DYNAMICS OF GALAXIES

Two-body problem

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Contents

Fundamentals

The fundamental equations

Take two masses m_1 and m_2 with position vectors $\vec{r}_1$ and $\vec{r}_2$.

Then there are two fundamental equations:

$$m_1 \ddot{\vec{r}}_1 = -G m_1 m_2 \frac{\vec{r}_1 - \vec{r}_2}{r^3} \quad (1)$$

$$m_2 \ddot{\vec{r}}_2 = -G m_1 m_2 \frac{\vec{r}_2 - \vec{r}_1}{r^3} \quad (2)$$

Here $\ddot{\vec{r}}$ denotes the second derivative with respect to time $d^2\vec{r}/dt^2$.

Center of gravity

The center of gravity has the position

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \equiv \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{M}$$

Add (1) and (2), then we get

$$m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2 = 0$$

Integrate this twice, then

$$m_1\vec{r}_1 + m_2\vec{r}_2 \equiv M\vec{R} = \vec{a}t + \vec{b} \tag{3}$$

So the center of gravity is in linear motion.

Now go to a co-moving coordinate system, in which the center of gravity is the origin:

$$m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0$$

Then

$$\ddot{\vec{r}}_1 = -G \frac{m_2 \vec{r}_1 - m_2 \vec{r}_2}{r^3} = -GM \frac{\vec{r}_1}{r^3}$$

$$\ddot{\vec{r}}_2 = -G \frac{m_1 \vec{r}_2 - m_1 \vec{r}_1}{r^3} = -GM \frac{\vec{r}_2}{r^3}$$

So for $\vec{r} = \vec{r}_1 - \vec{r}_2$ (the vector between the two bodies)

$$\ddot{\vec{r}} = -GM \frac{\vec{r}}{r^3} \tag{4}$$

We have three differential equations of the second order, so we will have in principle **6** constants of integration.

Angular momentum

Multiply eqn. (4) with $\vec{r}$. Then

$$\vec{r} \times \ddot{\vec{r}} = -GM \frac{\vec{r} \times \vec{r}}{r^3} = 0$$

Integrate this equation with respect to time. This gives

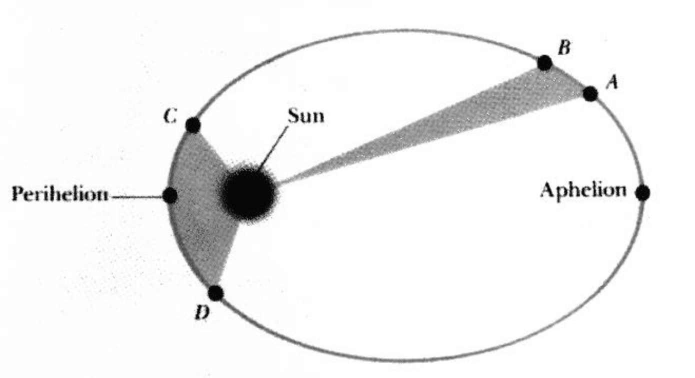
$$\vec{r} \times \dot{\vec{r}} = \mathrm{constant} = \vec{h} \tag{5}$$

This can be verified by differentiating and using $\dot{\vec{r}} \times \dot{\vec{r}} = 0$.

Eqn. (5) tells us that angular momentum is conserved (which is also Kepler's second law) and that the motions of the bodies is restricted to the plane

$$\vec{r} \cdot \vec{h} = 0$$

With $\vec{h}$ we already have identified three of the integration constants.


B
A
C
Sun
Perihelion
Aphelion
D

Solution of the orbit

Equation of motion

For what we will do next we need the following result from vector calculus:

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}) \cdot \vec{b} - (\vec{a} \cdot \vec{b}) \cdot \vec{c}$$

$$\vec{a} \cdot \dot{\vec{a}} = a\dot{a}$$

since $\vec{a} \cdot \vec{a} = a^2$.

Again start with eq. (4) and now multiply with $\vec{h}$.

$$
\begin{aligned}
\vec{h} \times \ddot{\vec{r}} &= -\frac{GM}{r^3}(\vec{h} \times \vec{r}) = -\frac{GM}{r^3}(\vec{r} \times \dot{\vec{r}}) \times \vec{r} \\
&= -\frac{GM}{r^3}\{r^2\dot{\vec{r}} - (\vec{r} \cdot \dot{\vec{r}})\vec{r}\} \\
&= -\frac{GM}{r^3}\{r^2\dot{\vec{r}} - (r\dot{r})\vec{r}\} \\
&= -GM\left(\frac{\dot{\vec{r}}}{r} - \frac{\dot{r}\vec{r}}{r^2}\right) \\
&= -GM\frac{d}{dt}\frac{\vec{r}}{r} = -GM\frac{d\hat{r}}{dt}
\end{aligned}
$$

Integrate this

$$\vec{h} \times \dot{\vec{r}} = -GM\hat{r} - \vec{P} \tag{6}$$

To find further information about $\vec{P}$ multiply with $\vec{h}$

$$\vec{h} \cdot (\vec{h} \times \dot{\vec{r}}) = -GM\hat{r} \cdot \vec{h} - \vec{P} \cdot \vec{h}$$

In this equation the term on the left and the first one on the right are equal to 0, so

$$0 = \vec{P} \cdot \vec{h} \tag{7}$$

So we find then that $\vec{P}$ is perpendicular to $\vec{h}$ and this gives two new constants of integration.

Now go back to eq. (6) and multiply by $\vec{r}$:

$$\begin{aligned}\vec{r}\cdot(\vec{h}\times\dot{\vec{r}}) &= -GM\vec{r}\cdot\hat{r}-\vec{P}\cdot\vec{r}\\ -\vec{h}\cdot(\vec{r}\times\dot{\vec{r}}) &= -GMr-\vec{p}\cdot\vec{r}\\ -\vec{h}\cdot\vec{h}=-h^2 &= -GMr-\vec{P}\cdot\vec{r}\end{aligned}$$

So

$$\frac{h^2}{GMr}=1+\frac{\vec{P}}{GM}\cdot\frac{\vec{r}}{r}=1+\frac{\vec{P}}{GM}\cdot\hat{r}$$

Denote the angle between $\hat{r}$ and $\vec{P}$ by ν (the so-called *true anomaly*), so that

$$\vec{P}\cdot\hat{r}=P\cos\nu$$

Then we get

$$\frac{h^2}{GMr} = 1 + \frac{P}{GM} \cos \nu \tag{8}$$

This is called the *equation of motion*.

Note that this is the general equation for a conic section in polar coordinates for the case that one of the foci is located at the origin:

$$r = \frac{q}{1 + e \cos \nu}$$

From this we see that the excentricity e is

$$e = \frac{P}{GM}$$

For the case of an ellipse ($e < 1$) we have

$$\frac{h^2}{GM} = q = a(1 - e^2)$$

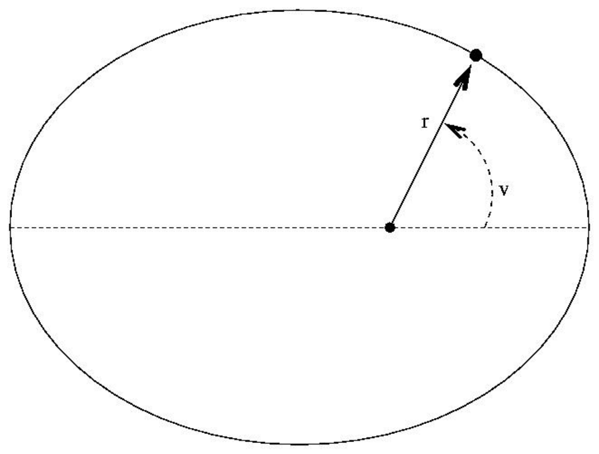
and for a hyperbola ($e > 1$)

$$\frac{h^2}{GM} = q = a(e^2 - 1)$$

A parabola has $e = 1$.

For elliptical planetary orbits the *perihelion* distance (the smallest r) is $a(1 - e)$ and the *aphelion* distance $a(1 + e)$.

From eq. (7) we see that $\nu = 0$ corresponds to the smallest r and that therefore $\vec{P}$ indicates the direction of the perihelion.


r
v

The energy integral

The total kinetic and potential energy (also called the energy integral) must be constant

$$\frac{V^2}{2} - \frac{GM}{r} = C$$

The angular momentum equals the angular velocity $d\nu/dt$ times the square of the radius.

So $h = r^2 d\nu/dt$.

The tangential velocity is

$$r\frac{d\nu}{dt} = \frac{h}{r} = \frac{h}{q}(1 + e\cos\nu) = \frac{GM}{h}(1 + e\cos\nu)$$

The equation for an elliptical orbit was

$$\frac{a}{r}(1-e^2) = \frac{h^2}{GMr} = 1 + e\cos\nu$$

Differentiate this and use $d\nu/dt = h/r^2$, then we find the radial velocity to be

$$\dot{r} = \frac{eGM}{h}\sin\nu$$

From this we find the total velocity to be

$$V^2 = \left(\frac{GM}{h}\right)^2 (1 + e^2 + 2e\cos\nu)$$

Substitute that in the energy integral, then we get

$$C = -\frac{1}{2}\left(\frac{GM}{h}\right)^2 (1-e^2) = -\frac{GM}{2a}$$

With this we can calculate the velocity in the orbit at any position:

$$V^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right) \tag{9}$$

For a parabolic orbit we have $C = 0$ and for a hyperbolic one $C = GM/2a$.

For a circular orbit

$$V_{\rm circ}^2 = \frac{GM}{r}$$

At each radius r the velocity of escape is that in a parabolic orbit through that position, so

$$V_{\rm escape}^2 = 2\frac{GM}{r}$$

Anomalies and Kepler's equation

Anomalies and Kepler's third law

The position in an orbit follows from the value of the true anomaly ν from

$$r = \frac{a(1-e^2)}{1+e\cos\nu} \tag{10}$$

Now define the *excentric anomaly* E as

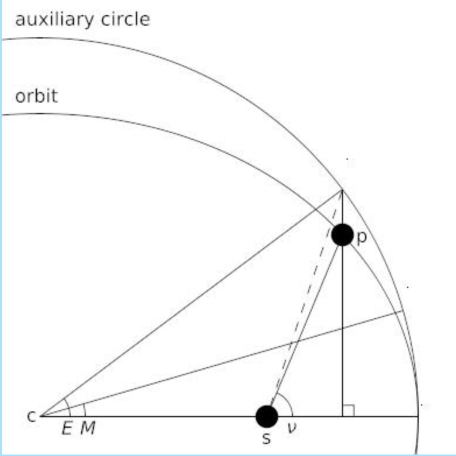
$$r = a(1-e\cos E) \tag{11}$$

The anomalies in the orbit:

true anomaly ν,

excentric anomaly E,

mean anomaly M.



Denoting the semi-major axis as a and the semi-minor axis as $b = a\sqrt{1-e^2}$.

From the figure it can be seen that

$$\begin{aligned} r^2 &= PD^2 + DF^2 = \left(\frac{b}{a}DQ\right)^2 + (CF - CD)^2 \\ &= \left(\frac{b}{a}a\sin E\right)^2 + (ae - a\cos E)^2 \\ &= a^2(1 - e\cos E)^2 \end{aligned}$$

Furthermore with eqn.'s (10) and (11) it can be shown that

$$\tan\left(\frac{\nu}{2}\right) = \sqrt{\frac{1+e}{1-e}}\tan\left(\frac{E}{2}\right) \tag{12}$$

E has been defined such that $dE/dt \geq 0$, since E is an angle measured from the center of the orbit and follows the projection of the planet as it goes along the circumcircle in the same direction as ν.

We had

$$\vec{r} \times \dot{\vec{r}} = \vec{h}$$

Take the square of that and substitute the energy integral:

$$GM\left(\frac{2}{r} - \frac{1}{a}\right) r^2 - r^2 \dot{r}^2 = GMa(1 - e^2)$$

Differentiate eq. (11)

$$\dot{r} = ae \frac{dE}{dt} \sin E$$

and substitute this

$$dt = \sqrt{\frac{a^3}{GM}}(1 - e\cos E)dE$$

Integrate this over the full orbit

$$\int_0^T dt = \int_0^{2\pi} \sqrt{\frac{a^3}{GM}}(1 - e\cos E)dE$$

So

$$T = 2\pi\sqrt{\frac{a^3}{GM}}$$

This is Kepler's third or harmonic law, which can also be written as

$$\frac{T^2}{a^3} = \frac{4\pi^2}{G(m_1 + m_2)} \tag{13}$$

Kepler's equation

E does not increase linearly with time, just as is the case for ν.

Therefore we use the *mean anomaly* M, which by definition increases linearly with time.

$$M = \frac{2\pi}{T}(t - t_\circ) = \sqrt{\frac{GM}{a^3}}(t - t_\circ) = n(t - t_\circ) \qquad (14)$$

Here n is the mean motion and $t_\circ$ the time of perihelion passage. Then

$$dt = \frac{1}{n}(1 - e\cos E)dE$$

Integrate this

$$n(t - t_\circ) = M = E - e \sin E \tag{15}$$

This is called *Kepler's equation*.

It shows how Kepler constructed and described planetary orbits.

The problem remains how to find for a given value of M what value E follows from this equation.

Kepler used tabular listings of the two for various values of the excentricity e.

Orbital parameters

We have identified in the above 6 integration constants:

• Three constants from the angular momentum vector $\vec{h}$.
This defines the plane of the orbit and the magnitude of the total angular momentum (and through this the size of the orbit).

• Two constants from the vector $\vec{P}$.
Its direction defines that of the perihelion and its magnitude de excentricity of the orbit.

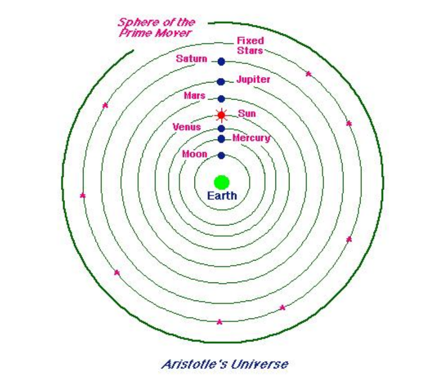
• The sixth constant is the time of perihelion passage $t_\circ$.

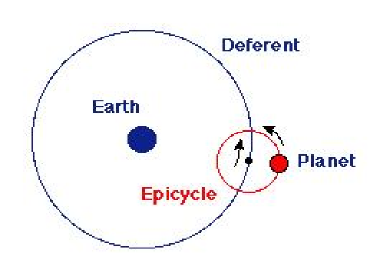
Geocentric versus heliocentric models

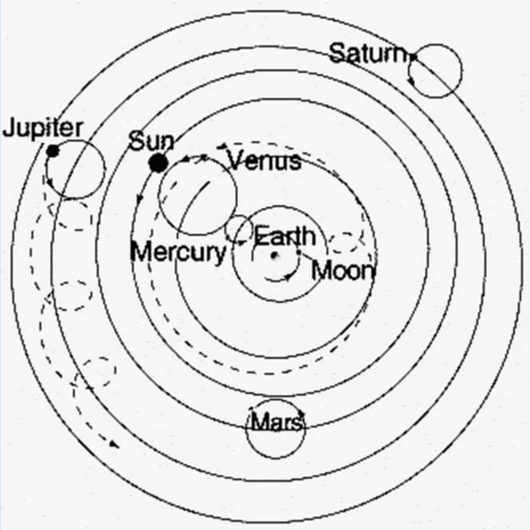
The geocentric model

- The philosophies of **Plato** (417 – 348 BC) and **Aristotle** (384 – 322 BC) separated the world in the imperfect one here on Earth (the 'Sublunary') and from the Moon onward the perfect spheres of the Sun and planets and the stellar sky.
- Consequently the motions of the Sun and planets had to be 'perfect', i.e. exactly on circles and with uniform angular velocity.

- **Claudius Ptolemy** ($\pm$100 – 170 AD) completed the geocentric model that was used to describe the motions of the planets.
- It was assumed that the (outer) planets moved on epicycles of which the center moved on a deferent.
- Obviously the epicyle is a reflection of the Earth's orbit and the deferent that of the planet's orbit.
- For inner planets this is the other way around.

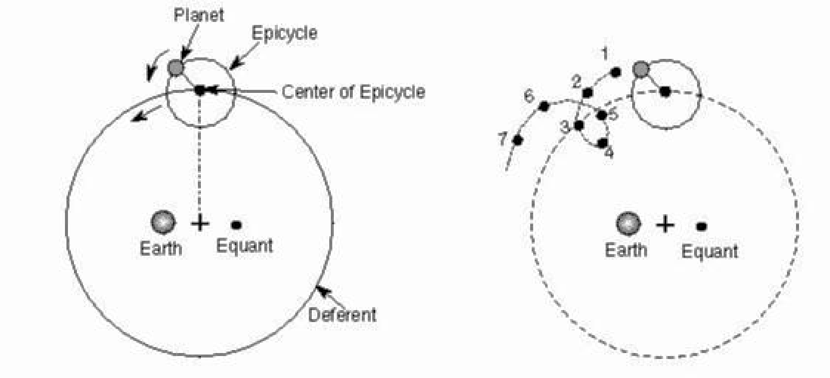

Sphere of the Prime Mover
Fixed Stars
Saturn
Jupiter
Mars
Sun
Venus
Mercury
Moon
Earth
Aristotle's Universe


Deferent
Earth
Planet
Epicycle


Saturn
Jupiter
Sun
Venus
Earth
Mercury
Moon
Mars

- In reality planets move with varying angular velocity in their orbits, so the model failed to predict position well.
- Ptolemy therefore assumed that the angular motion in the deferent was not uniform with respect to the center, but with respect to the equant.
- The equant is opposite and equidistant from the center with respect to the Earth.
- It is surprising that this works so well.
- This can actually be explained.[1]

[1] See Deeming et al., Observatory 97, 84 (1977).


Planet
Epicycle
Center of Epicycle
Earth
Equant
Deferent
1
2
3
4
5
6
7
Earth
Equant

- The angular velocity of the planet as seen from the Sun is

$$\frac{d\nu}{dt} \propto (1-e^2)^{1/2}(1-e\cos E)^{-2}$$

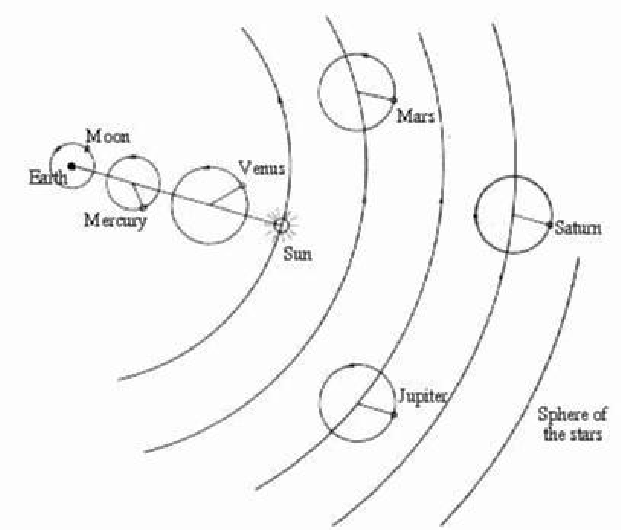
- This varies proportionally over a range from $(1-e)^{-2}$ at perihelion to $(1+e)^{-2}$ at aphelium.
- For the empty focus the appropriate formula is

$$\frac{d\nu'}{dt} \propto (1-e^2)^{1/2}(1-e^2\cos^2 E)^{-1}$$

- This variation is over a relative range from $(1-e^2)^{-1}$ at perihelion <u>and</u> aphelion to 1 for $E=90^\circ$ or $E=270^\circ$.

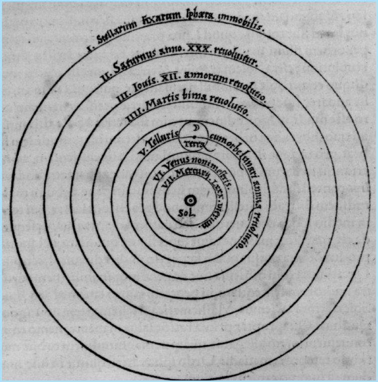
- For an excentricity of the Earth's orbit ($e = 0.017$) the variation is from -3.4% to +3.4% around the mean from the Sun, but only between -0.014% to +0.014% from the empty focus.
- For Mars with an orbital excentricity $e = 0.093$, the difference is substantial and remarkable. The range is from -17.7% to +19.5 % from the Sun, but only between -0.4% and +0.4% from the empty focus!
- This is exactly what Ptolemy's model needs.

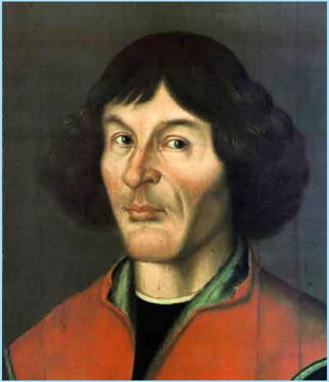
- There is a remarkable fact in the full picture of the solar system (which Ptolemy probably never drew).
- The directions in de the epicycles for the outer planets all *have to be the same*, while the epicycles of the inner planets both *have te be centered* on the line from the Earth to the Sun.
- Also the changing brightness of Venus was ignored.


Moon
Earth
Mercury
Venus
Sun
Mars
Jupiter
Saturn
Sphere of the stars

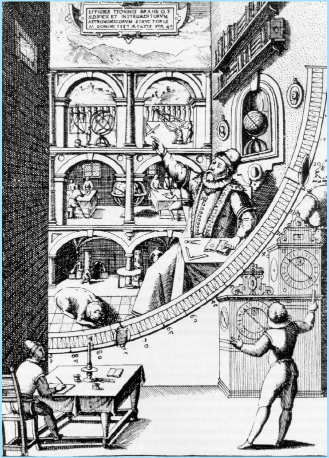
The Copernican revolution

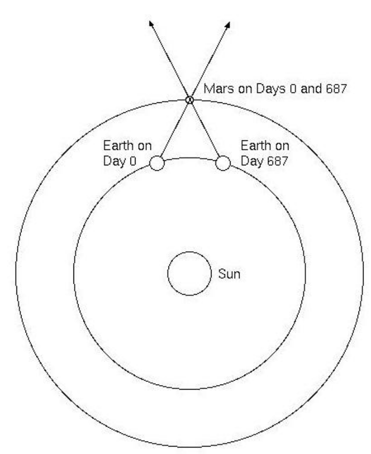
- **Nicolaus Copernicus** (1473 – 1543) made the model more complicated by approximating the orbits not by using the equant, but by superposing uniform circular motions, such that is corresponded more to what we now know te be Keplerian motion along ellipses.
- He then simplified the model by assuming that the Sun was in the center.
- Copernicus was more a man of the Middle Ages.


I. Stellarum fixarum sphæra immobilis.
II. Saturnus anno. XXX. revolvitur.
III. Iovis. XII. annorum revolutio.
IIII. Martis bima revolutio.
V. Telluris cum orbe lunari annus revolutio
terra
VI. Venus nonimestris.
VII. Mercury. LXXX. dierum.
Sol.

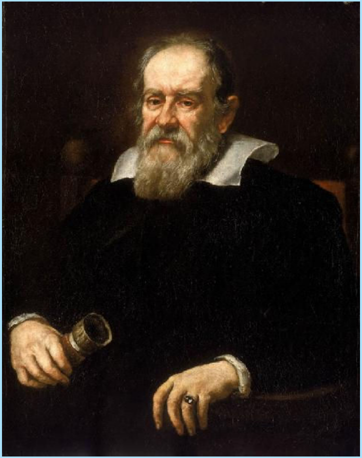
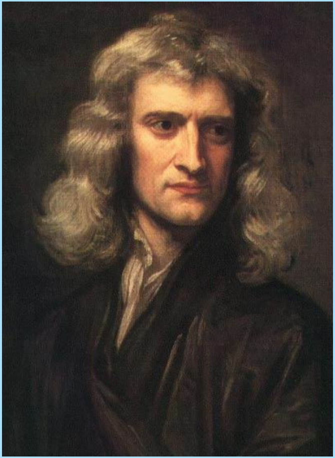


- **Johannes Kepler** (1571 – 1630) was the first to actually go and determine the shape of the orbits.
- He was the first to start from observations and questioned (for the first time in almost two millennia) the validity of the postulates of perfect motion by Plato and Aristoteles.
- Kepler used observations from **Tycho Brahe** (1546 – 1601) to do this first for the orbit of Mars.
- He used triangulation by selecting times when either Mars or the Earth were in the same position in their orbits.
- In this way he derived his first two laws and abandoned the concept of the perfect circular, uniform motion.


EFFIGIES TYCHONIS BRAHE O.F
ÆDIFICII ET INSTRUMENTORVM
ASTRONOMICORVM STRVCTORIS
A°. DOMINI 1587 ÆTATIS SVÆ 40

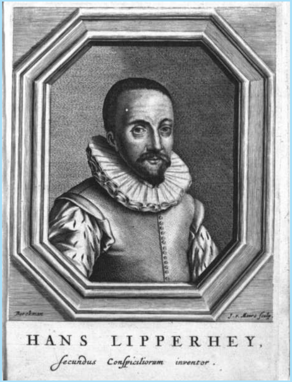

Mars on Days 0 and 687
Earth on
Day 0
Earth on
Day 687
Sun

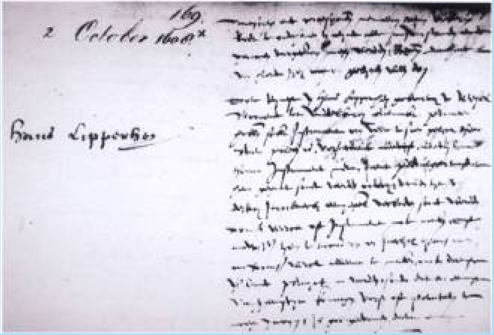
- The heliocentric model was confirmed by **Galilei Galileo** (1564 – 1642) with his observations of the satellites of Jupiter and the phases of Venus strarting in 1609.
- Eventually **Isaac Newton** (1642 – 1727) explained Kelper's laws (includig the harmonic law) as a prediction from his theory of gravity.

Galilei, telescopes and Neptune

- It is not certain who invented the telescope. Probably many lensmakers and opticians thought of the idea.
- Lenses then were made for spectacles, so were good only in central part close to the eyeball.
- Hans Lipperhey (1570 – 1619) of Middelburg designed a working telescope and demonstrated it in the Hague to Stadholder Maurits and many diplomats at an international peace conference in 1608.
- He documented it by submitting a request for a patent to the States General, which was refused.
- Vital (and innovative) in Lipperhey's design was that he stopped down the aperture.

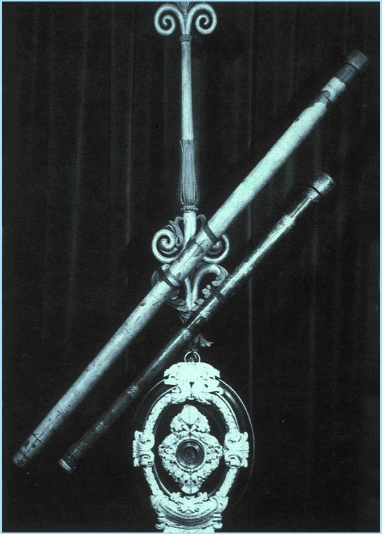




Hans Lipperhey and part of his request for a patent.

Galileo Galiei (probably with help of others) quickly improved telescope design and started observing the sky.

Here are some of his telescopes.



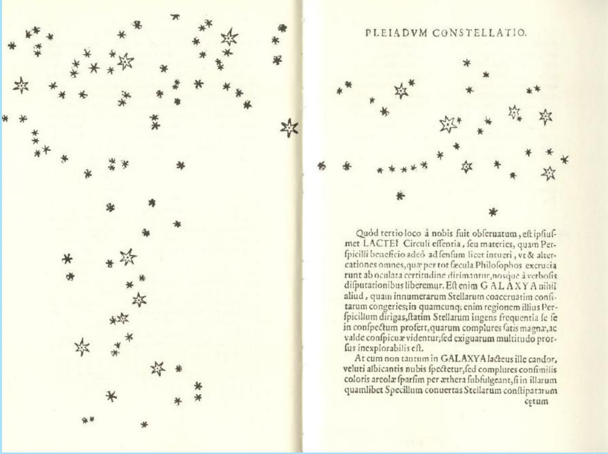
Galilei started observing the Moon on 30 November 1609.

He used initially a telescope with magnification about 3.

Here are some of Galilei's drawings.

In December 1609 he discovered that the Milky Way is full of stars.

On the left the lower part of Orion and on the right the Plejades.



PLEIADVM CONSTELLATIO.

Quòd tertio loco à nobis fuit obſeruatum, eſt ipſiuſmet LACTEI Circuli eſſentia, ſeu materies, quam Perſpicilli beneficio adeò ad ſenſum licet intueri, vt & altercationes omnes, quæ per tot ſæcula Philoſophos excruciarunt ab oculata certitudine dirimantur, noſque à verboſis diſputationibus liberemur. Eſt enim GALAXYA nihil aliud, quam innumerarum Stellarum coaceruatim conſitarum congeries; in quamcunq; enim regionem illius Perſpicillum dirigas, ſtatim Stellarum ingens frequentia ſe ſe in conſpectum profert, quarum complures ſatis magnæ, ac valde conſpicuæ videntur; ſed exiguarum multitudo prorſus inexplorabilis eſt.

At cum non tantum in GALAXYA lacteus ille candor, veluti albicantis nubis ſpectetur, ſed complures conſimilis coloris areolæ ſparſim per æthera ſubfulgeant, ſi in illarum quamlibet Specillum conuertas Stellarum conſtipatarum

cętum

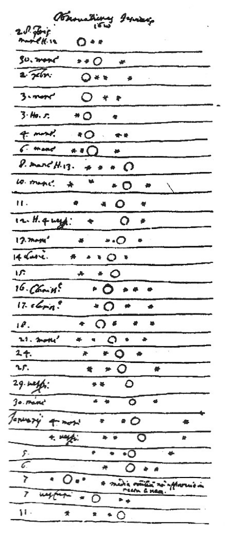
This is what Jupiter and the four brightests satellites look like through a small telescope.

On January 7, 1610 Galilei observed Jupiter with a telescope with magnification 30.

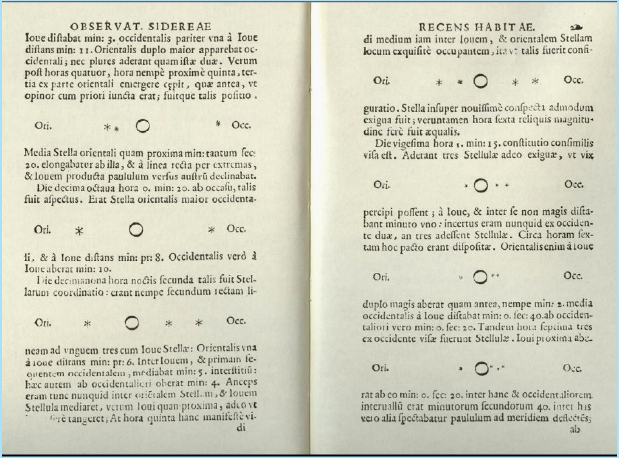
He discovered three 'stars" near Jupiter and on 10 January concluded that they circles around Jupiter.

On 13 January 1610 he discovered the fourth of the satellite (or 'moons').

Here is a page with Jupiter observations from Galilei's notebook for January 10.



Here two pages from Galilei's famous Sidereus Nuncius (the 'Starry Messenger'), published in 1610.


OBSERVAT. SIDEREAE

Ioue diſtabat min: 3. occidentalis pariter vna à Ioue diſtans min: 11. Orientalis duplo maior apparebat occidentali; nec plures aderant quam iſtæ duæ. Verum poſt horas quatuor, hora nempè proximè quinta, tertia ex parte orientali emergere cępit, quæ antea, vt opinor cum priori iuncta erat; fuitque talis poſitio.

Ori. * * ○ * Occ.

Media Stella orientali quam proxima min: tantum ſec: 20. elongabatur ab illa, & à linea recta per extremas, & Iouem producta paululum verſus auſtrũ declinabat.

Die decima octaua hora o. min: 20. ab occaſu, talis fuit aſpectus. Erat Stella orientalis maior occidentali, & à Ioue diſtans min: pr: 8. Occidentalis verò à Ioue aberat min: 10.

Ori. * ○ * Occ.

Die decimanona hora noctis ſecunda talis fuit Stellarum coordinatio: erant nempe ſecundum rectam lineam ad vnguem tres cum Ioue Stellæ: Orientalis vna à Ioue diſtans min: pr: 6. Inter Iouem, & primam ſequentem occidentalem, mediabat min: 5. interſtitiũ: hæc autem ab occidentaliori oberat min: 4. Anceps eram tunc nunquid inter orientalem Stellam, & Iouem Stellula mediaret, verum Ioui quam proxima, adeo vt ferè tangeret; At hora quinta hanc manifeſtè vi-

Ori. * ○ * * Occ.

di

RECENS HABITAE. 22

di medium iam inter Iouem, & orientalem Stellam locum exquiſitè occupantem, ita vt talis fuerit configuratio. Stella inſuper nouiſſimè conſpecta admodum exigua fuit; veruntamen hora ſexta reliquis magnitudine ferè fuit æqualis.

Ori. * * ○ * * Occ.

Die vigeſima hora 1. min: 15. conſtitutio conſimilis viſa eſt. Aderant tres Stellulæ adeo exiguæ, vt vix percipi poſſent; à Ioue, & inter ſe non magis diſtabant minuto vno: incertus eram nunquid ex occidente duæ, an tres adeſſent Stellulæ. Circa horam ſextam hoc pacto erant diſpoſitæ. Orientalis enim à Ioue duplo magis aberat quam antea, nempe min: 2. media occidentalis à Ioue diſtabat min: o. ſec: 40. ab occidentaliori vero min: o. ſec: 20. Tandem hora ſeptima tres ex occidente viſæ fuerunt Stellulæ. Ioui proxima aberat ab eo min: o. ſec: 20. inter hanc & occidentaliorem interuallũ erat minutorum ſecundorum 40. inter his vero alia ſpectabatur paululum ad meridiem deflectẽs;

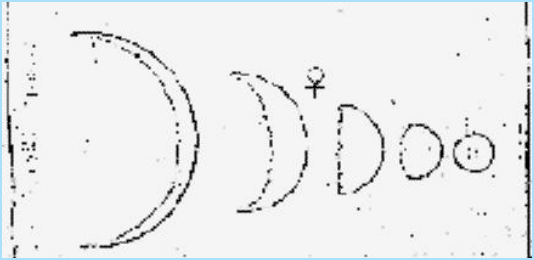
Ori. * ○ * * Occ.

Ori. * ○ * * Occ.

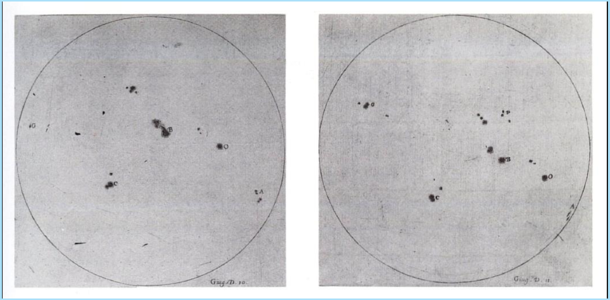
Ori. * ○ * * * Occ.

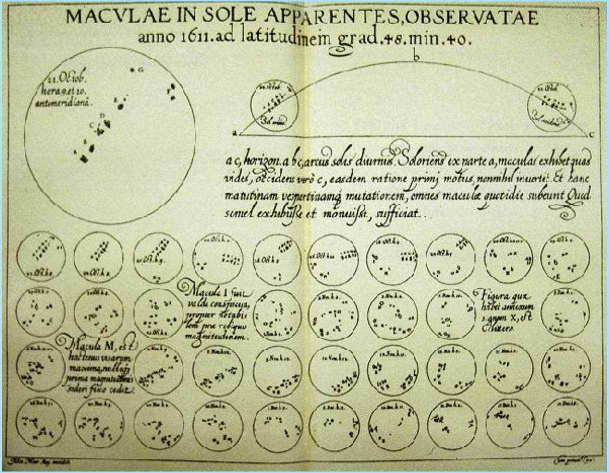
ab

Galilei's observations of the phases of Venus, starting in September 1610.

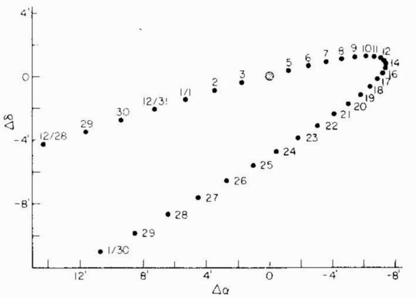


He later also observed sunspots, that were in disagreement with the perfectness of the sun in the Aristotelian philosophy.



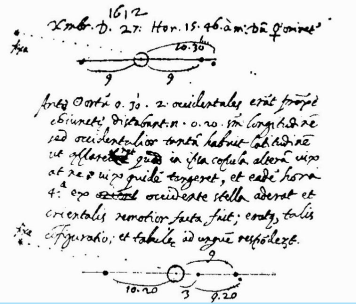

MACVLAE IN SOLE APPARENTES, OBSERVATAE
anno 1611. ad latitudinem grad. 48. min. 40.
a c, horizon. a b c, arcus solis diurnus. Sol oriens ex parte a, maculas exhibet quas vides, occidens verò c, easdem ratione primi motus, nonnihil inuertit. Et hanc matutinam vespertinamque mutationem, omnes maculae quotidie subeunt. Quod semel exhibuisse et monuisse, sufficiat.

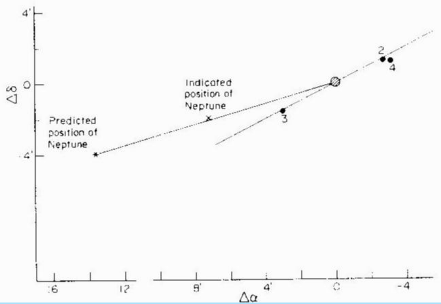
- Mutual occultations of planets are extremely rare.
- Between the superior planets there are only **9** between 1100 and 2500! (Next is Mars-Jupiter on December 2, 2223).
- On January 4, 1613 there was an occultation of Neptune by Jupiter.
- Galilei observed Jupiter at that time.



Here is Galilei's notebook for December 28, 1612.

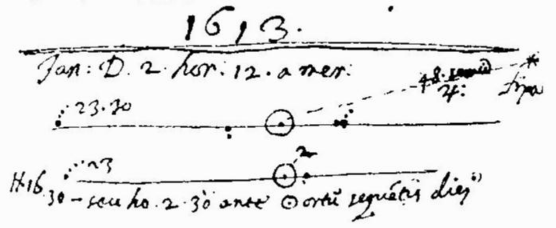
The 'star' indicated is not a known star and is in the correct direction to be Neptune, but at the edge of the page.



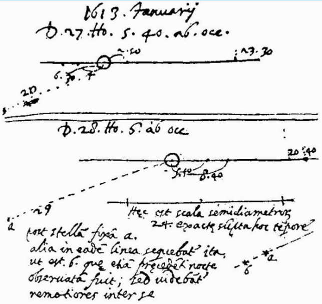


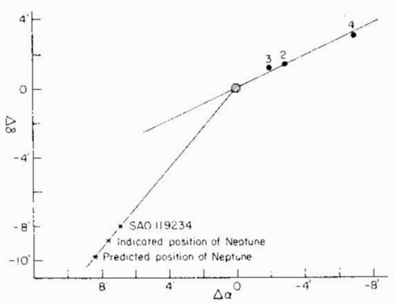
This is January 2, 1613 with a star (SAO119234), sketched at the edge of the page.

It says it is 48 jovian radii from Jupiter, while it actually is 52.



Again Neptune is present on the records of 27/28 January, 1613.





Galilei observed Neptune 234 years before it was discovered![2]

[2] Kowal & Drake, Nature **287**, 311 (1980)