# Dynamics of Galaxies 2008-2009

Tuesday 13:15 – 15:00; Friday 13:15–15:00

Lectures will be given by **Piet van der Kruit (PCK)** on the basic isssues and application to observations of spiral and elliptical galaxies.

**Amina Helmi (AH)** will treat a the problem of modeling numerically the evolution of a (non-self-gravitating) satellite system orbiting in a spherical gravitational potential.

| | Tuesday | | Friday | |
|---|---|---|---|---|
| PCK | November 11 | Fundamentals | November 14 | Practical work |
| PCK | November 18 | Timescales; stellar orbits | November 21 | Practical work |
| AH | November 25 | Numerical orbit integration | November 28 | Practical work |
| PCK | December 2 | Motions, instabilities, velocity ellipsoid | December 5 | Practical work |
| PCK | December 9 | Self-consistency problem, potential theory | December 12 | Practical work |
| PCK | December 16 | Measurements of structure | December 19 | .. of kinematics |
| PCK | January 6 | Application to spirals | January 9 | .. to ellipticals |
| AH | January 13 | The satellite on orbit | January 16 | Practical work |

More informaton on the course is available at www.astro.rug.nl/∼vdkruit/#Dynamics of galaxies.

The presentations of the lectures by Piet van der Kruit are available as .pdf files at www.astro.rug.nl/∼vdkruit/jea3/homepage/dynamics0$n$.pdf, where 0$n$ runs from 01 to 08.

## Practical Work I

**1.** The **collisionless Boltzman equation** in cartesian coordinates is

$$u\frac{\partial f}{\partial x}+v\frac{\partial f}{\partial y}+w\frac{\partial f}{\partial z}-\frac{\partial \Phi}{\partial x}\frac{\partial f}{\partial u}-\frac{\partial \Phi}{\partial y}\frac{\partial f}{\partial v}-\frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial w}=0.$$

where the distributiom function is $f(x,y,z,u,v,w)$ and the potential $\Phi(x,y,z)$.

In cylindrical coordinates $(R,z,V_{\rm R},V_\theta,V_{\rm z})$ and this becomes

$$\dot{R}\frac{\partial f}{\partial R}+\dot{\theta}\frac{\partial f}{\partial \theta}+\dot{z}\frac{\partial f}{\partial z}+\dot{V_{\rm R}}\frac{\partial f}{\partial V_{\rm R}}+\dot{V_\theta}\frac{\partial f}{\partial V_\theta}+\dot{V_{\rm z}}\frac{\partial f}{\partial V_{\rm z}}=0.$$

Assume axial symetry. In the radial direction the acceleration comes from both the radial potential gradient and the centrifugal force and in the tangential direction we have conservation of angular momentum.

$$\ddot{R}-R\dot{\theta}^2=-\frac{\partial \Phi}{\partial R}\quad ; \quad \frac{d}{dt}\left(R^2\dot{\theta}\right)=0$$

Use this to show that the collisionless Boltzmann equation then is

$$V_{\rm R}\frac{\partial f}{\partial R}+V_{\rm z}\frac{\partial f}{\partial z}+\left(\frac{V_\theta^2}{R}-\frac{\partial \Phi}{\partial R}\right)\frac{\partial f}{\partial V_{\rm R}}-\frac{V_{\rm R}V_\theta}{R}\frac{\partial f}{\partial V_\theta}-\frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial V_{\rm z}}=0$$

Derive from this the moment or Jeans equation by multiplying by $V_{\rm R}$ and integrating over all velocities (write the integral of $f$ over all velocities as $\nu$)

$$\frac{\partial}{\partial R}(\nu\langle V_{\rm R}^2\rangle)+\frac{\nu}{R}\{\langle V_{\rm R}^2\rangle-V_{\rm t}^2-\langle(V_\theta-V_{\rm t})^2\rangle\}+\frac{\partial}{\partial z}(\nu\langle V_{\rm R}V_{\rm z}\rangle)=\nu K_{\rm R}$$

**2.** The **second-order moment of the Boltzman equation** is

$$\frac{\partial}{\partial t}\left(\rho\bar{v}_j\right)+\frac{\partial}{\partial x_i}\left(\rho\overline{v_iv_j}\right)+\rho\frac{\partial \Phi}{\partial x_j}=0$$

The first order moment over spatial coordinates then is

$$\int x_k\frac{\partial\left(\rho\bar{v}_j\right)}{\partial t}d^3x=-\int x_k\frac{\partial}{\partial x_i}\left(\rho\overline{v_iv_j}\right)d^3x-\int x_k\rho\frac{\partial \Phi}{\partial x_j}d^3x$$

Define

| | |
|---|---|
| moment of inertia tensor | $I_{jk}=\int\rho x_jx_kd^3x$ |
| kinetic energy tensor | $K_{kj}=\frac{1}{2}\int\rho\overline{v_kv_j}d^3x$ |
| motions tensor | $T_{jk}=\int\rho\bar{v}_i.\bar{v}_jd^3x$ |
| velocity dispersion tensor | $\Pi_{jk}=\int\rho\sigma_{ij}^2d^3x$ |
| potential energy tensor | $W_{jk}=-\int x_j\frac{\partial\Phi}{\partial x_k}d^3x$ |

Show that this implies the virial equation

$$\tfrac{1}{2}\frac{d^2}{dt^2}I_{ij}=2K_{ij}+W_{ij}=2T_{ij}+\Pi_{ij}+W_{ij}$$

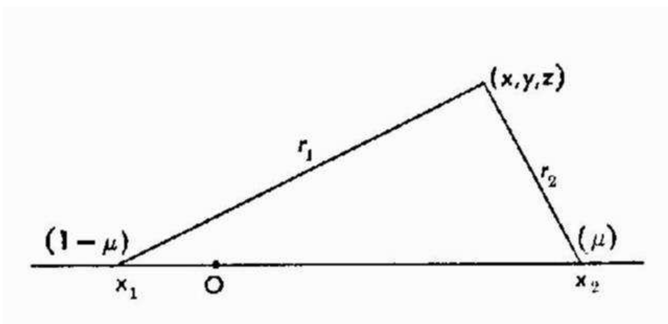
# Practical Work II

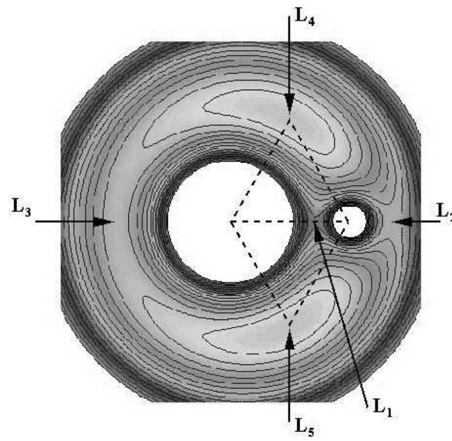
## The restricted three-body problem

This subject is important to understand the dynamics of stars in the case of a non-axisymmetrci potential such as a bar. However, we do a much simpler porblem, namely that of the restricted three-body problem, shich studies the equilibria in teh case of two primary masses orbiting each other in circular orbit and a thrid bvody with negligible mass.

The full treatment is in a presentation that is available through my homepage at: www.astro.rug.nl/∼vdkruit/jea3/homepage/three-body.pdf.

I will do give the presentation up to page 34. Briefly, it is shown that in a co-rotating frame one can identify tive *Lagrange libration points*, three on the axis through the two primary masses and two on equilateral trangles with it.




The stability of the points is treated as follows:
The equations of motion are

$$\ddot{x} - 2\dot{y} = -\frac{\partial\phi}{\partial x} \quad ; \quad \ddot{y} + 2\dot{x} = -\frac{\partial\phi}{\partial y} \quad ; \quad \ddot{z} = -\frac{\partial\phi}{\partial z}$$

We take coordinates

$$x = x_\circ + \xi \quad ; \quad y = y_\circ + \eta \quad ; \quad z = z_\circ + \zeta$$

Then

$$\begin{aligned}
\ddot{\xi} - 2\dot{\eta} &= -\xi\Phi_{\rm xx} - \eta\Phi_{\rm xy} - \zeta\Phi_{\rm xz} \\
\ddot{\eta} + 2\dot{\xi} &= -\xi\Phi_{\rm yx} - \eta\Phi_{\rm yy} - \zeta\Phi_{\rm yz} \\
\ddot{\zeta} \qquad &= -\xi\Phi_{\rm zx} - \eta\Phi_{\rm zy} - \zeta\Phi_{\rm zz}
\end{aligned}$$

with $\Phi_{\rm xy} = \partial^2\Phi/\partial x\partial y$, etc.
We have

$$\Phi = \tfrac{1}{2}(x^2 + y^2) + \frac{1-\mu}{\sqrt{(x-x_1)^2 + y^2 + z^2}} + \frac{\mu}{\sqrt{(x-x_2)^2 + y^2 + z^2}}$$

Now define

$$\alpha = \frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3} \quad ; \quad \beta = \frac{1-\mu}{r_1^5} + \frac{\mu}{r_2^5}$$

This gives

$$\Phi_{\rm xx} = -1 + \alpha - 3(1-\mu)\frac{(x-x_1)^2}{r_1^5} - 3\mu\frac{(x-x_2)^2}{r_2^5}$$

$$\Phi_{\rm yy} = -1 + \alpha - 3y^2\beta \quad ; \quad \Phi_{\rm zz} = \alpha - 3z^2\beta$$

$$\Phi_{\rm xy} = \Phi_{\rm yx} = -3xy\beta \quad ; \quad \Phi_{\rm xz} = \Phi_{\rm zx} = -3zx\beta \quad ; \quad \Phi_{\rm yz} = \Phi_{\rm zy} = -3yz\beta$$

In the presentation I treated the stability for the points on the $x$-axis:
$y = z = 0$. Write $x = x_\circ$ so that $r_1^2 = (x_\circ - x_1)^2$ and $r_2^2 = (x_\circ - x_2)^2$, then

$$\Phi_{\rm xx} = -1 - 2\alpha \quad ; \quad \Phi_{\rm yy} = -1 + \alpha \quad ; \quad \Phi_{\rm zz} = \alpha$$

$$\Phi_{\rm xy} = \Phi_{\rm yx} = \Phi_{\rm xz} = \Phi_{\rm zx} = \Phi_{\rm yz} = \Phi_{\rm zy} = 0$$

Then the equations of motion are

$$\ddot{\xi} - 2\dot{\eta} = \xi(1+2\alpha) \tag{1}$$

$$\ddot{\eta} + 2\dot{\xi} = \eta(1-\alpha) \tag{2}$$

$$\ddot{\zeta} \qquad = -\zeta\alpha \tag{3}$$

Eqn. (3) is easily solved; it gives $\zeta \propto e^{\sqrt{-\alpha}t} = e^{i\sqrt{\alpha}t}$. Now $\alpha > 0$, so $\sqrt{\alpha}$ is imaginary. Remembering that $e^{(a+ib)t} = e^{at}(\cos bt + i \sin bt)$ we see that *if and only if* the exponent is fully imaginary (or $a = 0$) we will have an oscillating solution.
This is the case, so we have a harmonic oscillation and these libration points are **stable** in the $z$-direction.

Say the solutions in the ($x$,$y$)-plane are $\xi = Ke^{\lambda t}$ and $\eta = Le^{\lambda t}$. Substitution, using $\dot{\xi} = \lambda Ke^{\lambda t}$, $\ddot{\xi} = \lambda^2 Ke^{\lambda t}$, etc. in eqn. (1) and (2) gives

$$K\lambda^2 - 2L\lambda = K(1+2\alpha) \quad ; \quad L\lambda^2 + 2K\lambda = L(1-\alpha)$$

Eliminate $K$ and $L$:

$$\frac{K}{L} = \frac{2\lambda}{\lambda^2 - (1+2\alpha)} = \frac{\lambda^2 - (1-\alpha)}{-2\lambda}$$

$$\lambda^4 + (2-\alpha)\lambda^2 + (1+2\alpha)(1-\alpha) = 0$$

Regard this as a quadratic polynomial equation in $\lambda^2$. We need for stability that $\lambda$ is purely imaginary so the two roots for $\lambda^2$ should both be real and negative.
Then for their product[1] we should have $(1+2\alpha)(1-\alpha) > 0$, or $(1-\alpha) > 0$.

Go back to the energy integral equation

$$x_\circ - (1-\mu)\frac{x-x_1}{|x-x_1|^3} - \mu\frac{x-x_2}{|x-x_2|^3} = 0$$

With the definition of $\alpha$ we can write

$$x_\circ(1-\alpha) + (1+\mu)\frac{x_1}{r_1^3} + \mu\frac{x_2}{r_2^3} = 0$$

[1]For $ax^2 + bx + c = 0$ the product of the roots is $c/a$.

With $x_1 = \mu$ and $x_2 = 1 - \mu$, this becomes

$$(1-\alpha) = \frac{\mu(1-\mu)}{x_\circ}\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right)$$

Now we have in the cases of the three points on the $x$-axis

L$_1$: $x_\circ > x_2 > 0$ and $r_1 > r_2 \Rightarrow (1-\alpha) < 0$
L$_2$: $0 < x_\circ < x_2$ and $r_1 > r_2 \Rightarrow (1-\alpha) < 0$
L$_3$: $x_\circ < x_1 < 0$ and $r_1 < r_2 \Rightarrow (1-\alpha) < 0$

Then we only have real solutions for $\lambda$. So all three Lagrangian points on the $x$-axis are **unstable**.

The practical work is to finalise the problem by doing the same analysis for the triangular points which have $r_1 = r_2 = 1$ and therefore

$$x = \tfrac{1}{2}(1-2\mu) \quad ; \quad y = \pm\frac{\sqrt{3}}{2} \quad ; z = 0$$

I will use the last 15 minutes to complete the presentation and show what you should have found.

# Practical Work III

## Orbits in the Galactic disk and the third integral problem

This excescise concerns the problem that orbits of stars in the disk of the Galaxy appeared to have an third isolating integral and the work we will study has been an important step towards the description of dynamics using so-called Stäckel potentials, that I will discuss in Lecture 4.

For this you will need to download some early papers on the subject, namely

- *The third integral of motion for low-velocity stars* by H.C. van de Hulst,
  Bull. Astr. Inst. Neth., Vol. 16, p. 235 (1962)
  ADS: adsabs.harvard.edu/abs/1962BAN....16..235V

- *Three-dimensional galactic stellar orbits* by A. Ollongren,
  Bull. Astr. Inst. Neth., Vol. 16, p. 241 (1962).
  ADS: adsabs.harvard.edu/abs/1962BAN....16..241O.
  This is a long paper (actually it is Ollongren's Ph.D. thesis); you don't have to print it completely, since you need only section 18a and 18b (pages 277 – 286) and page 255.

- *Theory of Stellar Orbits in the Galaxy* by A. Ollongren,
  Ann. Rev. Astron. Astroph., vol. 3, p. 113 (1965)
  Available through the ADS at: adsabs.harvard.edu/abs/1965ARA%26A...3..113O.

First you should read the section *Particle orbit theory in stellar dynamics* in the review of Ollongren (1965) on pages 113 –120. Most of this should be familiar from my lectures. It is not necessary to try and redo the full algebra of the derivations of the equations; just see if you can follow the arguments in the exposition.

Before we continue we look at the case of orbits with very small amplitudes. First remember from Lecture 1:
In the case of the Galaxy near the plane (at small $z$) the potential is separable and the $R$- and $z$-motions will then be decoupled

$$\Phi(R, z) = \Phi_1(R) + \Phi_2(z)$$

Then the decoupled $z$-energy is a third integral of motion:

$$I_3 = \tfrac{1}{2}V_{\mathrm{z}}^2 + \Phi_2(z)$$

So far Lecture 1. Now we look at the case

$$\Phi(x, y) = \Phi(0, 0) + \tfrac{1}{2}Px^2 + \tfrac{1}{2}Qx^2$$

The equations of motion then are separated in $x$ and $y$

$$\frac{d^2x}{dt^2} = -Px \quad ; \quad \frac{d^2y}{dt^2} = -Qy$$

These equations can be solved independently:

$$x = A\cos(\sqrt{P}t + a) \quad ; \quad y = B\cos(\sqrt{Q}t + b) \tag{4}$$

What are the expressions for $V_{\mathrm{x}}$ and $V_{\mathrm{y}}$? The energy in the $x$- and $y$-direction are integrals of motion; their values depend of course on $A$ and $B$; express them in $A$, $B$, $P$ and $Q$. What is the total energy?

The solution (4) we have here is similar to the case of the two-dimensional harmonic oscillator that I used in Lecture 1 to illustrate the concept of isolating integrals. What form do the zero-velocity curves have. Make a sketch for the case that (in arbitrary units) $P = 1$ and $Q = 2$ (for simplicity and without loss of generality, take $\Phi(0,0) = 0$) and $A = 6$, $B = 2$. Also draw the rectangle where the star can go on the basis of the solutions (5). Compare to fig. 5 of Ollongren (1962).

Then have a look at sections 18a and 18b of the thesis of Ollongren (1962) on a description of his calculated orbits, in particular make yourself familiar with the orbits that he found by looking carefully at the figures.

The final aim of all this is that I want you to understand the first three sections of the paper by Henk van de Hulst (1962)[2]. Again you don't have to do the full algebra of derivation of the equations; I just want you to understand the basic line of the argument.

In the days of this paper it was normal usage to denote the radial coordinate that we now usually denote by $R$, with the symbol $\varpi$; it was pronounced as 'curly pi' or sometimes 'pomega'. My thesis supervisor Prof. Oort always used it. This peculiar custom has disappeared and I don't regret it.

First familiarise yourself with *elliptic coordinates*; look at fig. 1 and determine where the foci are (what are the numerical values of $k$ and $c$?). Note that $\xi$ and $\eta$ are dimensionless; dimensions come in via eq. (2). In case you don't have experience with hyperbolic functions, here are some of their properties (I prefer to write 'sinh' rather than 'sh' as van de Hulst does):

$$\sinh z = \frac{e^z - e^{-z}}{2} \quad ; \quad \cosh z = \frac{e^z + e^{-z}}{2} \quad ; \quad \tanh z = \frac{\sinh z}{\cosh z}$$

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + ..... \quad ; \quad \cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + .....$$

Just for completeness: equivalently we have for geometric functions

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - ..... \quad ; \quad \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - .....$$

Can you follow how van de Hulst arrived at eq. (5) (even without doing the full algebra of the derivation)? If you followed Ollongren (1962), can you see how Ollongren's eq. (14) corresponds to eq. (6) of van de Hulst? Can you see how van de Hulst arrived at eq. (8)?

Van de Hulst uses a particular form for the potential, which is presented in his eq. (7):

$$2\Phi(x,y) = Px^2 + Qy^2 - \frac{2a}{3}x^3 - sbxy^2 - dx^2y^2 \tag{5}$$

Why are there not terms that are linear in $x$ or $y$? Note that this is the case we did above but now extended with higher order terms.

You can skip the rest of this section, except noting that eq. (13) as a result of his assumptions.

Then we proceed to section 3 in the paper of van de Hulst. Eq. (14) should be obvious. Do you understand why eq. (15) arises and how you get eq. (16)? For this look again at eq.

[2]Of course if you wish you may also read section 4 on the oscillating periods of that paper, but that is more involved.

(2) and then follow van de Hulst' remark below eq. (16) that you can understand eq. (15) by adding the two and applying eq. (6).

Read the rest of this section to understand how $\alpha$ defines a boundary curve and $\beta$ a box within that, just as we had for the case above of small amplitudes of trajectories. Now the potential has been separated in elliptic coordinates and the orbit can be described by a two-dimensional oscillator in this new coordinate system.

Finally have a look at fig. 31 in Ollongren (1962). This has all the envelopes of his orbits (see the previous figures). Can we use the theory of van de Hulst to reproduce this? For this try to estimate where the foci of the elliptic coordinate system should be (what are the approximate values of $c$ and $k$?). Draw the elliptic coordinate system for the value of $c$ you select (or some other nearby values) for the range $\xi = 0$ (0.25) 1.5; $\nu = 0$ (0.25) 1.5 and plot the coordinate system on the scqle of fig. 31. For this I will provide you with a version of the figure on which 1 kpc is precisely 3 cm.

You might want to consult a further paper of Alex Ollongren:

- *Construction of galactic stellar orbits similar to harmonic oscillators. I.* by A. Ollongren, Astron. J., Vol. 72, p. 436 (1967)
  ADS: adsabs.harvard.edu/abs/1967AJ.....72..436O

# Practical Work IV

## Rotation curve of an exponential disk

Download the following papers using ADS:

- *On the distribution of matter within highly flattened galaxies* by Alar Toomre, Ap.J. 138, 385 (1963)
- *On the disks of spiral and S0 galaxies* by K.C. Freeman, Ap.J. 160, 811 (1970)
- *Rotation curve of the edge-on spiral galaxy NGC 5907: disc and halo masses* by Stefano Casertano, Mon. Not. R.A.S. 203, 735 (1983)

For the rest of this exercise we need an equation that can give us the rotation curve of a disk in a galaxy when the density distribution is known. Look for this at the three papers (without trying to reproduce the algebra) and summarize for yourself in general terms what these papers addressed and what the results are. Compare equations (9) of Toomre, eq.s (8) and (9) of Freeman and eq. (4) of Casertano. We will continue with the last equation (of Casertano).

The vertical density distribution in a stellar disk can conveniently be approximated by an exponential function (although the isothermal sheet or the sech-function are more realistic representations) and the truncation with a change in radial scalelength

$$\rho(R,z) = \rho_\circ \exp\left(-R/h\right) \exp\left(-|z|/z_\circ\right) \quad \text{for} \quad R < R_{\max}$$

$$\rho(R,z) = \rho'_\circ \exp\left(-R/\alpha h\right) \exp\left(-|z|/z_\circ\right) \quad \text{for} \quad R > R_{\max}$$

We require $\alpha < 1$ for truncations and $\rho'_\circ$ has to be such that the two parts of the profile join at $R_{\max}$. So, if we write $\rho'_\circ = C\rho_\circ$, what is $C$? This is similar but somewhat different from the approach of Casertano; see his equations (2) and (3).

How would you write a code to investigate the effect of varying values for $z_\circ$, $R_{\max}$ and $\alpha$? In other words, what are the steps in such a code, what additional information do you need to collect, how do you treat the elliptical integrals, how would you perform the integrations, etc.?

If you actually do write such a code (which you may of course but do not have to) you could reproduce the results of Casertano in his fig.'s 2 and 3 and supplement it by studying the effect of the sharpness of the truncation by taking different values for $\alpha$.

Answer in any case the following question: Why do the curves for the cases of truncations tend towards Keplerian curves more quickly and why is there a rise in the rotation curve just before the truncation with respect to untruncated disks?