STRUCTURE OF GALAXIES

Lecture 6. Vertical dynamics in disks, secular evolution, surface densities, stellar kinematics, stability and Toomre's Q.

Vertical dynamics.

First I recall some parameters of disk kinematics.

The epicyclic frequency κ describes the motion of objects with velocitites small compared to rotation and follows from the rotation curve:

$$\kappa = 2\{B(B - A)\}^{1/2}$$

A and B are the Oort constants, which follow from

$$A = \frac{1}{2}\left(\frac{V_{\text{rot}}}{R} - \frac{dV_{\text{rot}}}{dR}\right)$$

$$B = \frac{1}{2}\left(\frac{V_{\text{rot}}}{R} + \frac{dV_{\text{rot}}}{dR}\right)$$

The continuity equation for the case of axial symmetry is

$$\frac{\partial K_R}{\partial R} + \frac{K_R}{R} + \frac{\partial K_z}{\partial z} = -4\pi G \rho(R, z)$$

At small z the first two terms on the right are equal to $2(A - B)(A + B)$ and this is zero for a flat rotation curve. So for practical purposes we may use the plane-parallel case:

$$\frac{dK_z}{dz} = -4\pi G \rho(z)$$

Poisson's equation for the axi-symmetric case is

$$\frac{d}{dz}\left[\rho(z)\langle V_z^2\rangle\right] = \rho(z) K_z$$

For an isothermal distribution this becomes

$$\frac{d\rho(z)}{dz} = \frac{\rho(z) K_z}{\langle V_z^2\rangle}$$

The equations for the isothermal sheet are the solutions of this set of equations.

$$\rho(z) = \rho_\circ \,\mathrm{sech}^{\,2}\left(\frac{z}{z_\circ}\right)$$

$$\sigma = 2 z_\circ \rho_\circ$$

$$K_\mathrm{z} = -4\pi G \rho_\circ z_\circ \tanh\left(\frac{z}{z_\circ}\right)$$

$$\langle V_\mathrm{z}^2 \rangle^{1/2} = z_\circ \sqrt{2\pi G \rho_\circ}$$

When a second isothermal component (II) with negligible mass moves in this force field we have

$$\frac{\partial \rho_{\mathrm{II}}}{\partial z} = \frac{\rho K_\mathrm{z}}{\langle V_\mathrm{z}^2 \rangle_{\mathrm{II}}}$$

$$\rho_{\mathrm{II}}(z) = \rho_{\mathrm{II}}(0) \,\mathrm{sech}^{\,2p}\left(\frac{z}{z_\circ}\right)$$

$$p = \frac{\langle V_\mathrm{z}^2 \rangle}{\langle V_\mathrm{z}^2 \rangle_{\mathrm{II}}}$$

Disks are not entirely isothermal, since velocity dispersions of the stellar generations increase with age. Therefore replace the solution by the set*

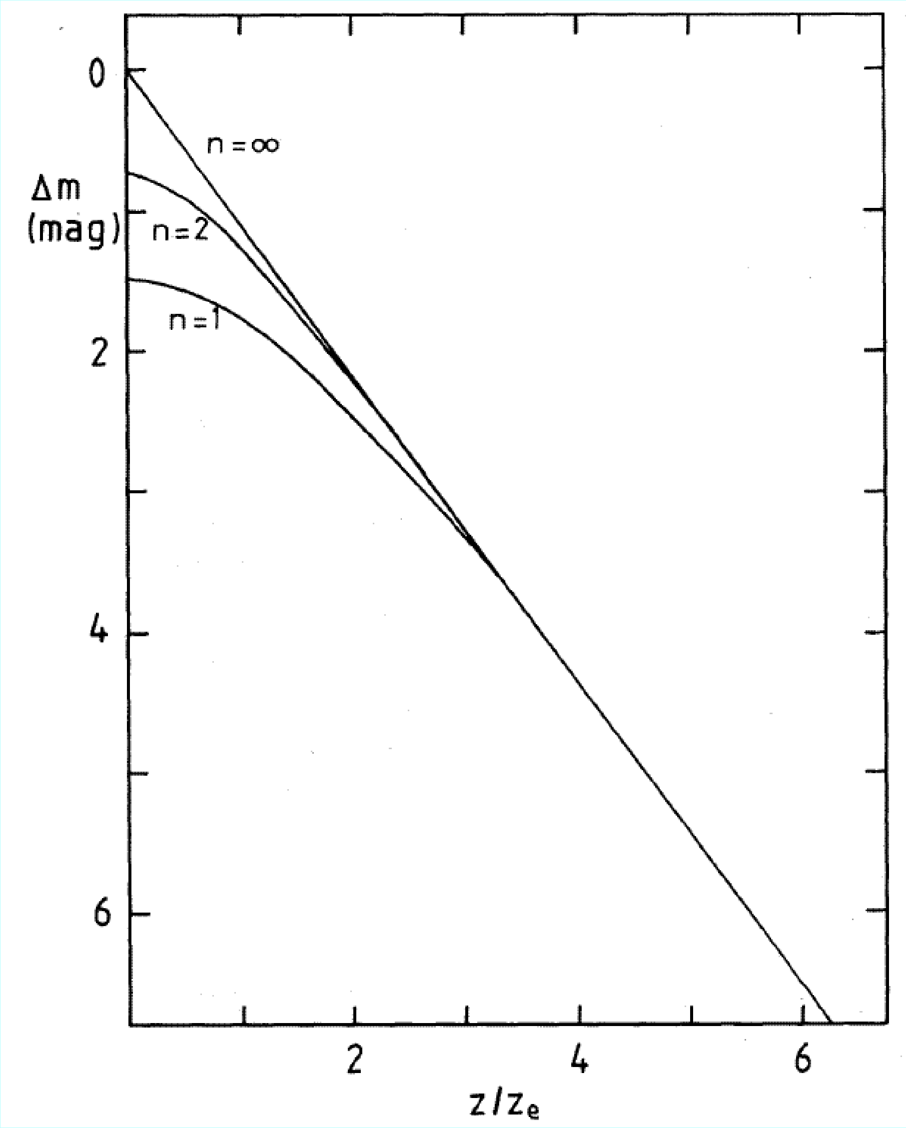
$$\rho(z) = 2^{-2/n} \rho_{\rm e} \, \mathrm{sech}^{\,2/n} \left(\frac{nz}{2z_{\rm e}}\right)$$

Consider the extremes $n = \infty$ (the exponential) and $n = 1$ (the isothermal) and one intermediate case $n = 2$.

$$\begin{array}{lll} n = 1 & \rho(z) = & \dfrac{\rho_{\rm e}}{4} \, \mathrm{sech}^{\,2} \left(\dfrac{z}{2z_{\rm e}}\right) \\ n = 2 & \rho(z) = & \dfrac{\rho_{\rm e}}{2} \, \mathrm{sech} \left(\dfrac{z}{z_{\rm e}}\right) \\ n = \infty & \rho(z) = & \rho_{\rm e} \exp - \left(\dfrac{z}{z_{\rm e}}\right) \end{array}$$

We then calculate various properties. Eventually this can be used to evaluate the effects of assuming isothermal distributions.

*van der Kruit, A.&A. 192, 117 (1988)



The surface densities are

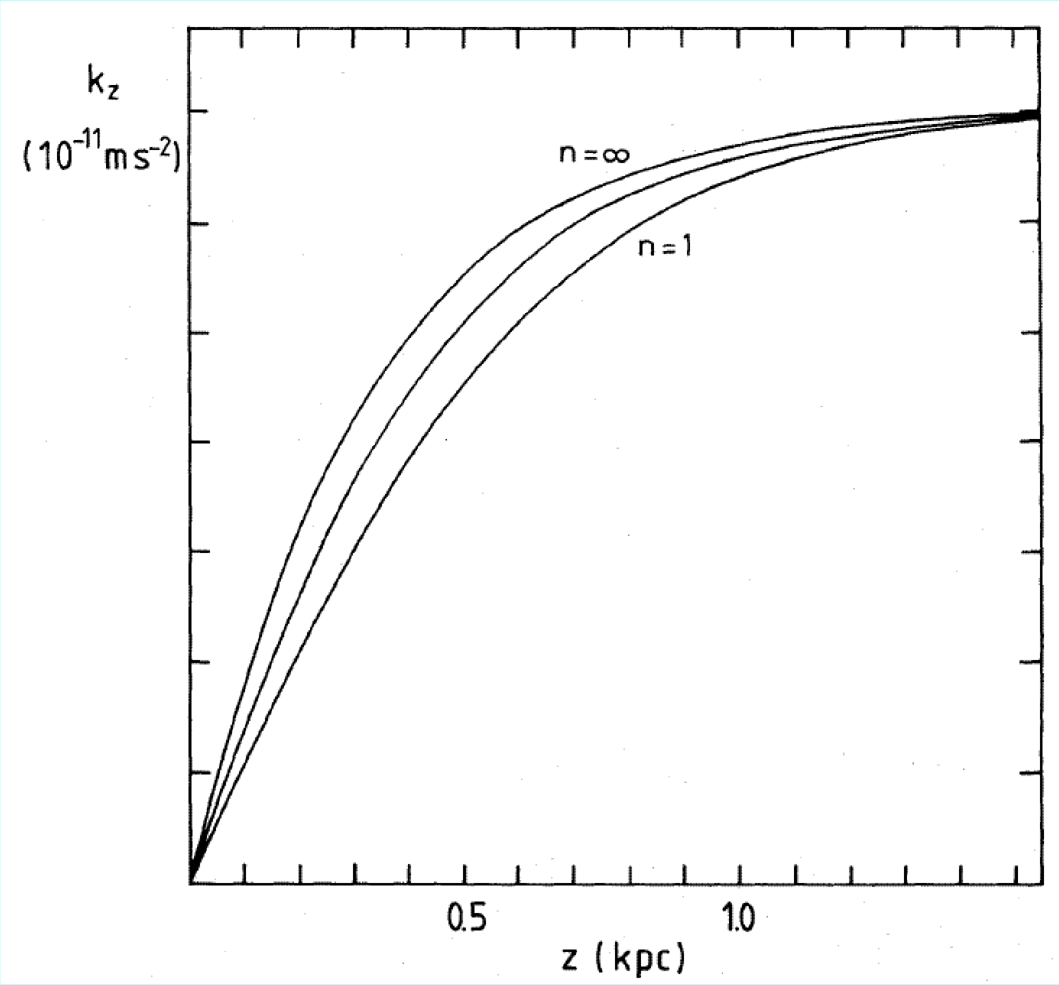
$$
\begin{array}{rcl}
n = 1 & \sigma = & \rho_e z_e \\
n = 2 & \sigma = & \frac{\pi}{2} \rho_e z_e \\
n = \infty & \sigma = & 2 \rho_e z_e
\end{array}
$$

The vertical force is

$$n = 1 \qquad K_z = -2\pi G\sigma \tanh\left(\frac{z}{2z_e}\right)$$

$$n = 2 \qquad K_z = -4G\sigma \arctan\left\{\sinh\left(\frac{z}{z_e}\right)\right\}$$

$$n = \infty \qquad K_z = -2\pi G\sigma \left\{1 - \exp - \left(\frac{z}{z_e}\right)\right\}$$



In the graph parameters approximately those of the solar neighborhood are chosen.

Velocity dispersion (squared) as a function of z

$$n = 1 \qquad \langle V_z^2 \rangle = 2\pi G \sigma z_e$$

$$n = 2 \qquad \langle V_z^2 \rangle = \left(\frac{\pi^2}{2}\right) G\sigma z_e \cosh\left(\frac{z}{z_e}\right) \left\{1 - \left(\frac{2}{\pi^2}\right) \arctan^2\left(\sinh\left(\frac{z}{z_e}\right)\right)\right\}$$

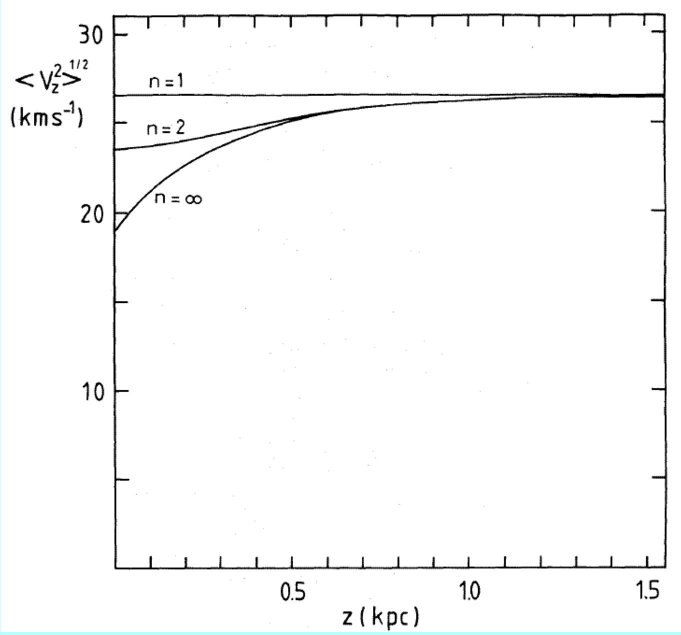
$$n = \infty \qquad \langle V_z^2 \rangle = \pi G \sigma z_e \left\{2 - \exp\left(\frac{z}{z_e}\right)\right\}$$

and integrated over all z

$$n = 1 \qquad \langle V_z^2 \rangle_{FO} = 2\pi G \sigma z_e$$

$$n = 2 \qquad \langle V_z^2 \rangle_{FO} = (1.705)\pi G \sigma z_e$$

$$n = \infty \qquad \langle V_z^2 \rangle_{FO} = (3/2)\pi G \sigma z_e$$



For a second isothermal component with

$$p = \frac{\langle V_z^2 \rangle_{z=0}}{\langle V_z^2 \rangle_{\rm II}}$$

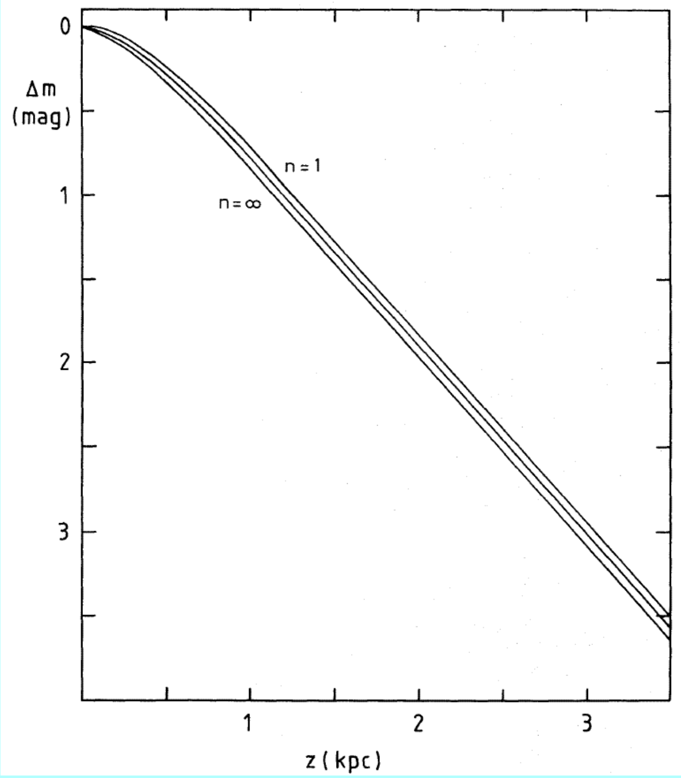
$$n = 1 \qquad \rho_{\rm II}(z) = \rho_{\rm II}(0)\ \mathrm{sech}^{\,2p}\left(\frac{z}{2z_e}\right)$$

$$n = 2 \qquad \rho_{\rm II}(z) = \rho_{\rm II}(0)\ \exp\left[-\left(\frac{8}{\pi^2}\right) pI\left(\frac{z}{z_e}\right)\right]$$

$$n = \infty \qquad \rho_{\rm II}(z) = \rho_{\rm II}(0)\ \exp\left[-\frac{2pz}{z_e} + 2p\left\{1 - \exp\left(\frac{z}{z_e}\right)\right\}\right]$$

where the function I for $n = 2$ is

$$I(y) = \int_0^y \arctan(\sinh x)\, dx$$



The thickness of an HI-layer can then be expressed in terms of

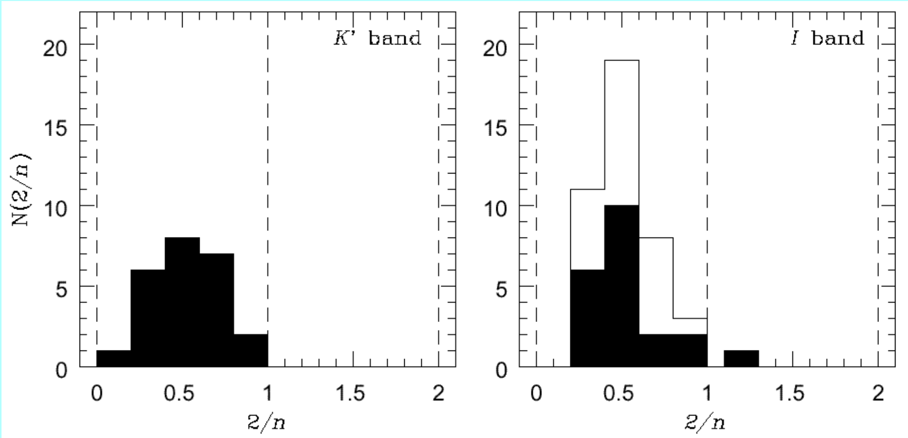
$$d_{\mathrm{HI}} = \left(\frac{\langle V_{\mathrm{z}}^2 \rangle_{\mathrm{HI}} z_{\mathrm{e}}}{G\sigma} \right)^{1/2}$$

and the full width at half maximum is

$$\begin{array}{rrl} n = 1 & W_{\mathrm{HI}} = & 1.33\, d_{\mathrm{HI}} \\ n = 2 & W_{\mathrm{HI}} = & 1.18\, d_{\mathrm{HI}} \\ n = \infty & W_{\mathrm{HI}} = & 0.94\, d_{\mathrm{HI}} \end{array}$$

From measurements of a sample of edge-on galaxies* the index n has been determined as

$$2/n = 0.54 \pm 0.20$$



*de Grijs, Peletier & van der Kruit, A.&A. 327, 996 (1997)

Next we need to consider Toomre's* criterion for local stability:

$$Q = \frac{\langle V_{\mathrm{R}}^2 \rangle^{1/2} \kappa}{3.36 G\sigma}$$

$\langle V_{\mathrm{R}}^2 \rangle^{1/2}$ is the stellar velocity dispersion in the R-direction, σ is the local disk surface density and κ is the epicyclic frequency.

An approximate derivation of Toomre's criterion can be made for an infinitesimally thin disk.

1. At small scales the Jeans instability needs to be considered.

Take an area with radius R and surface density σ. The equation of motion is

$$\frac{d^2 R}{dt^2} = -\pi G\sigma$$

Solve this and apply for $R = 0$; this gives the free-fall time

$$t_{\mathrm{ff}} = \left(\frac{2R}{\pi G\sigma}\right)^{1/2}$$

*Toome, Ap.J. 139, 1217 (1964)

A star moves out to radius R in a time

$$t = \frac{R}{\langle V^2 \rangle^{1/2}}$$

and this must for marginal stability be equal to the free-fall time.

This then gives the Jeans length

$$R_{\mathrm{Jeans}} = \frac{2\langle V^2 \rangle}{\pi G \sigma}$$

2. At large scale we need to consider stability resulting from differential rotation.

Take an area with radius $R_\circ$; the angular velocity from differential rotation is

$$\Omega = B$$

The centrifugal force is then

$$F_{\mathrm{cf}} = R_\circ \Omega^2$$

Let it contract to radius R, then the angular velocity becomes

$$\Omega = \frac{R_\circ^2 B}{R^2}$$

and the centrifugal force

$$F_{\text{cf}} = R\Omega^2 = \frac{R_\circ^4 B^2}{R^3}$$

If the contraction is dR then

$$\frac{dF_{\text{cf}}}{dR} = -\frac{3R_\circ^4 B^2}{R^4}$$

Now look at the gravitational force

$$F_{\text{grav}} = -\frac{G\pi R_\circ^2 \sigma}{R^2}$$

This is correct to within a factor 2 for a flat distribution. Then

$$\frac{dF_{\text{grav}}}{dR} = \frac{2\pi G R_\circ^2 \sigma}{R^3}$$

At $R = R_\circ$ these two must compensate each other, so

$$R_{\text{crit}} = \frac{2\pi G \sigma}{3B^2}$$

and the disk is stable for all $R > R_{\text{crit}}$.

3. Toomre's stability criterion.

The disk is stable at all scales if the minimum radius for stability by differential rotation is equal to or smaller than the maximum radius for stability by random motions (the Jeans radius).

Thus

$$\langle V^2 \rangle_{\mathrm{crit}}^{1/2} = \frac{\pi}{\sqrt{3}} \frac{G\sigma}{B}$$

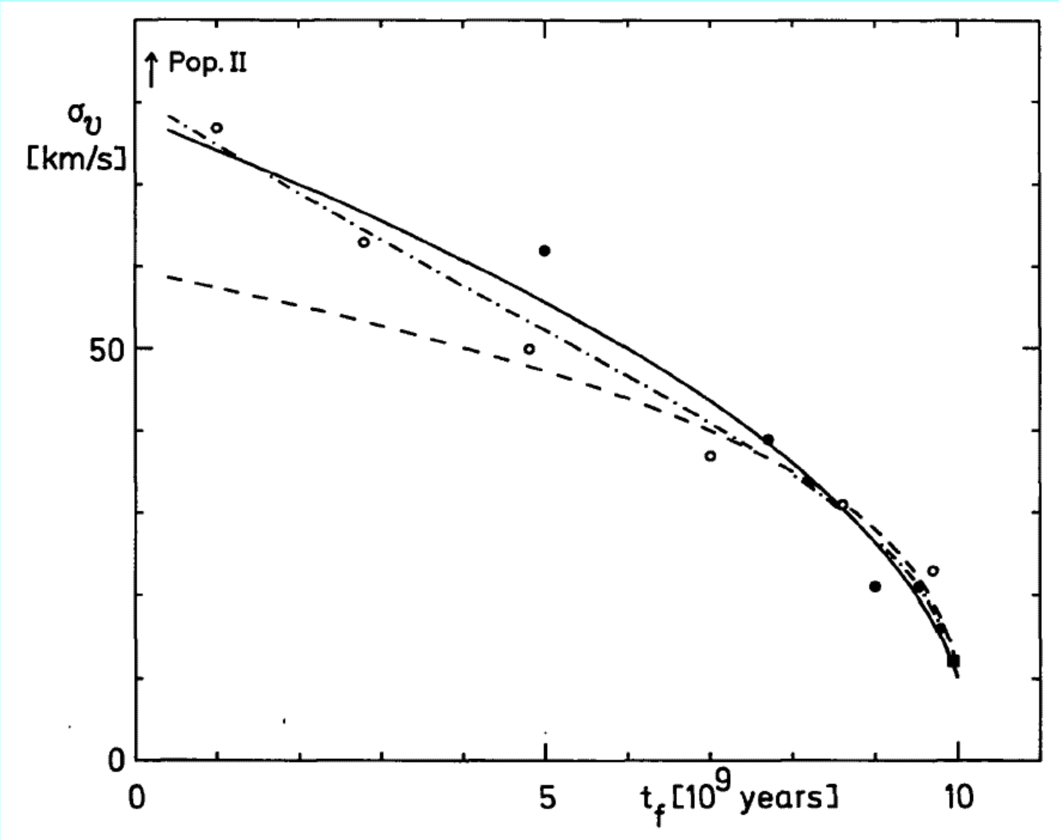
In practice $B \approx -A$ (for flat rotation curves), so we can write

$$\langle V^2 \rangle_{\mathrm{crit}}^{1/2} \sim 2\pi \left(\frac{2}{3}\right)^{1/2} \frac{G\sigma}{\kappa} = 5.13 \frac{G\sigma}{\kappa}$$

Toomre in his precise treatment found a constant of 3.36.

Stellar velocity dispersions in disks.

Stars increase their velocity dispersion with age*; this is referred to as the velocity dispersion – age relation.



Stars are formed with the motions of the interstellar medium (velocity dispersion about 10 km/s).

*Wielen, A.&A. 60, 262 (1977)

Then the velocity dispersion increases roughly as $\sigma \propto t^{\alpha}$ with $\alpha \sim 0.5$.

There are three general mechanisms proposed for this:

- Stars are in their orbits scattered by concentrations in the ISM*, now identified as Large Molecular Clouds.
- Spiral structure systematically increases the random motions of the stars†.
- Infall of small companion galaxies has the same effect on disks‡.

The scattering becomes less as the stars move outside the gas layer.

Th Spitzer-Schwarzschild mechanism appears to be incapable of explaining the ratio of the R- and z-velocity dispersions (the axis ratio of the velocity ellipsoid), scattering too little in z.

It is most likely that both processes contribute§.

*Spitzer & Schwarzschild, Ap.J. 114, 385 (1951)

†Barbanis & Woltjer, Ap.J. 150, 461 (1967); Carlberg & Sellwood, Ap.J. 292, 79 (1985)

‡Veázquez & White, Mon.Not.R.A.S. 304, 254 (1999)

§Jenkins & Binney, Mon.Not.R.A.S. 257, 305 (1990)

Observations of stellar velocity dispersions.

1. Z-velocity dispersion

If disks have constant mass-to-light ratios M/L, the density can be described by

$$\rho(R,z) = \rho(0,0)\ \mathrm{exp}\ (-R/h)\ \mathrm{sech}\ ^2(z/z_\circ)$$

The vertical velocity dispersion then is

$$\langle V_\mathrm{z}^2\rangle^{1/2} = \sqrt{2\pi G\rho(R,0)}z_\circ$$

and it is expected that

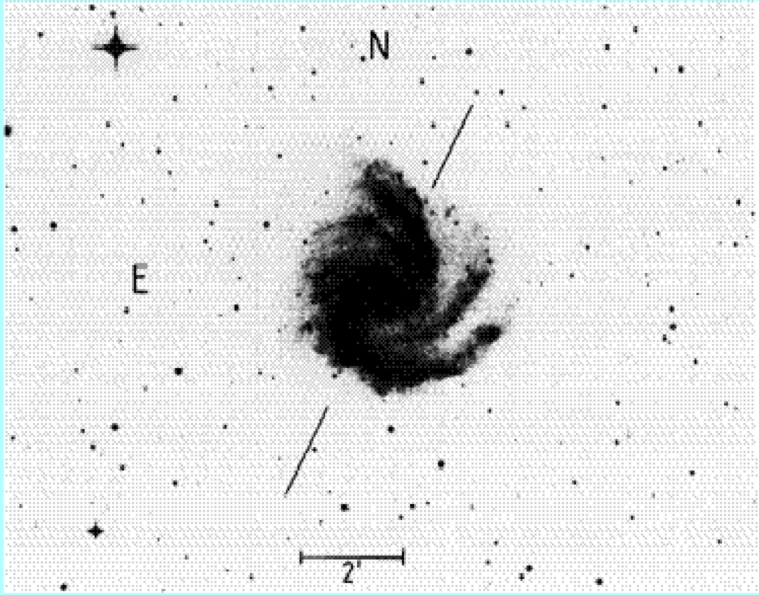
$$\langle V_\mathrm{z}^2\rangle^{1/2} \propto\ \mathrm{exp}\ (-R/2h)$$

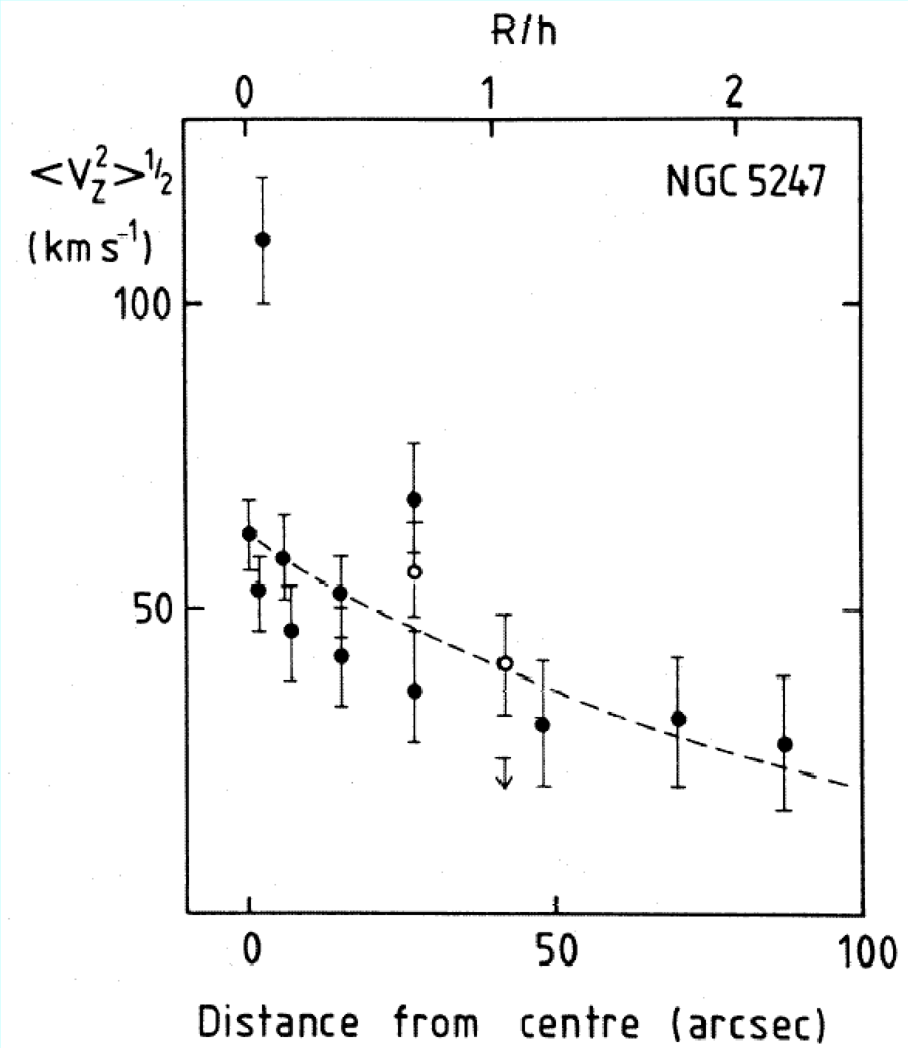
This can be tested by observations in face-on systems, e.g. NGC 5247*. The result is

$$\langle V_\mathrm{z}^2\rangle^{1/2} = (62{\pm}7)\ \mathrm{exp}\ [-(0.42{\pm}0.10)\ R/h]\mathrm{km\ s}^{-1}$$

This is consistent with M/L about constant.

*van der Kruit & Freeman 1986, Ap.J. 303, 556 (1968)


N
E
2'


R/h
0
1
2
$<V_z^2>^{1/2}$
(km s^{-1})
100
50
NGC 5247
0
50
100
Distance from centre (arcsec)

R- and θ -velocity dispersions

From fundamental kinematics we have

$$\frac{\langle (V_\theta - V_{\mathrm{t}})^2 \rangle}{\langle V_{\mathrm{R}}^2 \rangle} = \frac{B}{B-A}$$

So, if we know the rotation curve we know the ratio of the radial and tangential velocity dispersion.

The other property to consider is the asymmetric drift.

The continuity equation can be written as

$$\begin{aligned} -K_{\mathrm{R}} &= V_{\mathrm{t}}^2 - \langle V_{\mathrm{R}}^2 \rangle \frac{\partial}{\partial R} \ln(\nu \langle V_{\mathrm{R}}^2 \rangle) + \\ & \frac{1}{R} \Big\{ \langle V_{\mathrm{R}}^2 \rangle - \langle (V_\theta - V_{\mathrm{t}})^2 \rangle + \\ & \langle V_{\mathrm{z}} V_{\mathrm{R}} \rangle \frac{\partial}{\partial z} (\ln \nu \langle V_{\mathrm{z}} V_{\mathrm{R}} \rangle) \Big\} \end{aligned}$$

Poisson equation is

$$\frac{\partial K_{\mathrm{R}}}{\partial R} + \frac{K_{\mathrm{R}}}{R} + \frac{\partial K_{\mathrm{z}}}{\partial z} = -4\pi G \rho$$

For small z it can be shown that

$$\frac{\partial K_{\mathrm{R}}}{\partial R} + \frac{K_{\mathrm{R}}}{R} = 2(A-B)(A+B)$$

and for a flat rotation curve $A = -B$, so that

$$\frac{\partial K_{\mathrm{z}}}{\partial z} = -4\pi G\rho$$

Then

$$\langle V_{\mathrm{z}} V_{\mathrm{R}} \rangle = 0$$

Obviously we have

$$K_{\mathrm{R}} = V_{\mathrm{rot}}^2 / R$$

For an exponential disk with constant M/L

$$\frac{\partial}{\partial R} \ln \nu = -\frac{1}{h}$$

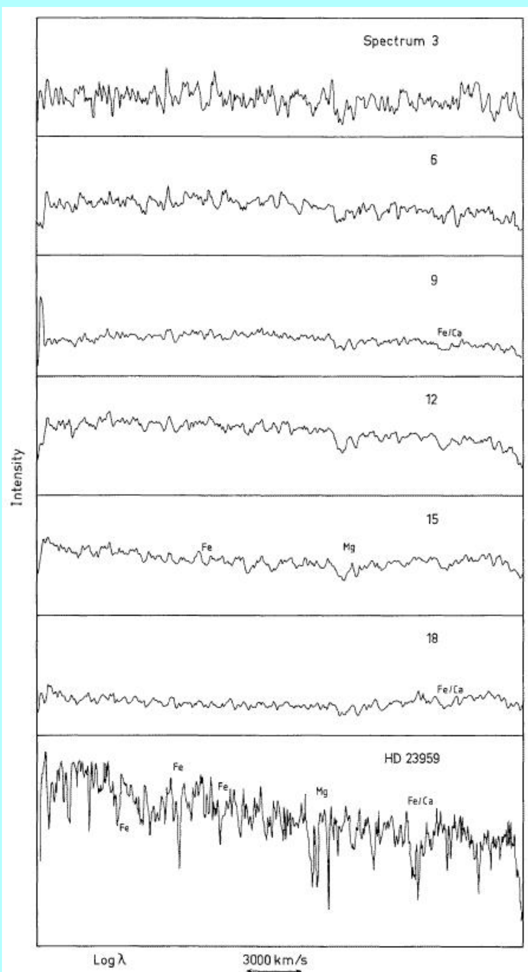
The asymmetric drift equation then becomes

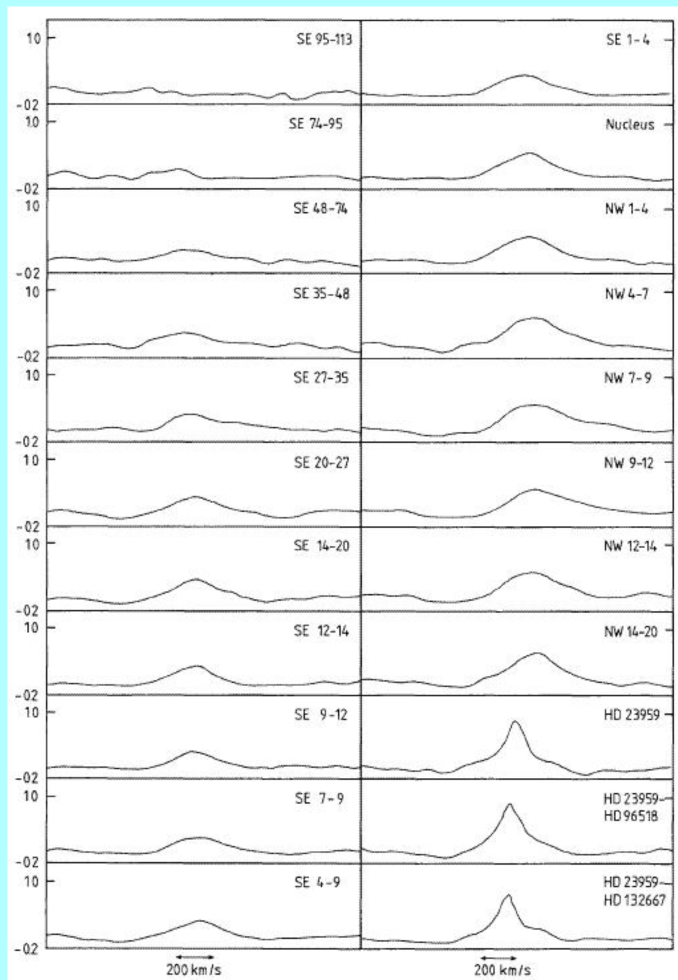
$$V_{\mathrm{rot}}^2 - V_{\mathrm{t}}^2 = \langle V_{\mathrm{R}}^2 \rangle \left[\frac{R}{h} - R\frac{\partial}{\partial R} \ln \langle V_{\mathrm{R}}^2 \rangle - \left\{ 1 - \frac{B}{B-A} \right\} \right]$$

There are now two possibilities for observing. The first is to measure $\langle V_{\mathrm{R}}^2 \rangle^{1/2}$ directly from spectra.

The difficulty is the line-of-sight integration. This has to be treated by modeling as was done in the edge-on galaxy NGC 5170*.

The profiles now have become asymmetric. We see here the spectra and the cross-correlation peaks between galaxy and template spectra.

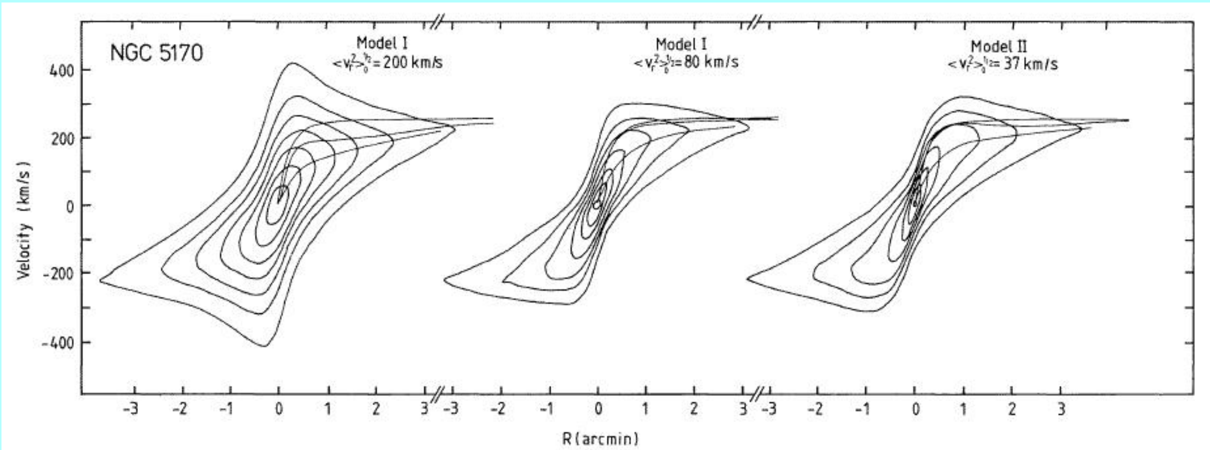




*Bottema, van der Kruit & Freeman, Ap.J. 178. 77 (1987)

Using an estimate of the circular motion from the HI-rotation curve one can calculate the profiles in a stellar "l,V-diagram".

To do this one needs an assumed radial variation of the velocity dispersion, the rotation curve (and from that the Oort constants) and the density distribution of the stars.



In the figure here we see a few such simulations. The three lines in each panel are form top to bottom: the circular motion from HI-observations, the stellar rotation velocity and peaks of Gaussians fitted to the resulting profiles.

The second option is to measure the asymmetric drift. The relevant equation was

$$V_{\rm rot}^2 - V_{\rm t}^2 = \langle V_{\rm R}^2 \rangle \left[\frac{R}{h} - R \frac{\partial}{\partial R} \ln \langle V_{\rm R}^2 \rangle - \left\{ 1 - \frac{B}{B-A} \right\} \right]$$

So we see that we need to measure:

• $V_{\rm rot}$, A and B from HI-synthesis or emission line spectroscopy.

• $V_{\rm t}$ from absorption line spectroscopy.

• h from surface photometry.

For a flat rotation curve:

$$\frac{B}{B-A} = 0.5 \text{ and } \kappa^2 = \frac{2V_{\rm rot}^2}{R^2}$$

For small asymmetric drift:

$$V_{\rm rot}^2 - V_{\rm t}^2 \approx 2V_{\rm rot}(V_{\rm rot} - V_{\rm t})$$

Consider two possibilities:

Model I with $\langle V_{\rm R}^2 \rangle / \langle V_{\rm z}^2 \rangle$ constant. Then

$$\langle V_{\rm R}^2 \rangle^{1/2} \propto \exp\left(-R/2h\right)$$

$$V_{\mathrm{rot}} - V_{\mathrm{t}} = \frac{\langle V_{\mathrm{R}}^2 \rangle}{2V_{\mathrm{rot}}} \left(\frac{2R}{h} - 0.5 \right)$$

- Model II with Q constant. Then

$$\langle V_{\mathrm{R}}^2 \rangle^{1/2} \propto R \exp\left(-R/h\right)$$

$$V_{\mathrm{rot}} - V_{\mathrm{t}} = \frac{\langle V_{\mathrm{R}}^2 \rangle}{2V_{\mathrm{rot}}} \left(\frac{3R}{h} - 2.5 \right)$$

How different are these models? For comparison calculate a Q (arbitrarily set to unity at one scalelength) for the first model:

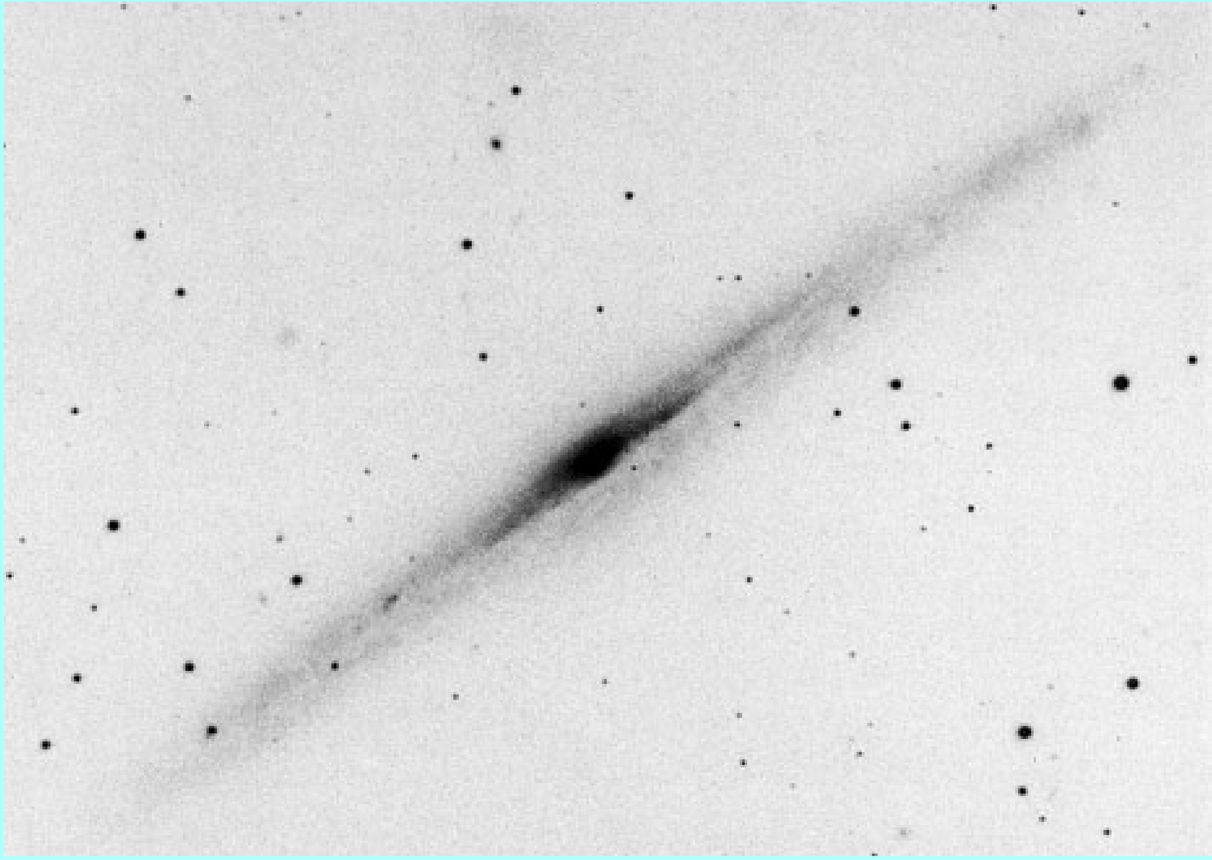
$R/h =$	$Q =$
1.0	1.17
1.5	**1.00**
2.0	0.96
3.0	1.06
4.0	1.31
5.0	1.73

We see that the models are really not different up to three or four scalelengths.

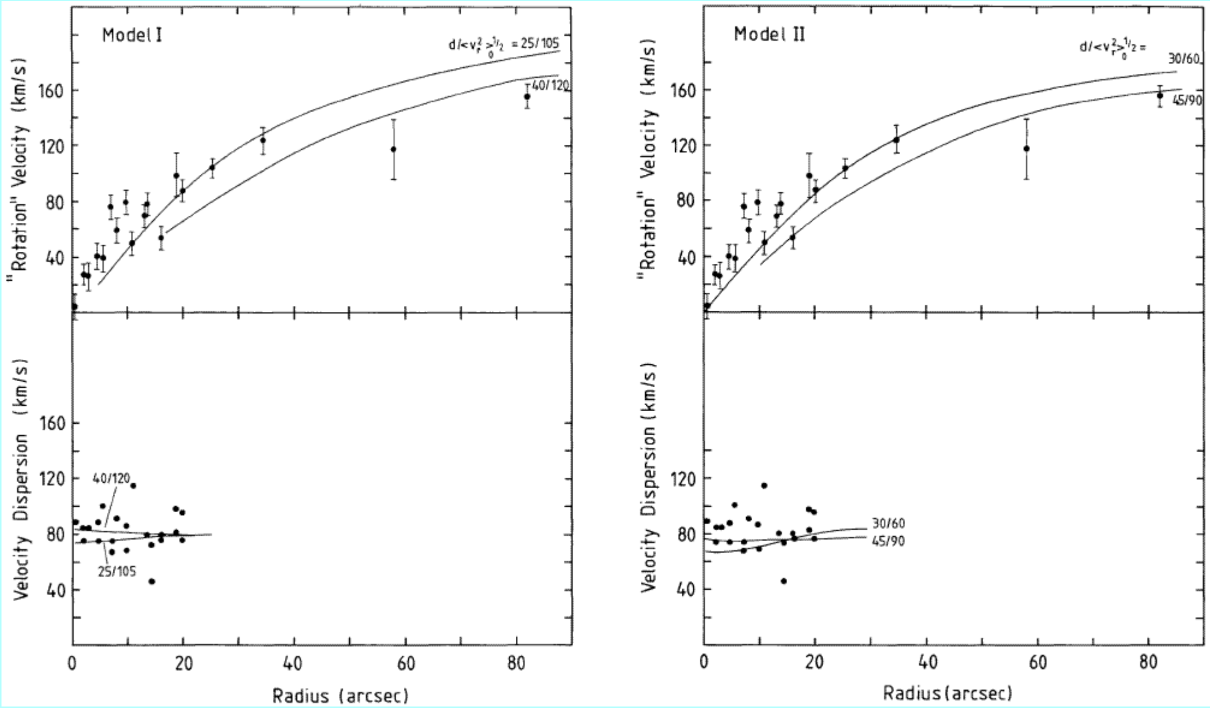
But at large R there is a contribution from the gas, which lowers the total velocity dispersion and increases the surface density and therefore lowers Q.

Numerical experiments on dynamics of stellar disks give $Q \sim 1.5 - 2.0$ at all radii.

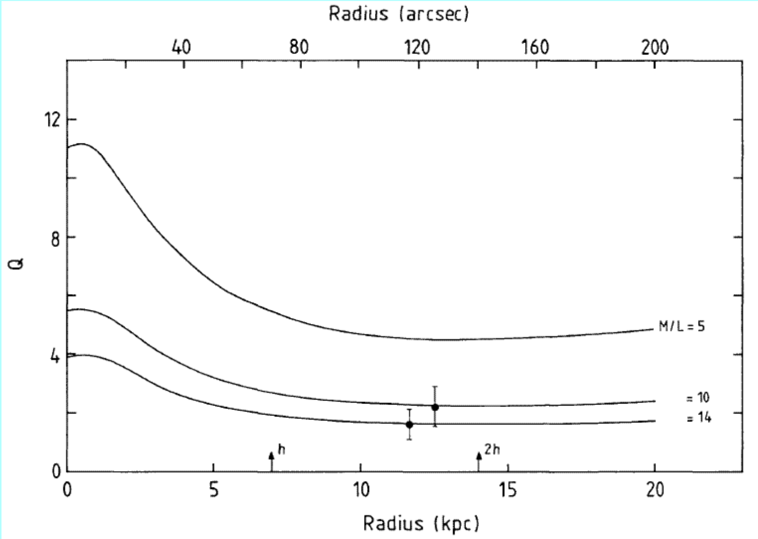
Back to the data on NGC 5170.



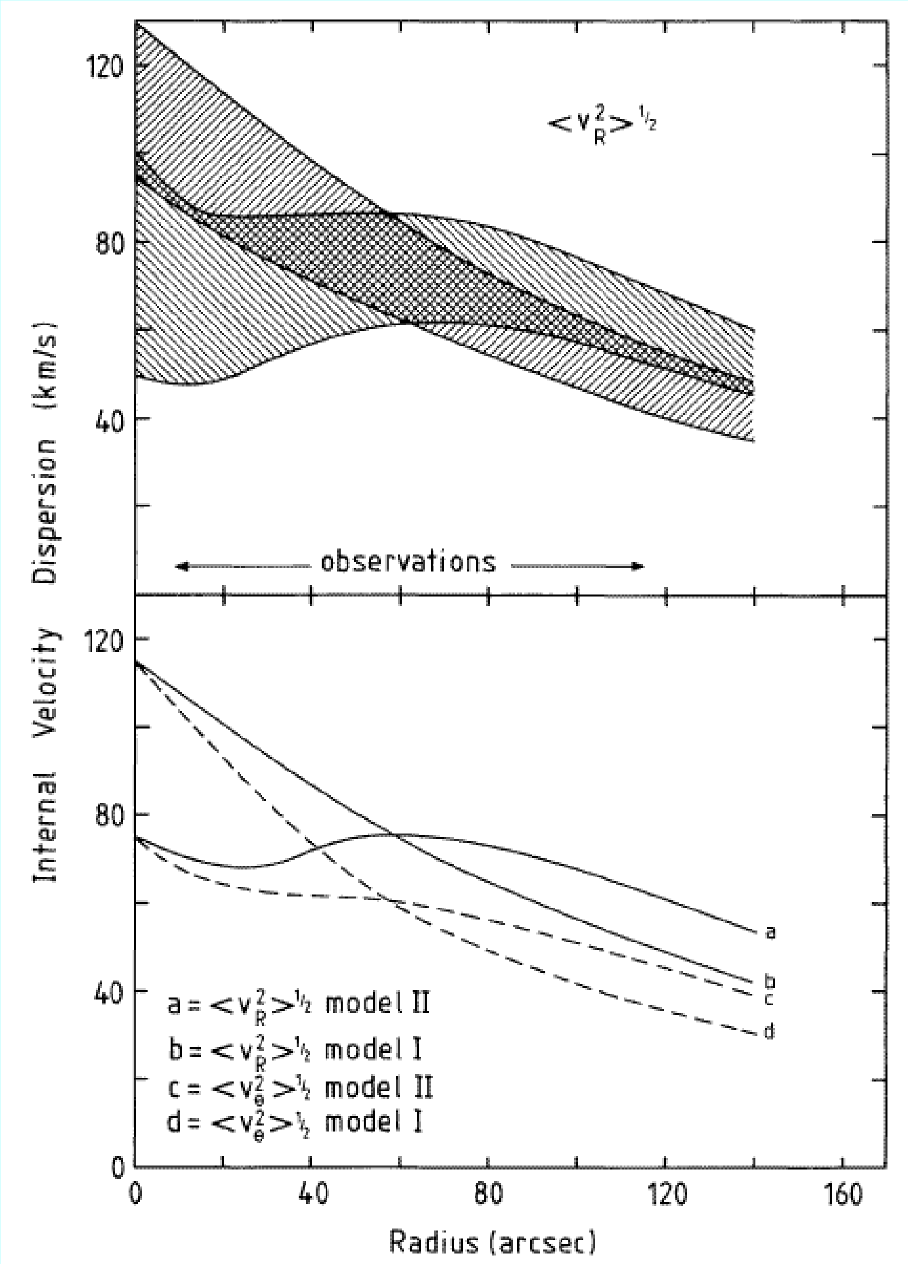
The fits to the data of NGC 5170 are as follows.



The resulting Q is as the lines in the figure below for Model I and the dots plus error bars in Model II for various assumed values for the disk M/L.



The velocity dispersions have the following radial distribution.



There is little difference between the two models.

The Bottema relations.

Bottema* observed stellar velocity dispersions in a set of 12 galaxies.

He then defined as fiducial values the radial velocity dispersion at one scalelength for inclined systems and the vertical velocity dispersion in the center for face-on systems.

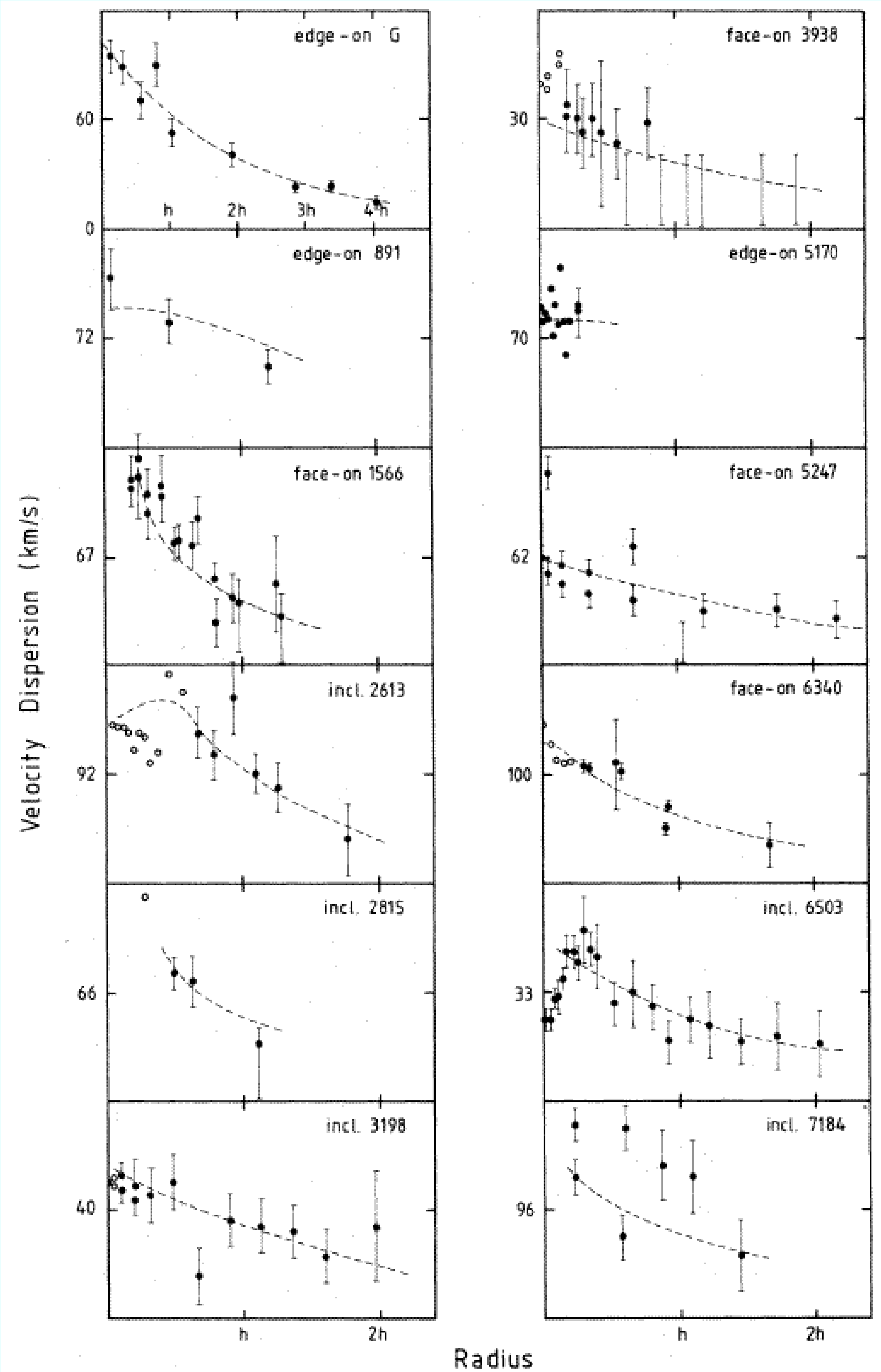
For the velocity ellipsoid of the solar neighborhood those should be comparable.

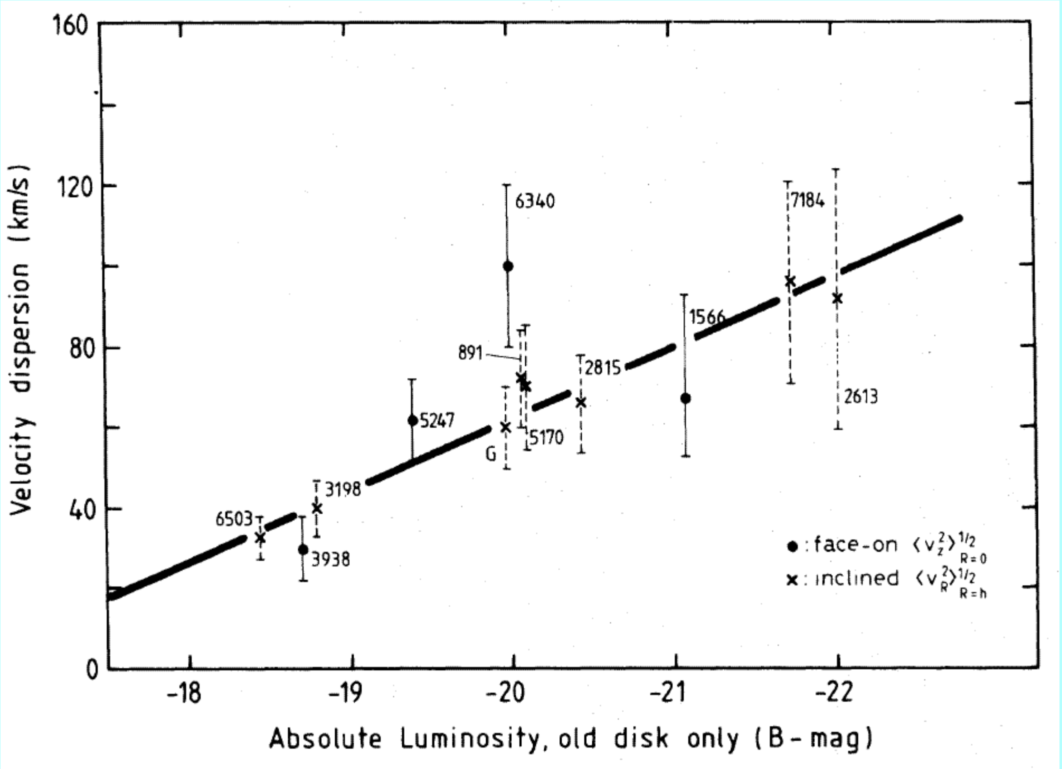
He then found the following relations

$$\langle V_{\rm R}^2 \rangle^{1/2}_{\rm R=h} = \langle V_{\rm z}^2 \rangle^{1/2}_{\rm R=0} = -17 \times M_{\rm B} - 279 \ \ {\rm km/s}$$

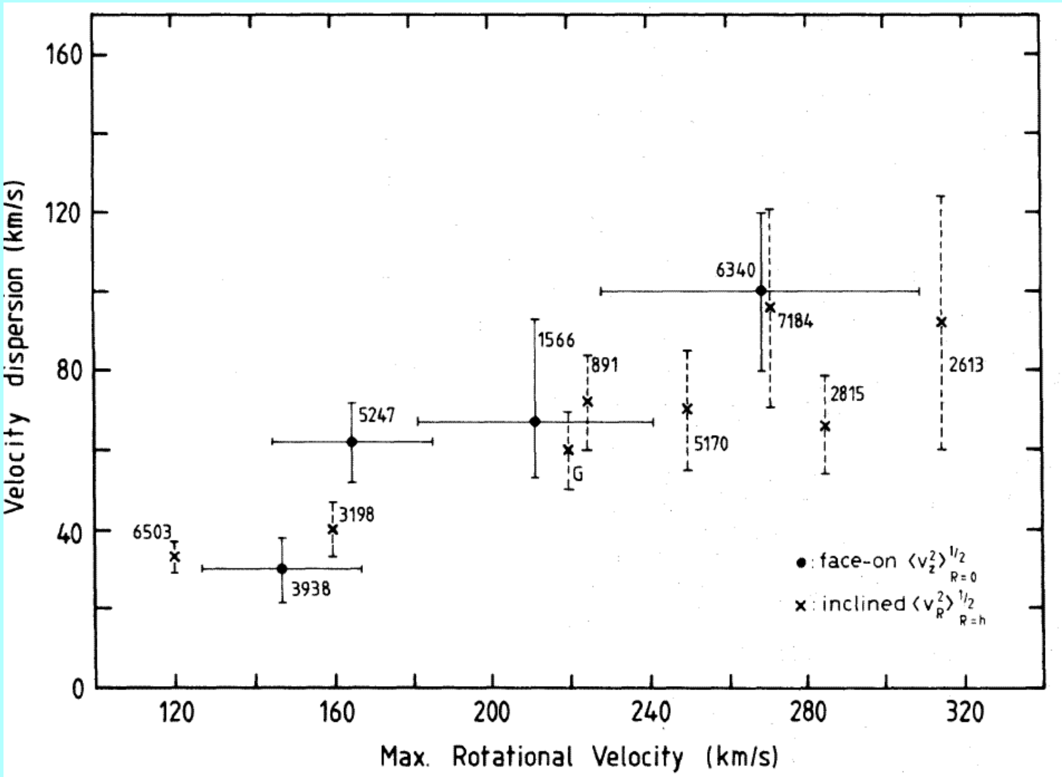
$$\langle V_{\rm R}^2 \rangle^{1/2}_{\rm R=h} = \langle V_{\rm z}^2 \rangle^{1/2}_{\rm R=0} = 0.29 V_{\rm rot} \ \ {\rm km/s}$$

*Ph.D. thesis (1995); Bottema, A.&A. 275, 16 (1993)


edge-on G
face-on 3938
edge-on 891
edge-on 5170
face-on 1566
face-on 5247
incl. 2613
face-on 6340
incl. 2815
incl. 6503
incl. 3198
incl. 7184
60
0
30
72
70
67
62
92
100
66
33
40
96
h
2h
3h
4h
Velocity Dispersion (km/s)
Radius



Velocity dispersion (km/s)
Absolute Luminosity, old disk only (B-mag)
6340
7184
1566
891
2815
2613
5247
5170
G
3198
6503
3938
•: face-on $\langle v_z^2\rangle^{1/2}_{R=0}$
×: inclined $\langle v_R^2\rangle^{1/2}_{R=h}$



Velocity dispersion (km/s)
Max. Rotational Velocity (km/s)
6340
7184
2613
1566
891
2815
5247
5170
G
3198
6503
3938
•: face-on $\langle v_z^2\rangle^{1/2}_{R=0}$
×: inclined $\langle v_R^2\rangle^{1/2}_{R=h}$

Can we understand these relations?

From the definition of Q we have

$$Q \propto \langle V_{\mathrm{R}}^2 \rangle^{1/2} \kappa \sigma^{-1}$$

For a flat rotation curve

$$\kappa \propto V_{\mathrm{rot}} R^{-1}$$

An exponential disk has

$$\sigma \propto \mu_\circ (M/L) \exp\left(-R/h\right)$$

Combining these equations gives

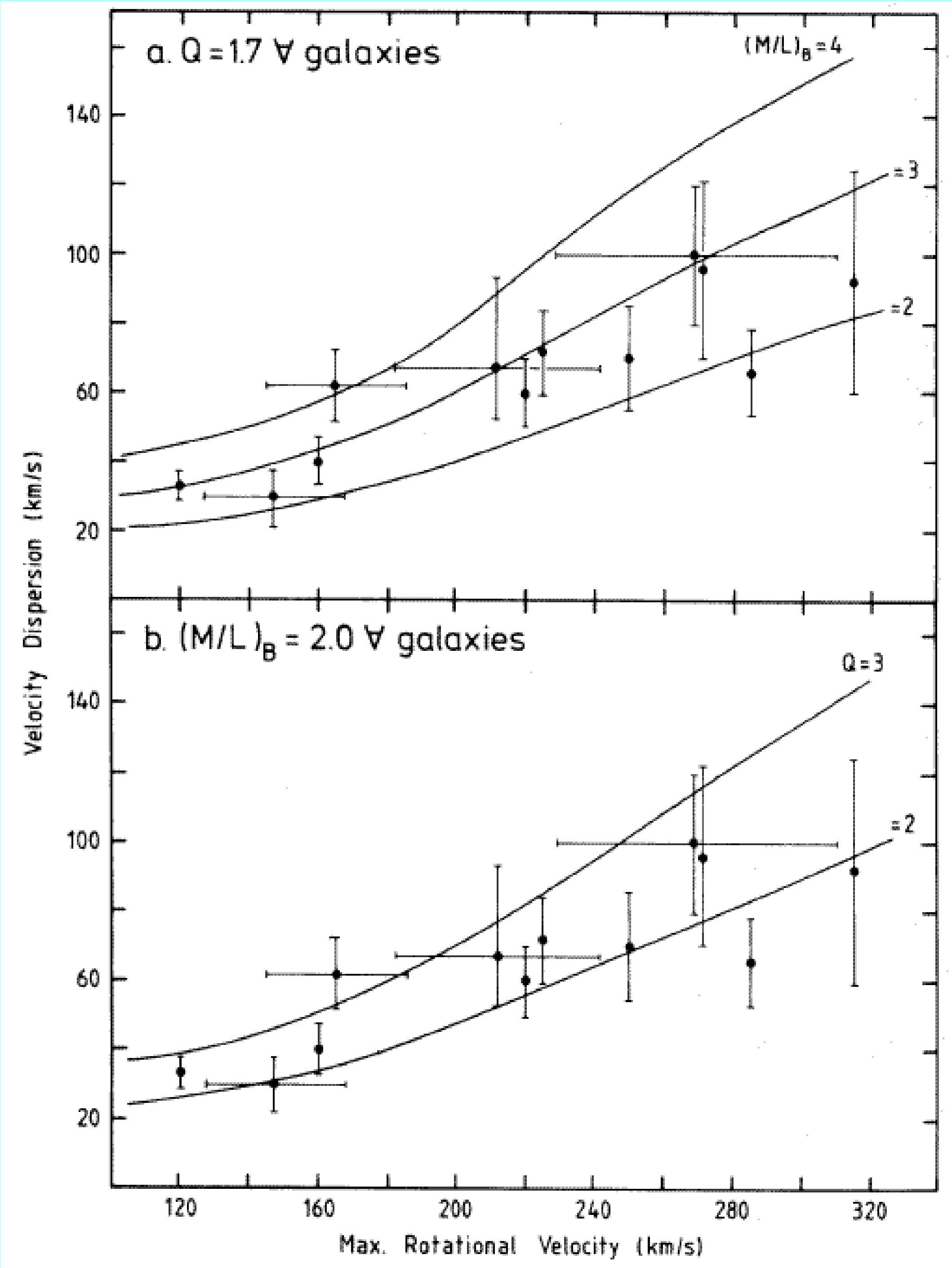
$$\langle V_{\mathrm{R}}^2 \rangle_{\mathrm{h}}^{1/2} \propto \mu_\circ (M/L) Q h V_{\mathrm{rot}}^{-1}$$

Now $L \propto \mu_\circ h^2$ and the Tully-Fisher relation gives $L \propto V_{\mathrm{rot}}^n$ with $n \approx 4$, so

$$\langle V_{\mathrm{R}}^2 \rangle_{\mathrm{h}}^{1/2} \propto \mu_\circ (M/L) Q V_{\mathrm{rot}} \propto \mu_\circ (M/L) Q L^{1/4}$$

So we expect that $\mu_\circ$, M/L and Q or at least their product are constant between disks.

With the actual observed central surface brightness the following curves result for either constant Toomre Q or constant M/L.



We had for hydrostatic equilibrium at the center

$$\langle V_{\mathrm{z}}^2 \rangle_{\mathrm{R=0}}^{1/2} = (2.3 \pm 0.1)\sqrt{G\sigma_\circ z_{\mathrm{e}}}$$

$\sigma_\circ$ is the central surface density and the range in the constant results from the choice of n.

The maximum rotation velocity of the exponential disk then is

$$_{\mathrm{disk}} = 0.88\sqrt{\pi G\sigma_\circ h} = (0.69 \pm 0.03)\langle V_{\mathrm{z}}^2 \rangle_{\mathrm{R=0}}^{1/2}\sqrt{\frac{h}{z_{\mathrm{e}}}}$$

With the Bottema relation between this central velocity dispersion and the maximum observed rotation velocity we get

$$\frac{V_{\mathrm{disk}}}{V_{\mathrm{rot}}} = (0.21 \pm 0.08)\sqrt{\frac{h}{z_{\mathrm{e}}}}$$

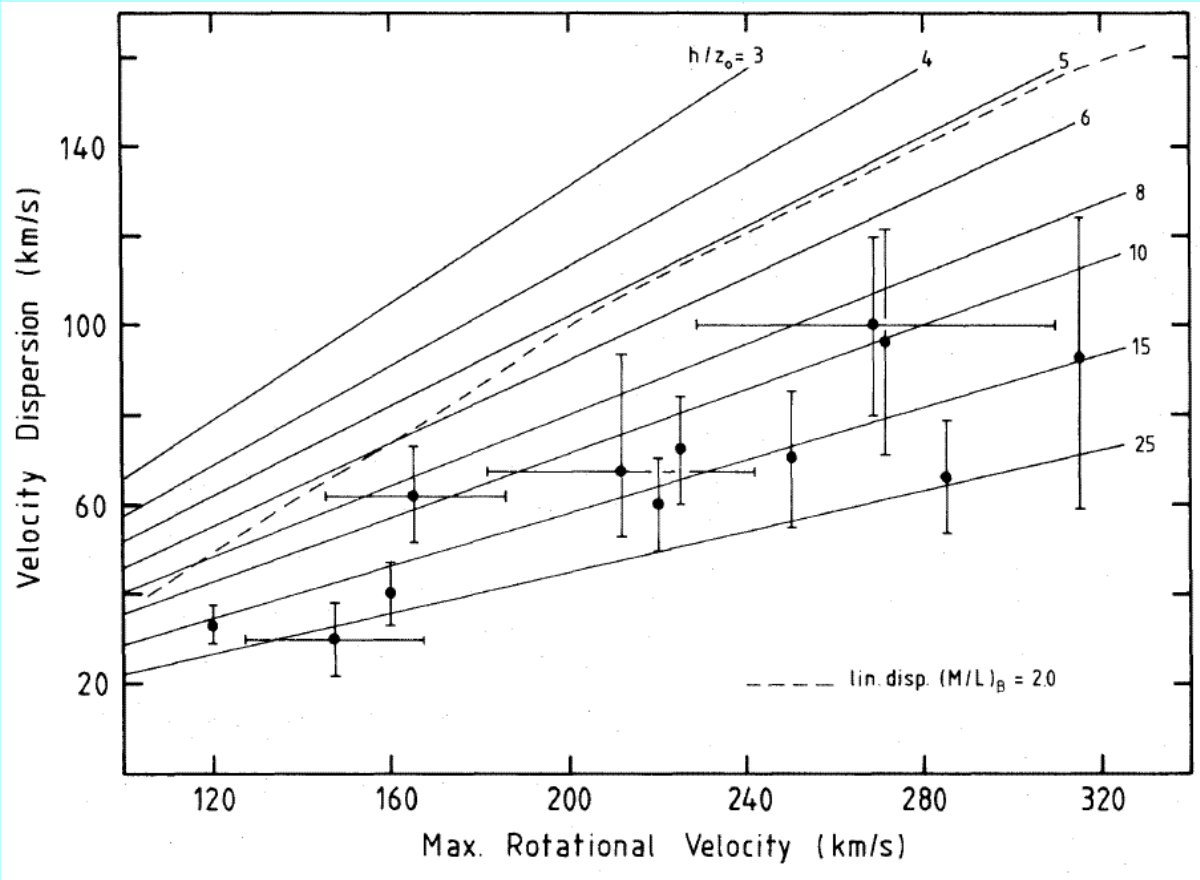
Analysis of a sample of edge-on galaxies gives for the ratio of scaleparameters 7.3 ± 2.2*, so that

$$\frac{V_{\mathrm{disk}}}{V_{\mathrm{rot}}} = (0.57 \pm 0.22)$$

So disks in general are not maximum disk.

*Kregel, van der Kruit & de Grijs, Mon.Not.R.A.S. 334, 646 (2002)

Bottema* first showed with this argument that his relations implied that for maximum disk situations the stellar disks should be much flatter than observed.



At that time observations indicated that the ratio $h/z_\circ$ (where the latter is for the isothermal sheet) was about 5 and this indicated a value of 0.63 for $V_{\rm disk}/V_{\rm rot}$.

*A.&A. 275, 16 (1993)

For a flat rotation curve we have

$$\kappa = 2\sqrt{B(B-A)} = \sqrt{2}\frac{V_{\rm rot}}{R}$$

From the definition of Q and applying at $R = h$ we get

$$\langle V_{\rm R}^2 \rangle_{\rm R=h}^{1/2} = \frac{3.36G}{\sqrt{2}} Q \frac{\sigma(R=h)h}{V_{\rm rot}}$$

Using hydrostatic equilibrium (also at $R = h$) gives*

$$\frac{\langle V_{\rm z}^2 \rangle^{1/2}}{\langle V_{\rm R}^2 \rangle^{1/2}} = \sqrt{\frac{(7.2 \pm 2.5)}{Q}\frac{z_{\rm e}}{h}}$$

In the solar neighborhoud this axis ratio of the velocity ellisoid is ~ 0.5† and for the Galaxy we have $z_{\rm e} \sim 0.35$ kpc and $h \sim 4$ kpc, so that

$$Q \sim 2.5.$$

Taking all data and methods together it is found that this applies in all galaxies; disks are locally stable according to the Toomre criterion.

*van der Kruit & de Grijs, A.&A. 352, 129 (1999)

†Dehnen & Binney, Mon.Not.R.A.S. 298, 387 (1998)

Swing amplification and global stability.

Swing amplification* of disturbances occurs as a result of the shear in rotating disks and turns these disturbances into growing trailing spiral waves.

It can be formulated in a criterion for prevention of this instability†

$$X = \frac{R\kappa^2}{2\pi G m \sigma(R)} \gtrsim 3$$

Here m is the number of spiral arms.

For a flat rotation curve this can be rewritten as

$$\frac{Q V_{\rm rot}}{\langle V_{\rm R}^2 \rangle^{1/2}} \gtrsim 3.97m$$

and with Bottema's relation it translates into

$$Q \gtrsim 1.1m$$

To prevent strong asymmetric $m = 1$ or bar-like $m = 2$ instabilities we require $Q \gtrsim 2$.

*Toomre, in a Cambridge conference on Structure and Evolution of Galaxies (1981)

†Sellwood, IAU Symp. 100, 197 (1983)

Numerical studies have indicated that disks with velocity dispersions as observed show global instabilities when evolving by themselves.

Disks can be stabilised by massive halos and therefore global stability requires that the disk mass has to be less than a certain fraction of the total mass, according to the criterion*

$$Y = V_{\mathrm{rot}} \left(\frac{h}{G M_{\mathrm{disk}}} \right)^{1/2} \gtrsim 1.1$$

This implies that within R_{max} the mass in the halo $M_{\mathrm{halo}} > 75\%$. This is also not true for maximum disk.

The criterion can be rewritten as

$$Y = 0.615 \left[\frac{Q R V_{\mathrm{rot}}}{h \langle V_{\mathrm{R}}^2 \rangle^{1/2}} \right]^{1/2} \exp\left(-\frac{R}{2h} \right) \gtrsim 1.1$$

Evaluating this at $R = h$ and using the Bottema relation gives

$$Q \gtrsim 2$$

*Efstathiou, Lake & Negroponte, Mon.Not.R.A.S. 199, 1069 (1982)