DYNAMICS OF GALAXIES

4. The self-consistency problem and potential theory

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The self-consistency problem

Ideally, one would like to construct self-consistent, self-gravitating models for galaxies, by solving the two coupled, fundamental equations:

$$u\frac{\partial f}{\partial x}+v\frac{\partial f}{\partial y}+w\frac{\partial f}{\partial z}-\frac{\partial \Phi}{\partial x}\frac{\partial f}{\partial u}-\frac{\partial \Phi}{\partial y}\frac{\partial f}{\partial v}-\frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial w}=0.$$

and

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}\equiv\nabla^2\Phi=4\pi G\rho(x,y,z)$$

Unfortunately, in general this is not possible.

There are two possible apporaches:

- The direct method. Assume a potential Φ on the basis of the density distribution, inferred from observations. Then use the observed kinematics to derive further properties of the distribution function.
- The inverse method. Make a guess for the dependence of the distribution function on the isolating integrals and calculate the density, potential, motions and velocity distrubutions.

The direct approach is straightforward in e.g. the case of the vertical distributions in a galactic disk (where it reduces to a one-dimensional treatment).

The inverse method makes use of functional solutions of well-defined cases, such a isothermal models.

First we turn to the direct method.

The Schwarzschild method

Schwarzschild[1] proceeds as follows:

- Choose a density distribution for the system you want to model.
- Solve Poission's equation (usually numerically).
- Compute a library (many hundreds) of orbits in this potential and calculate the density distribution that each orbit generates.
- Add these with appropriate weights to recover the density distribution started from (usually this involves "linear or quadratic programming").

[1] M. Schwarzschild, Ap.J. 232, 236 (1979)

Often it is possible to use constraints as the observations of the kinematics of the stars, i.e. their motions and velocity dispersions.

There is uncertainty whether any outcome is unique.

But it is an extremely powerful approach.

Isothermal solutions and related results

For simple geometries full semi-analytical solutions for the distribution function to the set of two fundamental equations can be obtained.

These solutions refer to self-gravitating systems, which means that ρ and ν are the same.

Examples are spherical density distributions or density distributions on stratified layers with isothermal velocity distributions (equal velocity dispersions at all positions),

Isothermal sphere and King models

The Poisson equations for spherical symmetry was

$$\frac{1}{R^2}\frac{\partial}{\partial R}(R^2 K_{\rm R}) = -4\pi G\rho(R)$$

and the Jeans equation

$$\frac{\partial}{\partial R}(\nu\langle V_{\rm R}^2\rangle) + \frac{\nu}{R}\{2\langle V_{\rm R}^2\rangle - V_{\rm t}^2 - \langle (V_\theta - V_{\rm t})^2\rangle - \langle V_\phi^2\rangle\} = \nu K_{\rm R}$$

If the velocity distribution is isotropic and if there is no rotation this reduces to

$$\langle V^2\rangle \frac{\partial \rho}{\partial R} = \rho K_{\rm R}$$

Here V is the radial velocity.

The equations can be combined to give

$$\frac{\langle V^2 \rangle}{R^2} \frac{\partial}{\partial R} \left(R^2 \frac{\partial \ln \rho}{\partial R} \right) = -4\pi G \rho$$

The solution is

$$\rho(R) = \frac{\langle V^2 \rangle}{2\pi G} R^{-2}$$

This is called the singular isothermal sphere, since the density is infinite at the center.

Note that we have not constrained the functional form of the velocity distribution.

A well-behaved solution is obtained by assuming that the velocity distribution is Gaussian.

There is in this spherical, non-rotating case only one isolating integral of motion, namely the energy E.

According to Jeans' theorem then the distribution function is only a function E.

So take the distribution function to be

$$f(E) = \mathrm{const.} \times \mathrm{e}^{-E/\langle V^2 \rangle}$$

With $E = -\Phi + \frac{1}{2}V^2$ integration over all V gives

$$\rho(R) = \rho(0)\mathrm{e}^{-\Phi(R)/\langle V^2\rangle}$$

Now set the boundary conditions $\rho(0) = \rho_\circ$ and $(d\rho/dR)_{\mathrm{z}=0} = 0$. Then the solution

$$\rho(R) = \rho_0\mathrm{e}^{-\Phi}$$

can be found from a numerical integration where Φ follows from

$$\mathrm{e}^{-\Phi} = \frac{1}{\chi^2}\frac{d}{d\chi}\left(\chi^2\frac{d\Phi}{d\chi}\right) \quad ; \quad \chi = \left(\frac{\langle V^2\rangle}{4\pi G\rho_0}\right)^{1/2} R$$

For large R this becomes

$$\rho(R) = \frac{\langle V^2 \rangle}{2\pi G} R^{-2}$$

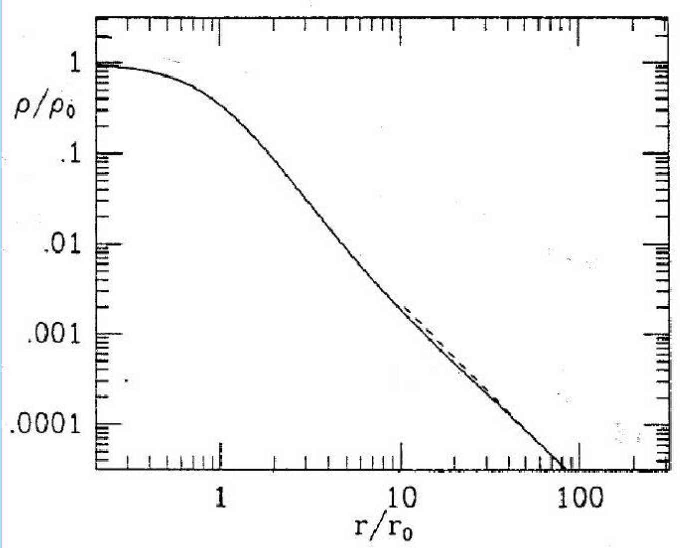
and thus approaches the singular isothermal sphere.

This solution has a natural length-scale that is called the core radius (also King radius)

$$R_0 = \left(\frac{4\pi G \rho_0}{9 \langle V^2 \rangle} \right)^{-1/2}$$

At this core radius the projected surface density is roughly half the central one.

The next slides show the density distribution and the logarithmic density slope.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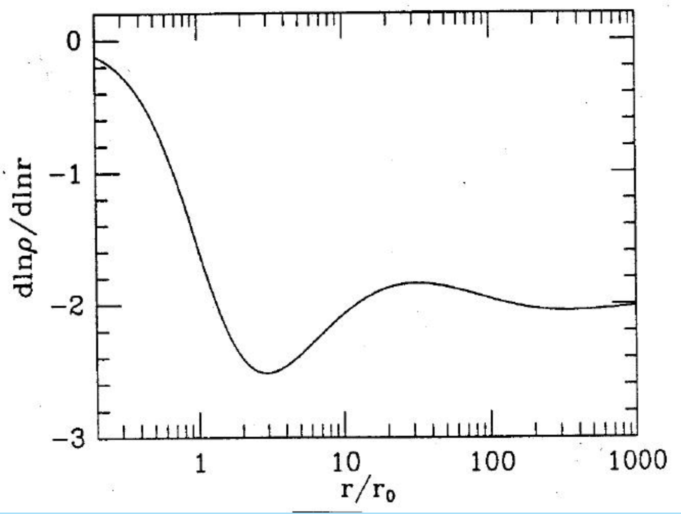
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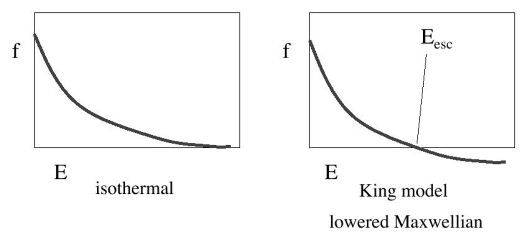
King models

King models are adapted isothermal spheres with a tidal radius R_t and a corresponding upper boundary in the velocity distribution.



The distribution function is

$$f(E) = \mathrm{const.}\left[\mathrm{e}^{-E/\langle V^2\rangle} - \mathrm{e}^{-E_\mathrm{esc}/\langle V^2\rangle}\right] \quad \mathrm{for}\ E < E_\mathrm{esc}$$
$$0 \quad \mathrm{for}\ E > E_\mathrm{esc}$$

Using again $E = -\Phi + \frac{1}{2}V^2$ and defining the zero-point of Φ such that $E_{\rm esc} = 0$ we may write this as

$$f(E) = \mathrm{const.} \left[\mathrm{e}^{-E/\langle V^2 \rangle} - 1 \right] \quad \text{for } E > 0$$

Integrating over all velocities then gives

$$\rho(R) = \rho_\circ \left[e^{\Phi(R)/\langle V^2 \rangle} \mathrm{erf} \left(\sqrt{\frac{\Phi}{\langle V^2 \rangle}} \right) - \sqrt{\frac{4\phi}{\pi \langle V^2 \rangle}} \left(1 + \frac{2\Phi}{3 \langle V^2 \rangle} \right) \right]$$

Here erf is the Error Function.

Then we get

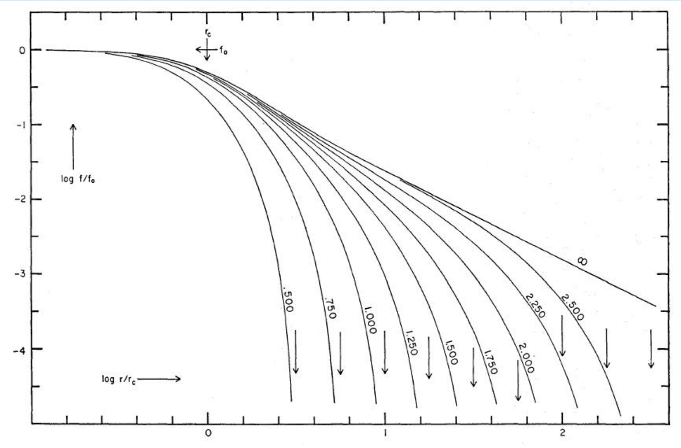
$$\frac{d}{dR}\left(R^2\frac{d\Phi}{dR}\right) = -4\pi G\rho_\circ R^2\left[e^{\Phi(R)/\langle V^2\rangle}\mathrm{erf}\left(\sqrt{\frac{\Phi}{\langle V^2\rangle}}\right) - \sqrt{\frac{4\phi}{\pi\langle V^2\rangle}}\left(1+\frac{2\Phi}{3\langle V^2\rangle}\right)\right]$$

This again has to be numerically integrated from the center outwards.

At the tidal radius R_t the density drops to zero.

The ratio $c = \log(R_\mathrm{t}/R_\circ)$ is called the concentration.

Here are some models in projected surface density[2].



[2] I.R. King, A.J. 71, 64 (1966)

The total mass is

$$M(R_{\rm t}) = \frac{2}{G}\langle V^2\rangle r_\circ \; f\left(\frac{R_{\rm t}}{R_\circ}\right)$$

and the central surface density

$$\sigma_\circ = \rho_0 r_0 \; g\left(\frac{R_{\rm t}}{R_\circ}\right)$$

The functions f and g can only be calculated numerically and are given in the literature. The velocity dispersion is

$$\langle V^2\rangle^{1/2} \propto \frac{\rho_\circ M(R_{\rm t})}{f\left(R_{\rm t}/R_\circ\right) g\left(R_{\rm t}/R_\circ\right)}$$

King models are useful to describe globular clusters and to some extent elliptical galaxies.

Isothermal sheet and other vertical distributions

For a self-gravitating isothermal sheet the basic equations become

$$\frac{\partial K_{\rm z}}{\partial z} = -4\pi G\rho(z)$$

and

$$\langle W^2\rangle \frac{\partial \nu}{\partial z} = \nu K_{\rm z}$$

The two basic equations can be combined into

$$-4\pi G\rho(z) = \langle W^2\rangle \frac{d^2}{dz^2}\left\{\ln\frac{\rho(z)}{\rho(0)}\right\}$$

The solution is

$$\rho(z) = \frac{\langle W^2 \rangle}{2\pi G z_0^2} \,\mathrm{sech}^2 \left(\frac{z}{z_\circ} \right)$$

The corresponding surface density is

$$\sigma = 2 z_0 \rho_\circ$$

and the relation to the velocity dispersion

$$\langle W^2 \rangle = \pi G \sigma z_\circ$$

The vertical force results from integration of Poisson's equation as

$$K_{\mathrm{z}} = -2 \frac{\langle W^2 \rangle}{z_\circ} \tanh \left(\frac{z}{z_\circ} \right)$$

Usefull approximations are

$$\mathrm{sech}^2\left(\frac{z}{z_0}\right) = \exp\left(-\frac{z^2}{z_0^2}\right) \qquad \text{for} \quad z \ll z_0$$

$$\mathrm{sech}^2\left(\frac{z}{z_0}\right) = 4\exp\left(-\frac{2z}{z_0}\right) \qquad \text{for} \quad z \gg z_0$$

The isothermal sheet is used to describe vertical distributions in stellar disks.[3]

[3] P.C. van der Kruit & L. Searle, A.&A. 95, 105 (1981)

For a second isothermal component of negligible mass and different velocity dispersion in this force-field we find

$$\rho_{\mathrm{II}}(z) = \rho_{\mathrm{II}}(0)\mathrm{sech}^{2p}\left(\frac{z}{z_\circ}\right)$$

where

$$p = \frac{\langle W^2\rangle}{\langle W^2\rangle_{\mathrm{II}}}$$

An application of this is for example the HI-gas layer inside a stellar disk that contains most of the surface density.

Exponential and sech-distributions

The isothermal sheet is only an approximate description of the vertical distribution of stars in disks of galaxies. There is a range of generations of stars, each with their own velocity dispersion.

Often used is the exponential distribution, since it is a convenient fitting function.

Since the velocity dispersion now varies with z we have to write the equation in terms of the velocity dispersion in he plane $\langle W^2\rangle_\circ^{1/2}$. The equations corresponding to this case are[4]:

$$\rho(z) = \frac{\langle W^2\rangle_\circ}{2\pi G Z_{\mathrm{e}}^2} \exp\left(-\frac{z}{z_{\mathrm{e}}}\right)$$

[4]P.C. van der Kruit, A.&A., 192, 117 (1988)

$$\sigma = 2 z_{\mathrm{e}} \rho_{\circ}$$

$$\langle W^2 \rangle_{\circ} = \pi G \sigma z_{\mathrm{e}}$$

$$K_{\mathrm{z}} = -2\pi G \sigma \left\{ 1 - \exp\left(-\frac{z}{z_{\mathrm{e}}} \right) \right\}$$

If an isothermal component of negligible mass moves in this force field, then

$$\rho_{\mathrm{II}}(z) = \rho_{\mathrm{II}}(0) \exp\left[-\frac{2pz}{z_{\mathrm{e}}} + 2p \left\{ 1 - \exp\left(-\frac{z}{z_{\mathrm{e}}} \right) \right\} \right]$$

where now

$$p = \frac{\langle W^2 \rangle_0}{\langle W^2 \rangle_{\mathrm{II}}}$$

As an intermediate case between the isothermal solution and the exponetial it is also possible to use the sech-distribution[5].

This corresponds probably closest to reality. The equations then are:

$$\rho(z) = \frac{2\langle W^2\rangle_{\rm II}}{\pi^3 G z_{\rm e}^2}\,\mathrm{sech}\left(\frac{z}{z_{\rm e}}\right)$$

$$\sigma = \pi \rho_\circ z_{\rm e}$$

$$\langle W^2\rangle_{0\circ} = \frac{\pi^2}{2} G\sigma z_{\rm e}$$

$$K_{\rm z} = -4G\sigma \arctan\left\{\sinh\left(\frac{z}{z_{\rm e}}\right)\right\}$$

[5] P.C. van der Kruit, A.&A. 192, 127 (1988)

For the second isothermal component we now get

$$\rho_{\mathrm{II}}(z) = \rho_{\mathrm{II}}(0)\mathrm{exp}\left\{ -\frac{8}{\pi^2} pI\left(\frac{z}{z_{\mathrm{e}}}\right)\right\}$$

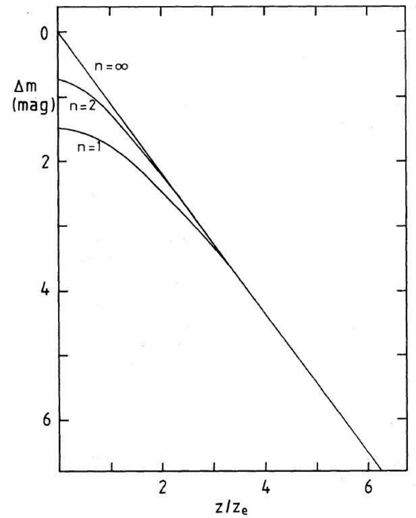
where

$$I(y) = \int_0^y \mathrm{arctan}(\mathrm{sinh}x)dx$$

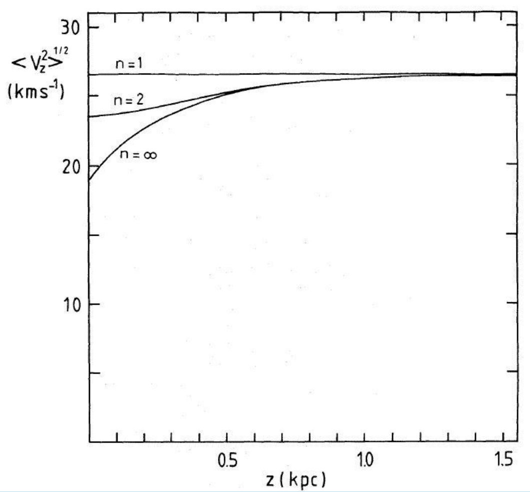
This integral can be evaluated easily by numerical methods or through a series expansion.

The properties are illustrated in the following figures, where properties appropriate for the Solar Neighborhood have been chosen.

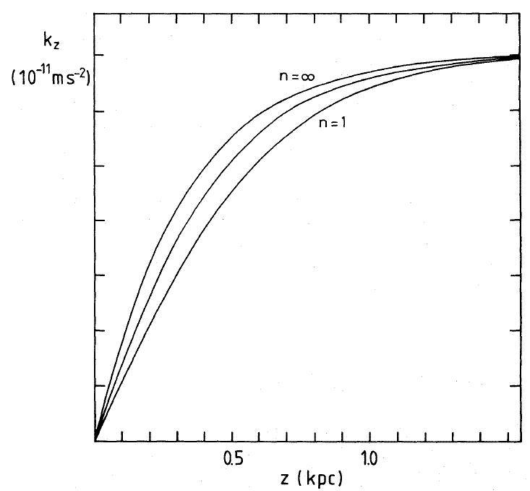
The density distributions as a function of z expressed in magnitudes.



The velocity dispersions as a function of z.



The vertical force K_z as a function of z.



Potential theory

General axisymmetric theory

Much attention has been paid to inverting Poisson's equation. For the axisymmetric case:

$$\frac{\partial^2\Phi}{\partial R^2} + \frac{1}{R}\frac{\partial\Phi}{\partial R} + \frac{\partial^2\Phi}{\partial z^2} = 4\pi G\rho(R,z)$$

so that the potential (and the forces) can be calculated when the density distribution is given.

This is a limited problem in that it does not involve the continuity equation and the distribution function and therefore is not a general solution for a dynamical system, such as the isothermal solutions above.

At the basis lies the Hankel (or Fourier-Bessel) transform, which in the radial direction for the density is

$$\tilde{\rho}(k,z) = \int_0^\infty uJ_0(ku)\rho(u,z)du$$

J_0 is the Bessel function of the first kind.

The important property, why this is useful, is that the transform can be inverted:

$$\rho(R,z) = \int_0^\infty kJ_0(kR)\tilde{\rho}(k,z)dk$$

Now, if we take this transform in the radial direction for both sides of the Poisson equation we get[6]

$$-k^2\tilde{\Phi}(k,z) + \frac{\partial^2}{\partial z^2}\tilde{\Phi}(k,z) = 4\pi G\tilde{\rho}(k,z)$$

This linear non-homogeneous ordinary differential equation can be solved to give

$$\tilde{\Phi}(k,z) = -\frac{2\pi G}{k}\int_{-\infty}^{\infty} \exp\left(-k|z-v|\right)\tilde{\rho}(k,v)dv$$

[6] S. Casertano, MNRAS 203, 735 (1983)

Using this, Poisson's equation can then be inverted to

$$\Phi(R,z) = -2\pi G \int_0^\infty \int_{-\infty}^\infty J_0(kR)\tilde{\rho}(k,v)e^{-k|z-v|}dv\,dk$$

Then

$$\Phi(R,z) = -2\pi G \int_0^\infty \int_0^\infty \int_{-\infty}^\infty uJ_0(kR)J_0(ku)\rho(u,v)e^{-k|z-v|}dv\,du\,dk$$

The integrations are simpler when the density is separable

$$\rho(R,z) = \sigma_{\mathrm{R}}(R)\rho_{\mathrm{z}}(z)$$

The forces follow by taking the negative derivatives of the potential in the radial and vertical directions.

$$K_{\rm R}(R,z) = -\frac{\partial\Phi(R,z)}{\partial R} = -2\pi G \int_0^\infty \int_0^\infty \int_{-\infty}^\infty ukJ_1(kR)J_0(ku)\rho(u,v)e^{-k|z-v|}dv\,du\,dk$$

and

$$K_{\rm z}(R,z) = -\frac{\partial\Phi(R,z)}{\partial z} = -2\pi G \int_0^\infty \int_0^\infty \int_{-\infty}^\infty uJ_0(kR)J_0(ku)\rho(u,v){\rm sign}(z-v)e^{-k|z-v|}dv\,du\,dk$$

Exponential disk

There are various ways of proceeding from here. The first is by taking an analytical form for the density distribution.

Kuijken and Gilmore[7] have done this for exponential disks.

If the radial density distribution is exponential

$$\sigma_{\mathrm{R}}(R) = \sigma_0 \exp\left(-R/h\right)$$

then the Hankel transform becomes

$$\int_0^\infty \sigma_0 J_0(ku) u e^{-u/h} du = \frac{\sigma_0 h^2}{(k^2 h^2 + 1)^{3/2}}$$

[7] K. Kuijken & G. Gilmore, MNRAS vol. 239, 571 (1989)

The potential can then be written as

$$\Phi(R,z) = -2\pi G h^2 \int_0^\infty \int_{-\infty}^\infty \frac{J_0(kR)}{(k^2h^2+1)^{3/2}} \rho_z(v) e^{-k|z-v|} dv\, dk$$

First note that if $\rho_z(z)$ is symmetric around $z=0$, then

$$I_z(k,z) = \int_{-\infty}^\infty \rho_z(v) e^{-k|z-v|} dv$$

$$= 2e^{k|z|} \int_0^{|z|} \rho_z(v) \cosh(kv) dv + 2\cosh(kz) \int_{|z|}^\infty \rho_z(v) e^{-kv} dv$$

$$= e^{-k|z|} \int_0^{|z|} \rho_z(v) e^{kv} dv + e^{k|z|} \int_{|z|}^\infty \rho_z(v) e^{-kv} dv + e^{-k|z|} \int_0^\infty \rho_z(v) e^{-kv} dv$$

Kuijken and Gilmore first solve for an exponential z-distribution:

$$\rho_{\mathrm{z}} = \exp\left(-|z|/z_{\mathrm{e}}\right)$$

Solving for this gives

$$\Phi(R,z) = -4\pi G\sigma_0 h^2 z_{\mathrm{e}} \int_0^\infty \frac{J_0(kR)}{(k^2h^2+1)^{3/2}} \frac{e^{-k|z|} - z_{\mathrm{e}} k e^{-|z|/z_{\mathrm{e}}}}{1-k^2 z_{\mathrm{e}}^2} dk$$

The possible term for which the denominator is zero ($kz_{\mathrm{e}} = 1$) is still finite; the last quotient is in that case

$$\frac{1}{2z_{\mathrm{e}}k}(1+k|z|)e^{-k|z|}$$

The forces are

$$K_{\mathrm{R}}(R,z) = -4\pi G\sigma_0 h^2 z_{\mathrm{e}} \int_0^\infty k \frac{J_1(kR)}{(k^2h^2+1)^{3/2}} \frac{e^{-k|z|} - z_{\mathrm{e}} k e^{-|z|/z_{\mathrm{e}}}}{1-k^2 z_{\mathrm{e}}^2} dk$$

and

$$K_{\mathrm{z}}(R,z) = -4\pi G\sigma_0 h^2 z_{\mathrm{e}} \int_0^\infty k \frac{J_0(kR)}{(k^2h^2+1)^{3/2}} \,\mathrm{sign}(z) \frac{e^{-k|z|} - e^{-|z|/z_{\mathrm{e}}}}{1-k^2 z_{\mathrm{e}}^2} dk$$

Next they assume that the density distribution is given by

$$\rho(R,z) = \rho_0 \exp\left(-R/h\right) \mathrm{sech}^n (z/nz_{\mathrm{e}})$$

For $n=0$ we have again the exponential z-distribution with vertical, exponential scaleheight z_{e}.
For $n=2$ we have the locally isothermal disk[8] and for $n=1$ the "sech-disk"[9].

Kuijken and Gilmore show that the potential can be written as

$$\Phi(R,z) = -4\pi G \rho_0 h^2 z_{\mathrm{e}} 2^n \int_0^\infty J_0(kR)(k^2h^2+1)^{-3/2} \times$$

$$\sum_{m=0}^{\infty} \binom{-n}{m} \frac{(1+2m/n)\exp\left(-k|z|\right) - z_{\mathrm{e}} k \exp\left[-(1+2m/n)|z|/z_{\mathrm{e}}\right]}{(1+2m/n)^2 - k^2 z_{\mathrm{e}}^2} dk$$

[8] P.C. van der Kruit & L. Searle, A.&A. 95, 105 (1981)
[9] P.C. van der Kruit, A.&A. 192, 117 (1988)

The possible term, for which $m = n(kz_\mathrm{e} - 1)/2$, has a zero denominator and must be written as

$$\frac{1}{2z_\mathrm{e} k} \binom{-n}{m} (1 + k|z|) e^{-k|z|}$$

The binomial with the upper coefficient negative can be written as follows

$$\binom{-n}{m} = \frac{(-n)(-n-1)........(-n-m+1)}{m!}$$

$$= (-1)^m \binom{m+n-1}{n-1} = (-1)^m \frac{(m+n-1)!}{(n-1)!m!}$$

So the potential is in this case expressed as a sum of those for exponential z-distributions.

This is essentially related to the fact that the sech is written as a sum of exponentials:

$$\operatorname{sech} x = 2\sum_{j=0}^{\infty}(-1)^j e^{-(2j+1)|x|}$$

This well-known expansion suffers from the fact that it does not work for $x = 0$, because the terms are alternatingly $+1$ and -1.

This does not necessarily make it unsuitable, because after integration each term gets divided by $-(2j+1)$ and the series will converge even for $x = 0$.

However, it may remain slow for small x. For example the sum for $x = 0$

$$2\sum_{j=0}^{\infty}\frac{(-1)^j}{2j+1} = \frac{\pi}{2}$$

takes 32 steps to reach an accuracy of 1%.

Similar expressions as above can be found for the forces, but this will not be fully written out here.

Rotation curves

Casertano[10] has derived an expression for the potential in the plane for an arbitrary density distribution in order to find the rotation curve of a disk with a density distribution derived from surface photometry.

He uses the radial force in the plane and performs the integration over k first (rather than over u).

The equation for the radial force in the plane for a symmetrical z-distribution is

$$K_{\mathrm{R}}(R,0) = -4\pi G \int_0^\infty \int_0^\infty \int_0^\infty ukJ_1(kR)J_0(ku)\rho(u,v)e^{-kv}\,dv\,du\,dk$$

[10] S. Casertano, MNRAS 203, 735 (1983)

It helps to have the same order Bessel functions and get rid of the linear factor k by integrating in parts

$$\int_0^\infty u J_0(ku)\rho(u,v)du = \frac{u}{k} J_1(uk)\rho(u,v)\Big|_0^\infty - \frac{1}{k}\int_0^\infty u J_1(uk)\frac{\partial\rho(u,v)}{\partial u}du$$

Then

$$K_{\mathrm{R}}(R,0) = -4\pi G\int_0^\infty\int_0^\infty\int_0^\infty u J_1(kR)J_1(uk)\frac{\partial\rho(u,v)}{\partial u}e^{-kv}dv\,dk\,du$$

and this can be solved to give

$$K_{\mathrm{R}}(R,0) = 8G\int_0^\infty\int_0^\infty\sqrt{\frac{u}{Rp}}\,\frac{\partial\rho(u,v)}{\partial u}[K(p)-E(p)]du\,dv$$

where

$$p = x-\sqrt{x^2-1},\quad x = \frac{R^2+u^2+v^2}{2Ru}$$

K and E are the complete elliptic integrals of the second and first kind respectively for which good approximations are known. For the z-dependence of the density one can take an exponential or the isothermal distribution.

Casertano's work can be extended to the potential, vertical force and the radial force out of the plane. First start with $K_{\rm R}$ at arbitrary z.

At a general position we had

$$K_{\rm R}(R,z) = -2\pi G \int_0^\infty \int_0^\infty \int_{-\infty}^\infty ukJ_1(kR)J_0(ku)\rho(u,v)e^{-k|z-v|}\,dv\,du\,dk$$

As Casertano we can do the integration over k (after integration by parts) and obtain

$$\int_0^\infty J_1(kR)J_1(uk)e^{-k|z-v|}dk = \frac{(2-p^2)K(p) - 2E(p)}{\pi p\sqrt{Ru}}$$

where

$$p = 2\frac{\sqrt{Ru}}{\sqrt{(z-v)^2+(R+u)^2}}$$

This is the same as Casertano found (except that he had $z=0$), but he chose to rework it further to the form above.

The formula for p has a singularity at $R=u=z=0$. Note however that for $R=u=0$ we already have $p=0$ for all z, so that we should take $p=0$ also for $z=0$. Of course this only occurs when evaluating the force in the center.

The radial force now becomes

$$K_{\mathrm{R}}(R,z) = 2G \int_0^{\infty} \int_{-\infty}^{\infty} \frac{(2-p^2)K(p) - 2E(p)}{p\sqrt{Ru}} \frac{\partial \rho(u,v)}{\partial u} du\, dv$$

For the vertical force and the potential itself we have a product of Bessel functions of equal order before the integration by parts, but this of different order after that.

When then the integration over k is done, we get expressions which contain the Heuman Lambda function. This can be rewritten only in forms that involve incomplete elliptic integrals of the first and second kind or the elliptic integral of the third kind, but these are much more difficult to evaluate numerically.

Also the integrals over u must then be written as the sum of two different integrals, one from 0 to R and one from R to ∞. So it is better to start with the forms before the integration by parts.

For the vertical force we start with

$$K_{\rm z}(R,z) = -2\pi G \int_0^\infty \int_0^\infty \int_{-\infty}^\infty u J_0(kR) J_0(ku) \rho(u,v) \mathrm{sign}(z-v) e^{-k|z-v|} dv\, du\, dk.$$

The integration over k yields

$$\int_0^\infty k J_0(kR) J_0(ku) e^{-k|z-v|} dk = \frac{|z-v| p^3}{4\pi(1-p^2)\sqrt{(uR)^3}} E(p)$$

and we get

$$K_{\rm z}(R,z) = -\frac{G}{2} \int_0^\infty \int_{-\infty}^\infty \mathrm{sign}(z-v) \frac{u|z-v| p^3 E(p)}{(1-p^2)\sqrt{(uR)^3}} \rho(u,v) dv\, du$$

For the potential we start with

$$\Phi(R,z) = -2\pi G \int_0^\infty \int_0^\infty \int_{-\infty}^\infty u J_0(kR) J_0(ku) \rho(u,v) e^{-k|z-v|} dv\, du\, dk$$

The integration over k now yields

$$\int_0^\infty J_0(kR) J_0(ku) e^{-k|z-v|} dk = \frac{p}{\pi\sqrt{uR}} K(p)$$

The potential then is given by

$$\Phi(R,z) = -2G \int_0^\infty \int_{-\infty}^\infty \frac{upK(p)}{\sqrt{uR}} \rho(u,v) dv\, du$$

Various potentials

There are in the literature many particular potentials that can be used to describe galaxies, but are not isothermal.

The most important ones will be summarized here.

These are **not** solutions of the Liouville and Poisson equation. Rather they are convenient expressions for the potential or density distribution that can be inserted analytically in Poisson's equation.

Plummer model

This was originally used to describe globular clusters.

The potential has the simple spherical form

$$\Phi(R) = -\frac{GM}{\sqrt{R^2 + a^2}}$$

The corresponding density distribution is

$$\rho(R) = \left(\frac{3M}{4\pi a^3}\right)\left(1 + \frac{R^2}{a^2}\right)^{-5/2}$$

Kuzmin model

This derives from the potential

$$\Phi(R,z) = -\frac{GM}{\sqrt{R^2+(a+|z|)^2}}$$

This is an axisymmetric potential that can be used to describe very flat disks.

The corresponding surface density is

$$\sigma(R) = \frac{aM}{2\pi(R^2+a^2)^{3/2}}$$

Toomre models

These are models that derive from the Kuzmin model by differentiating with respect to a^2.

The n-th model follows after $(n-1)$ differentiations:

$$\sigma_{\mathrm{n}}(R) = \sigma(0)\left(1 + \frac{R^2}{4n^2a^2}\right)$$

The corresponding potential can be derived by differentiating the potential an equal number of times.

It can be seen that Toomre's model 1 (which has $n = 1$) is Kuzmin's model.

The limiting case of $n \longrightarrow \infty$ becomes a Gaussian surface density model.

Logarithmic potentials

These are made to provide rotation curves that are not Keplerian for large R.

Since these can be flattened they provide an alternative to the simple isothermal sphere. The potential is

$$\Phi(R,z) = \frac{V_\circ^2}{2} \ln \left(r_\circ^2 + R^2 + \frac{z^2}{c^2} \right)$$

$V_\circ$ is the rotation velocity for large radii and c controls the flattening of the isopotential surfaces ($c \leq 1$).

The density distribution is

$$\rho(R,z) = \frac{V_\circ^2}{4\pi G c^2} \frac{(2c^2+1)r_\circ^2 + R^2 + 2z^2[1-1/(2c^2)]}{(r_\circ^2 + R^2 + z^2/c^2)^2}$$

At large radii $R \gg r_\circ$ the isodensity surfaces have a flattening

$$\left(\frac{b}{a}\right)^2 = c^4(2 - c^{-2})$$

In the inner regions $R \ll r_\circ$ it is

$$\left(\frac{b}{a}\right)^2 = \frac{1+4c^2}{2+3c^{-2}}$$

The rotation curve is

$$V_{\rm rot} = \frac{V_\circ R}{\sqrt{r_\circ^2 + R^2}}$$

Oblate spheroids

Assume that all iso-density surfaces are confocal ellipsoids with axis ratio c/a and therefore excentricity

$$e = \sqrt{1 - \frac{c^2}{a^2}}$$

Let the density along the major axis be $\rho(R)$. Define

$$\alpha(R, z) = R^2 + \frac{z^2}{1 - e^2}$$

The forces and the potential can then be calculated. I will not treat the full derivation[11], but simply list the equations.

[11] See Binney & Tremaine, section 2.5

Inside the spheroid the forces and potential are

$$K_{\mathrm{R}} = -\frac{4\pi G\sqrt{1-e^2}}{e^3} R \int_0^{\sin^{-1} e} \rho(\alpha)\sin^2\beta d\beta$$

$$K_{\mathrm{z}} = -\frac{4\pi G\sqrt{1-e^2}}{e^3} z \int_0^{\sin^{-1} e} \rho(\alpha)\tan^2\beta d\beta$$

$$\Phi(R,z) = \frac{4\pi G\sqrt{1-e^2}}{e}\left[\int_0^{\delta} \rho(\alpha)\alpha\beta d\alpha + \sin^{-1} e \int_{\delta}^{a} \rho(\alpha)\alpha d\alpha\right]$$

Here

$$\delta^2 = R^2 + \frac{z^2}{1-e^2}$$

$$\alpha^2 = \frac{R^2\sin^2\beta + z^2\tan^2\beta}{e^2}$$

Outside the spheroid ($\alpha > a$) we have

$$K_{\mathrm{R}} = -\frac{4\pi G\sqrt{1-e^2}}{e^3} R \int_0^{\gamma} \rho(\alpha) \sin^2 \beta d\beta$$

$$K_{\mathrm{z}} = -\frac{4\pi G\sqrt{1-e^2}}{e^3} z \int_0^{\gamma} \rho(\alpha) \tan^2 \beta d\beta$$

$$\Phi(R,z) = \frac{4\pi G\sqrt{1-e^2}}{e} \int_0^{a} \rho(\alpha)\alpha\beta d\alpha$$

Here γ follows from

$$R^2 \sin^2 \gamma + z^2 \tan^2 \gamma = a^2 e^2$$

Infinitesimally thin disks

This is analogous to the treatment of general disk potentials above, but now the vertical distribution is a δ-function.

The equation we had before based on the Hankel-transform was

$$\Phi(R,z) = -2\pi G \int_0^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty} u J_0(kR) J_0(ku) \rho(u,v) e^{-k|z-v|} dv\, du\, dk$$

The potential can be written for the infinitesimally thin disk as

$$\Phi(R,z) = -2\pi G \int_0^{\infty} \exp\left(-k|z|\right) J_0(kR) \int_0^{\infty} \sigma(r) J_0(kr) r\, dr\, dk$$

The rotation velocity then becomes

$$V_{\mathrm{c}}^2(R) = -R \int_0^\infty S(k) J_1(kR)\; k\; dk$$

where

$$S(k) = -2\pi G \int_0^\infty J_0(kR)\sigma(R) dR$$

It may be useful to calculate the surface density corresponding to a known rotation curve $V_{\mathrm{c}}(R)$.

Using the inversion of the first equation above it can be shown that

$$\sigma(R) = \frac{1}{\pi^2 G}\left[\frac{1}{R}\int_0^R \frac{dV_c^2}{dr} K\left(\frac{r}{R}\right) dr \; + \int_R^\infty \frac{1}{r}\frac{dV_c^2}{dr} K\left(\frac{R}{r}\right) dr\right]$$

where K is the complete elliptic integral.

There is a contribution from the part of the disk beyond R.

This also holds for disks with finite thickness as long as the density distribution is not described by spheroids.

In general the rotation curve of a disk depends on the surface density at all radii.

Mestel disk

This has the surface density distribution

$$\sigma(R) = \sigma_\circ \frac{R_\circ}{R}$$

The corresponding rotation curve is flat and has

$$V_{\mathrm{c}}^2(R) = 2\pi G \sigma_\circ R_\circ = \frac{GM(R)}{R}$$

where $M(R)$ is the mass interior to R.

Exponential disk

This is treated in a famous paper by Freeman[12]. The surface density is

$$\sigma(R) = \sigma_\circ \exp\left(-\frac{R}{h}\right)$$

The corresponding potential from the equation above for a infinitessimaly thin disk is

$$\Phi(R,0) = -\pi G \sigma_\circ R \left[I_\circ\left(\frac{R}{2h}\right) K_1\left(\frac{R}{2h}\right) - I_1\left(\frac{R}{2h}\right) K_0\left(\frac{R}{2h}\right)\right]$$

Here I and K are the modified Bessel functions.

[12] K.C. Freeman, Ap.J. 160, 811 (1970)

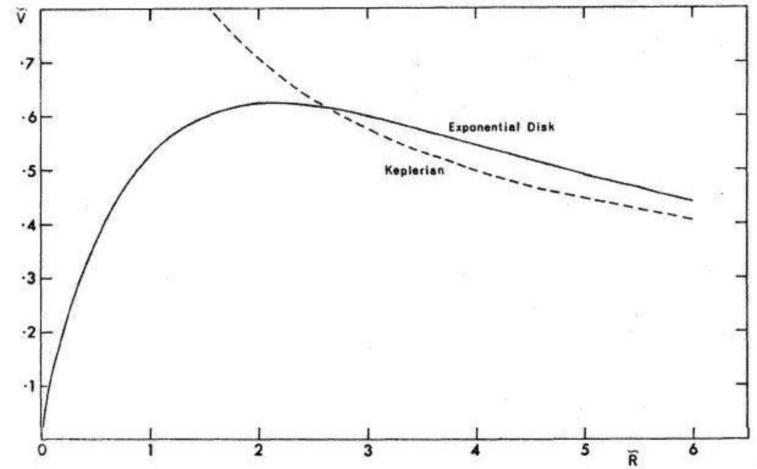
The rotation curve is (again with the equation above for infinitessimally thin disks)

$$V_c^2(R) = 4\pi G\sigma_\circ h \left(\frac{R}{2h}\right)^2 \left[I_0\left(\frac{R}{2h}\right)K_0\left(\frac{R}{2h}\right) - I_1\left(\frac{R}{2h}\right)K_1\left(\frac{R}{2h}\right)\right]$$

The total potential energy of the disk is

$$\Omega \approx -11.6 G\sigma_\circ^2 h^3$$

The rotation curve and the corresponding resonances are shown in the next figures. Note the approximate constancy of $\Omega - \kappa/2$ with radius.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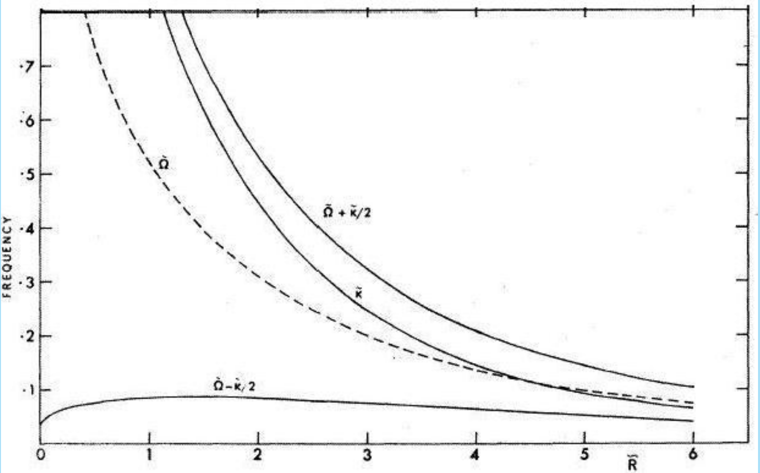
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Ω̃ − κ̃/2

Stäckel potentials

Stäckel potentials are potentials that can be written as separable functions in ellipsoidal coordinate systems.

They are defined as follows[13].

If (x, y, z) is a cartesian coordinate system, then the ellipsoidal coordinates (λ, μ, ν) are the three roots for τ of

$$\frac{x^2}{\tau+\alpha}+\frac{y^2}{\tau+\beta}+\frac{z^2}{\tau+\gamma}=1$$

where $\alpha < \beta < \gamma$ are three constants.

[13] P.T. de Zeeuw, MNRAS 236, 273 (1985)

The coordinate system is illustrated in the picture below.

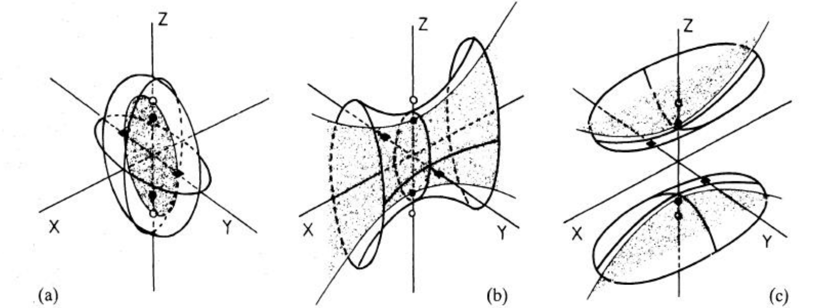

Figure 2. Ellipsoidal coordinates. The three pairs of foci are denoted by the open and filled circles and the filled squares. (a) Surfaces of constant λ are ellipsoids. The degenerate ellipsoid $\lambda = -\alpha$, inside the focal ellipse, is shaded. (b) Surfaces of constant μ are hyperboloids of one sheet. The degenerate hyperboloid $\mu = -\beta$, between the two branches of the focal hyperbola, is shaded. (c) Surfaces of constant ν are hyperboloids of two sheets. The degenerate hyperboloid $\nu = -\beta$ is shaded.

I will here only treat the axisymmetric case with oblate density distributions (which means a prolate potential distribution), which applies to disk galaxies[14].

In that case the coordinate system is spheroidal and it can be seen as a further generalisation of the axisymmetric, plane-parallel case, where the potential is separable in R and z.

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[14] See also H. Dejonghe & P.T. de Zeeuw, Ap.J. 333, 90 (1988); S.M. Kent & P.T. de Zeeuw, A.J. 102, 1994 (1991)

Coordinate system

The new coordinate system is (λ, ϕ, ν). The relation with the axisymmetric system (r, ϕ, z) is, that λ and ν are the two roots for τ of

$$\frac{r^2}{\tau + \alpha} + \frac{z^2}{\tau + \gamma} = 1$$

with

$$0 \leq \nu \leq \lambda$$

The constants α and γ are sometimes also given in the form

$$\alpha = -a^2, \quad \gamma = -c^2$$

These correspond to a focal distance

$$\Delta = (|\gamma - \alpha|)^{1/2} = (|a^2 - c^2|)^{1/2}$$

Note that λ and ν have a dimension of length^2.

The coordinate surfaces are spheroids for constant λ and hyperboloids for constant ν with the z-axis as rotation axis.

The case for flattened disks obtains, when $-\alpha > -\gamma$, so that $-\gamma = c^2 \leq \nu \leq -\alpha = a^2 \leq \lambda$.

Spheroids of constant λ then are prolate, while the hyperboloids of constant ν have two sheets.

On each meridional plane of constant ϕ we then have elliptical coordinates (λ, ν) with foci on the z-axis at $z = \pm\Delta$.

Note that the mass distribution is oblate, although the coordinate system is prolate.

Other relations between the two coordinate systems are

$$r^2 = \frac{(\lambda+\alpha)(\nu+\alpha)}{\alpha-\gamma} \quad ; \quad z^2 = \frac{(\lambda+\gamma)(\nu+\gamma)}{\gamma-\alpha}$$

and

$$\lambda, \nu = \tfrac{1}{2}(r^2+z^2-\gamma-\alpha) \pm \tfrac{1}{2}\sqrt{(r^2-z^2+\gamma-\alpha)^2+4r^2z^2}$$

Also

$$\lambda+\nu = r^2+z^2-\alpha-\gamma \quad ; \quad \lambda\nu = \alpha\gamma-\gamma r^2-\alpha z^2$$

Note that ν and λ occupy different, but contiguous parts of the positive real line.

- In the plane we have $\nu = -\gamma$, $\lambda = r^2 - \alpha$
- On the z-axis
  - $\nu = z^2 - \gamma$, $\lambda = -\alpha$ for $0 \leq |z| \leq \Delta$
  - $\nu = -\alpha$, $\lambda = z^2 - \gamma$ for $|z| \geq \Delta$.

The potential and the density distribution

Suppose that the potential Φ, which is minus the usual potential Φ and therefore always positive, can be separated as follows

$$\Phi(\lambda,\nu) = \frac{(\lambda+\gamma)G(\lambda) - (\nu+\gamma)G(\nu)}{\lambda-\nu}$$

Such potentials are called (axi-symmetric) Stäckel potentials.

For models with a finite mass M the potential should tend to zero for large radii, which means that for $\lambda \to \infty$ we get

$$G(\lambda) \sim \frac{GM}{\lambda^{1/2}}$$

The density ρ, which is defined such that $\rho\, dx\, dy\, dz$ is the mass in the volume element $dx\, dy\, dz$, can be calculated from Poisson's equation, which has the complicated form

$$\pi G\rho(\lambda,\nu)(\nu-\lambda)=(\lambda+\alpha)(\lambda+\gamma)\frac{\partial^2\Phi}{\partial\lambda^2}+\left(\tfrac{3}{2}\lambda+\tfrac{1}{2}\alpha+\gamma\right)\frac{\partial\Phi}{\partial\lambda}-$$

$$(\nu+\alpha)(\nu+\gamma)\frac{\partial^2\Phi}{\partial\nu^2}-\left(\tfrac{3}{2}\nu+\tfrac{1}{2}\alpha+\gamma\right)\frac{\partial\Phi}{\partial\nu}$$

The Kuzmin equation gives the properties, when the density on the z-axis are given:
Assume that this density is $\varphi(\tau)$, where $\tau=\lambda,\nu$ and note from above that on the z-axis we always have $\tau=z^2-\gamma$ for all z.

Then the density is

$$\rho(z) = \varphi(z^2 - \gamma) = \varphi(\tau)$$

Define the primitive function of $\varphi(\tau)$ as

$$\psi(\tau) = \int_{-\gamma}^{\tau} \varphi(\sigma)\, d\sigma$$

Then the density follows from

$$\rho(\lambda, \nu) = \left(\frac{\lambda + \alpha}{\lambda - \nu}\right)^2 \varphi(\lambda) -$$

$$2\frac{(\lambda + \alpha)(\nu + \alpha)}{(\lambda - \nu)^2}\frac{\psi(\lambda) - \psi(\nu)}{\lambda - \nu} + \left(\frac{\nu + \alpha}{\lambda - \nu}\right)^2 \varphi(\nu)$$

The total mass is

$$M = 2\pi \int_{-\gamma}^{\infty} \frac{\sigma + 2\gamma - \alpha}{\sqrt{\sigma + \gamma}} \varphi(\sigma)\, d\sigma = 4\pi \int_0^{\infty} (z^2 + \Delta^2)\varphi(z)\, dz$$

The potential follows from

$$G(\tau) = 2\pi G\psi(\infty) - \frac{2\pi G}{\sqrt{\tau + \gamma}} \int_{-\gamma}^{\tau} \frac{\sigma + \alpha}{2(\sigma + \gamma)^{3/2}} \psi(\sigma)\, d\sigma$$

Velocities, angular momentum and integrals of motion

In order to convert velocities we write

$$\cos\Theta = \left[\frac{(\nu+\alpha)(\lambda+\gamma)}{(\alpha-\gamma)(\lambda-\nu)}\right]^{1/2} \quad ; \quad \sin\Theta = \left[\frac{(\lambda+\alpha)(\nu+\gamma)}{(\gamma-\alpha)(\lambda-\nu)}\right]^{1/2}$$

Velocities are related for the oblate mass models $(\gamma-\alpha>0)$ as

$$V_{\mathrm{r}} = V_\lambda \cos\Theta - V_\nu \sin\Theta \quad ; \quad \mathrm{sign}\,(z)\; V_{\mathrm{z}} = V_\lambda \sin\Theta + V_\nu \cos\Theta$$

and

$$V_\lambda = V_{\mathrm{r}} \cos\Theta + \mathrm{sign}\,(z)\; V_{\mathrm{z}} \sin\Theta \quad ; \quad V_\nu = -V_{\mathrm{r}} \sin\Theta + \mathrm{sign}\,(z)\; V_{\mathrm{z}} \cos\Theta$$

Note that V_λ and V_ν are velocities in the local Cartesian system and do *not* describe the changes in λ and ν.

For the momenta we need the coefficients of the coordinate system

$$P^2 = \frac{\lambda - \nu}{4(\lambda + \alpha)(\lambda + \gamma)} \quad ; \quad R^2 = \frac{\nu - \lambda}{4(\nu + \alpha)(\nu + \gamma)}$$

The momenta then are

$$p_\lambda = PV_\lambda, \quad p_\phi = rV_\phi, \quad p_\nu = RV_\nu.$$

The angular momenta are

$$L_{\rm x} = y\dot{z} - z\dot{y} = rV_{\rm z}\sin\phi - z(V_{\rm r}\sin\phi + V_\phi\cos\phi)$$
$$L_{\rm y} = z\dot{x} - x\dot{z} = -rV_{\rm z}\cos\phi + z(V_{\rm r}\cos\phi - V_\phi\sin\phi)$$
$$L_{\rm z} = x\dot{y} - y\dot{x} = rV_\phi$$

The total angular momentum L is

Integrals of motion

It can then be shown that there are three integrals of motion, namely

$$I_1 = E = -\left(\frac{p_\lambda^2}{2P^2} + \frac{p_\phi^2}{2r^2} + \frac{p_\nu^2}{2R^2}\right) + \Phi(\lambda,\nu)$$

$$I_2 = \tfrac{1}{2} L_{\rm z}^2$$

$$I_3 = \tfrac{1}{2}(L_{\rm x}^2 + L_{\rm y}^2) + (\gamma - \alpha)\left[\tfrac{1}{2} V_{\rm z}^2 - z^2 \frac{G(\lambda) - G(\nu)}{\lambda - \nu}\right]$$

The equations of motion then are

$$p_\lambda^2 = \frac{1}{2(\lambda+\alpha)}\left[G(\lambda) - \frac{I_2}{\lambda+\alpha} - \frac{I_3}{\lambda+\gamma} - E\right]$$

$$p_\phi^2 = 2I_2$$

$$p_\nu^2 = \frac{1}{2(\nu+\alpha)}\left[G(\nu) - \frac{I_2}{\nu+\alpha} - \frac{I_3}{\nu+\gamma} - E\right]$$

In the meridional plane the orbits are restricted to the area defined by

$$-\gamma \leq \nu \leq \nu_0, \quad \lambda_1 \leq \lambda \leq \lambda_2$$

where the turning points ν_0, λ_1 and λ_2 are the values for ν and λ for which respectively V_ν and V_λ are zero.

The case $\nu = -\gamma$ corresponds to $z = 0$.

The turning points are the three solutions $\tau_1 \leq \tau_2 \leq \tau_3$ of

$$G(\tau) - \frac{I_2}{\tau + \alpha} - \frac{I_3}{\tau + \gamma} - E = 0$$

where in general there should be

- one solution $\tau_1 \leq -\alpha$, which is ν_0, and
- two solutions $-\alpha \leq \tau_2 \leq \tau_3$, which are λ_1 and λ_2.

In the case of an oblate mass distribution (prolate coordinate system) all orbits are "short axis tubes", bounded by two prolate spheroids and one hyperboloid of one sheet.