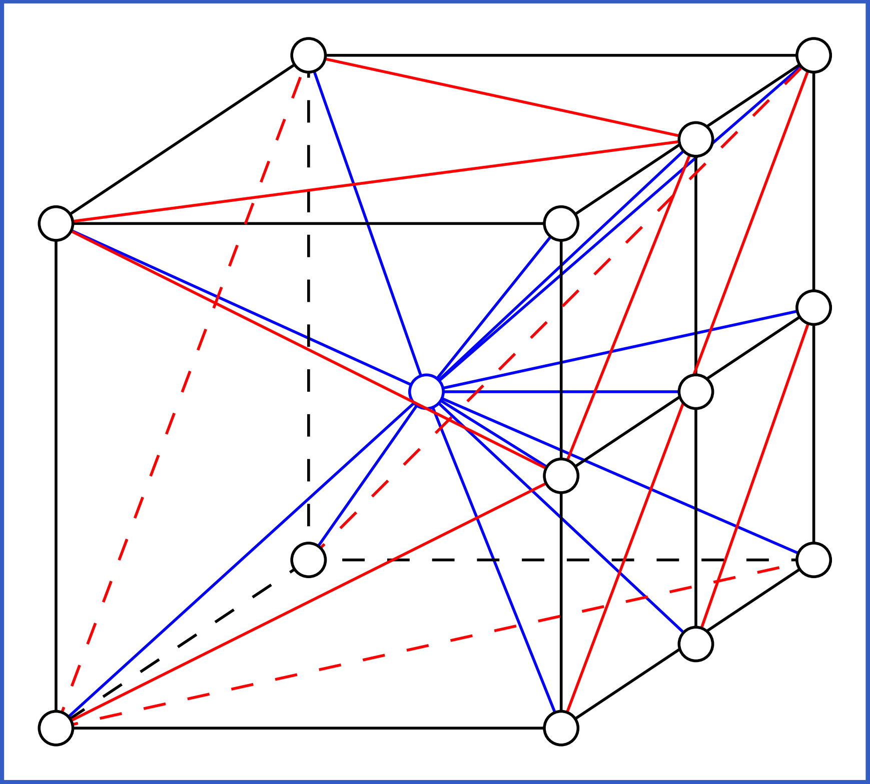
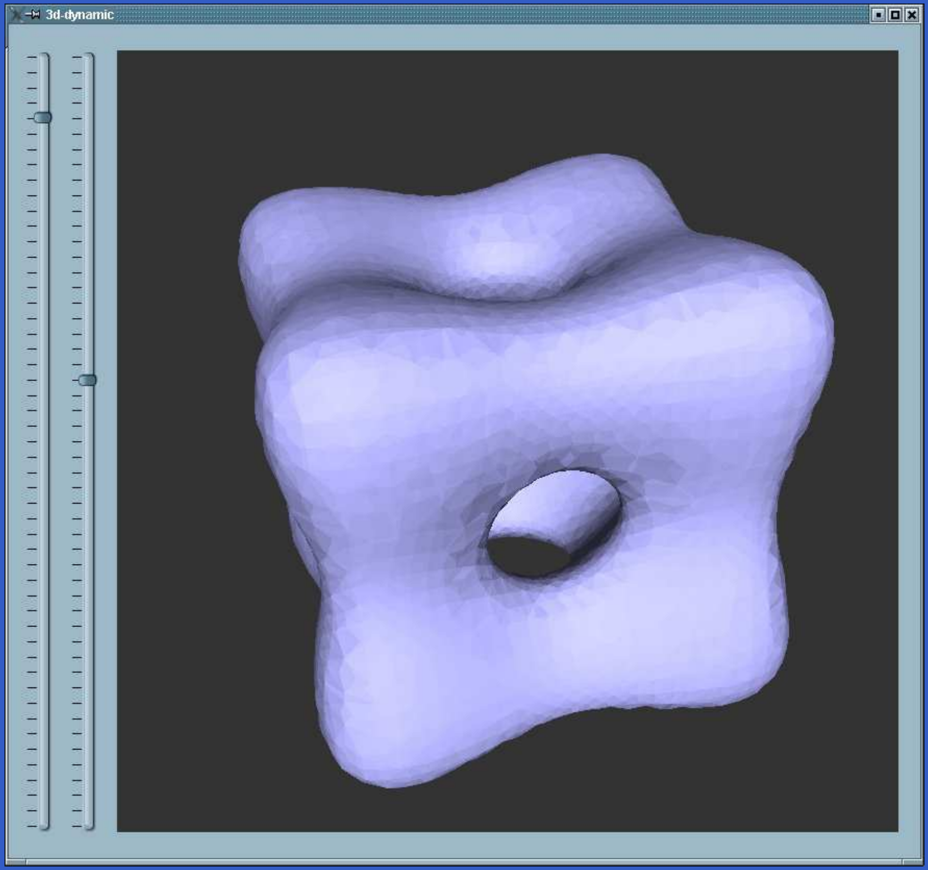


Level sets



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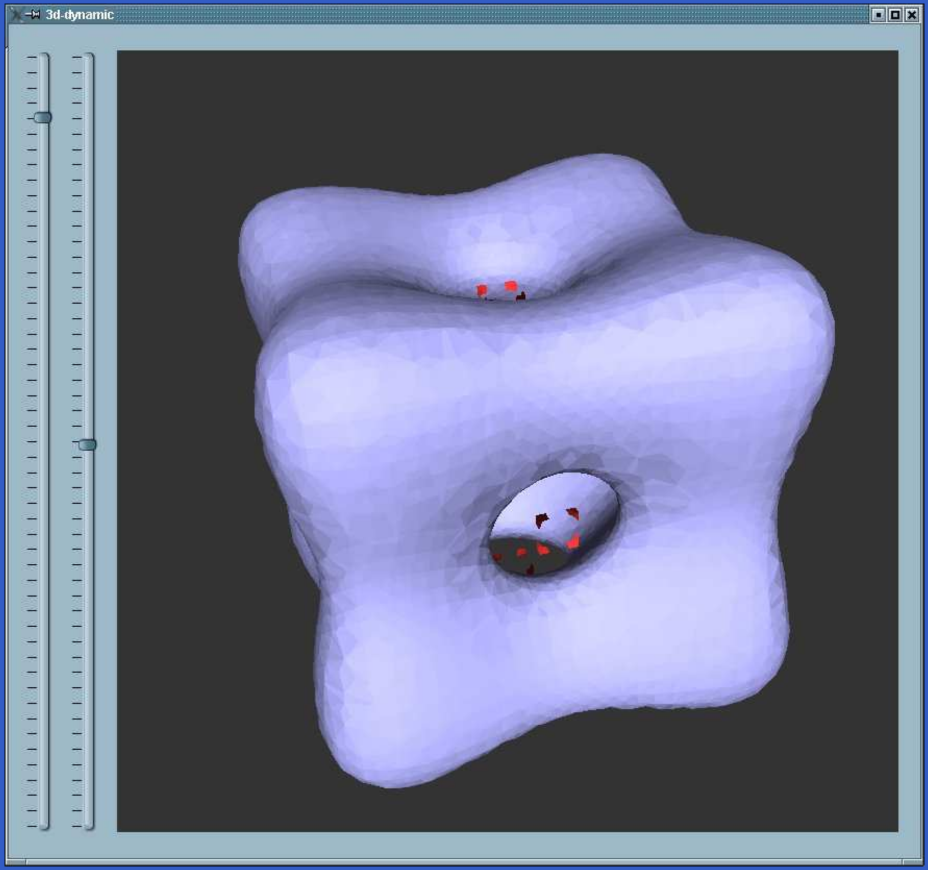

Improvements

1. Maintain list of octree vertices sorted on function value.
2. Perform extra subdivision for more accurate computation of $\Box F(C)$.
3. Trade-off between precomputation (speed) and memory use.

Conclusion

- Fast algorithm to mesh implicit surfaces
- Isotopic approximation (correct topology)
- Can be adapted to handle singularities and unbounded curves and surfaces