DYNAMICS OF GALAXIES

3. Motions, instabilities and the velocity ellipsoid

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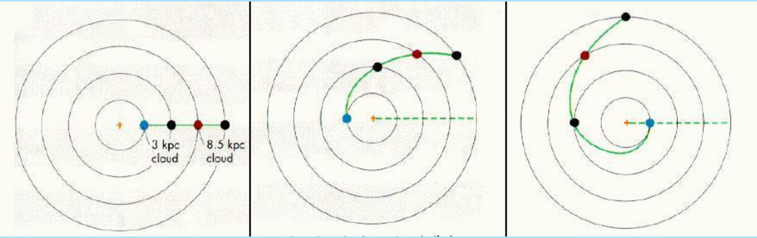
Kinematics of the Galaxy

Differential rotation

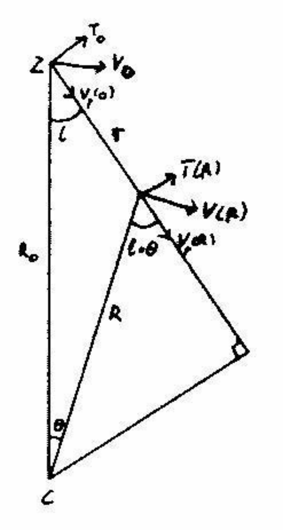
The Galaxy does not rotate like a solid wheel. The period of revolution varies with distance from the center. This is called differential rotation.

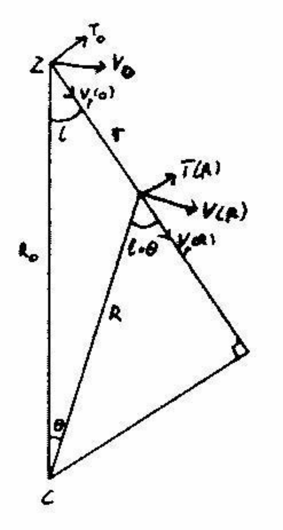


Each part moves with respect to those parts that do not happen to be at the same galactocentric distance.

Say, the rotation speed is $V(R)$ and in the solar neighborhood it is $V_\circ$.

If the Sun Z is at a distance $R_\circ$ from the center C, then an object at distance r from the Sun at Galactic longitude l has a radial velocity w.r.t. the Sun $V_{\rm rad}$ and a tangential velocity T.





$$V_{\rm rad} = V_{\rm r}(R) - V_{\rm r}(0) = V(R)\sin(l+\theta) - V_\circ \sin l$$

$$T = T(R) - T(0) = V(R)\cos(l+\theta) - V_\circ \cos l$$

$$R\sin(l+\theta) = R_\circ \sin l$$

$$R\cos(l+\theta) = R_\circ \cos l - r$$

Substitute this and we get

$$V_{\rm rad} = R_\circ \left(\frac{V(R)}{R} - \frac{V_\circ}{R_\circ} \right) \sin l \qquad (1)$$

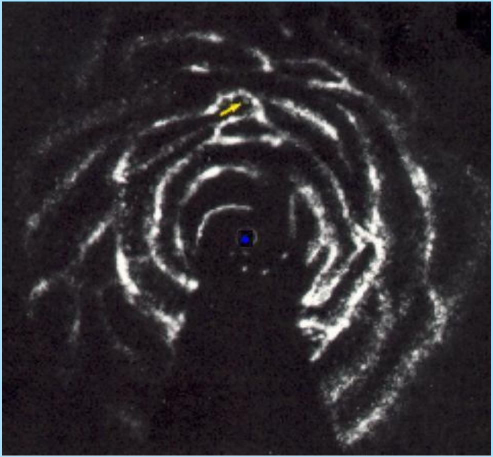
$$T = R_\circ \left(\frac{V(R)}{R} - \frac{V_\circ}{R_\circ} \right) \cos l - \frac{r}{R} V(R) \qquad (2)$$

So, if we would know the rotation curve $V(R)$ we can calculate the distance R from observations of $V_{\rm rad}$. From this follows r with an ambiguity symmetric with the sub-central point.

The latter is that point along the line-of-sight that is closest to the Galactic Center.

$V(R)$ can be deduced in each direction l by taking the largest observed radial velocity. This will be the rotation velocity at the sub-central point.

With the 21-cm line of HI, the distribution of hydrogen in the Galaxy has been mapped[a]. This was the first indication that the Galaxy is a spiral galaxy.



[a] K.K. Kwee, C.A. Muller & G. Westerhout, Bull. Astron. Inst. Neth. 12, 211 (1954); J.H. Oort, F.J. Kerr & G. Westerhout, Mon.Not.R.A.S. 118, 379 (1958) and J.H. Oort, I.A.U. Symp. 8, 409 (1959)

We now make local approximations; that is $r \ll R_\circ$.

Change to angular velocities $\omega(R) = V(R)/R$ and $\omega_\circ = V_\circ/R_\circ$ and make a Tayler expansion

$$f(a+x) = f(a) + x\frac{df(a)}{da} + \frac{1}{2}x^2\frac{d^2 f(a)}{d^2 a} + \ldots.$$

for the angular rotation velocity

$$\omega(R) = \omega_\circ + (R - R_\circ)\left(\frac{d\omega}{dR}\right)_{R_\circ} + \tfrac{1}{2}(R - R_\circ)^2\left(\frac{d^2\omega}{dR^2}\right)_{R_\circ}$$

The cosine-rule gives

$$R = R_\circ\left[1 + \left(\frac{r}{R_\circ}\right)^2 - \frac{2r}{R_\circ}\cos l\right]^{1/2}$$

Make a Tayler expansion for this expression and ignore terms of higher order than $(r/R_\circ)^3$.

$$R = R_\circ \left[1 - \frac{r}{R_\circ}\cos l + \frac{1}{2}\left(\frac{r}{R_\circ}\right)^2 (1-\cos^2 l)\right]$$

$$R - R_\circ = -r\cos l + \frac{1}{2}\frac{r^2}{R_\circ}(1-\cos^2 l)$$

$$(R-R_\circ)^2 = r^2\cos^2 l$$

Substitute this in the equation for ω

$$\begin{aligned}\omega(R) &= \omega_\circ + \left(\frac{d\omega}{dR}\right)_{R_\circ} R_\circ \left[-\frac{r}{R_\circ}\cos l + \frac{1}{2}\left(\frac{r}{R_\circ}\right)^2 (1-\cos^2 l)\right] \\ &+ \frac{1}{2}\left(\frac{d^2\omega}{dR^2}\right)_{R_\circ} R_\circ^2 \left(\frac{r}{R_\circ}\right)^2 \cos^2 l\end{aligned}$$

or in linear velocity

$$V_{\rm rad} = \left(\frac{r}{R_\circ}\right)^2 \left(\frac{d\omega}{dR}\right)_{R_\circ} \frac{R_\circ^2}{2} \sin l - \frac{r}{R_\circ} \left(\frac{d\omega}{dR}\right)_{R_\circ} R_\circ^2 \sin l \cos l + \frac{1}{2} \left(\frac{r}{R_\circ}\right)^2 \left[- \left(\frac{d\omega}{dR}\right)_{R_\circ} R_\circ^2 + \left(\frac{d^2\omega}{dR^2}\right)_{R_\circ} R_\circ^3 \right] \sin l \cos^2 l$$

Use $2 \sin l \cos l = \sin 2l$ and ignore terms with $(r/R_\circ)^2$ and higher orders. Then

$$V_{\rm rad} = -\tfrac{1}{2} R_\circ \left(\frac{d\omega}{dR}\right)_{R_\circ} r \sin 2l \equiv A r \sin 2l$$

So, stars at the same distance r will show a systematic pattern in the magnitude of their radial velocities accross the sky with Galactic longitude.

For stars at Galactic latitude b we have to use the projection of the velocities onto the Galactic plane:

$$V_{\rm rad} = Ar \sin 2l \cos b$$

For the tangential velocities we make a change to proper motions μ. In equivalent way we then find

$$\begin{aligned} \frac{T}{r} = 4.74\mu \quad &= \quad -\omega_\circ + \frac{3}{2}\left(\frac{d\omega}{dR}\right)_{R_\circ} r\cos l - \left(\frac{d\omega}{dR}\right)_{R_\circ} R_\circ \cos^2 l \\ &+ \quad \frac{r}{2R}\left[-\left(\frac{d\omega}{dR}\right)_{R_\circ} + \left(\frac{d^2\omega}{dR^2}\right)_{R_\circ} R_\circ^2\right]\cos^3 l \end{aligned}$$

Now use $\cos^2 l = \frac{1}{2} + \frac{1}{2}\cos 2l$ and ignore all terms $(r/R_\circ)$ and higher order.

$$\begin{aligned} 4.74\mu &= -\omega_\circ - \frac{1}{2}\left(\frac{d\omega}{dR}\right)_{R_\circ} R_\circ - \frac{1}{2} R_\circ \left(\frac{d\omega}{dR}\right)_{R_\circ} \cos 2l \\ &\equiv B + A\cos 2l \end{aligned}$$

Now the distance dependence has of course disappeared.

For higher Galactic latitude the right-hand side will have to be multiplied by $\cos b$.

The constants A and B are the Oort constants. Oort first made the derivation above (in 1927) and used this to deduce the rotation of the Galaxy from observations of the proper motions of stars.

The Oort constanten can also be written as

$$A = \frac{1}{2}\left[\frac{V_\circ}{R_\circ} - \left(\frac{dV}{dR}\right)_{R_\circ}\right]$$

$$B = -\frac{1}{2}\left[\frac{V_\circ}{R_\circ} + \left(\frac{dV}{dR}\right)_{R_\circ}\right]$$

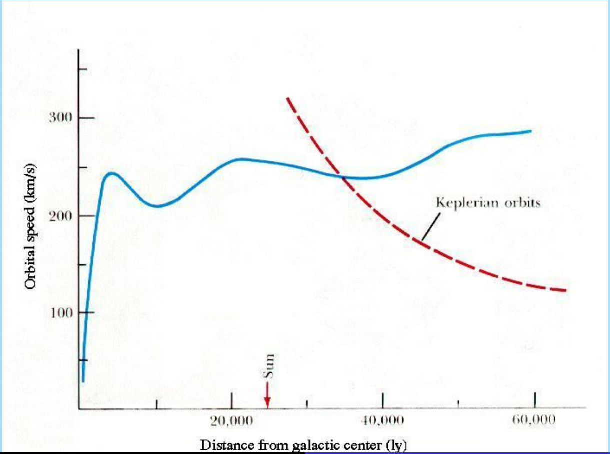
Furthermore

$$A + B = -\left(\frac{dV}{dR}\right)_{R_\circ} \quad ; \quad A - B = \frac{V_\circ}{R_\circ}$$

Current best values are

$R_\circ \sim 8.5$ kpc $\qquad A \sim 13$ km s^{-1} kpc^{-1}
$V_\circ \sim 220$ km s^{-1} $\qquad B \sim -13$ km s^{-1} kpc^{-1}

The rotation curve $V(R)$ is difficult to derive beyond $R_\circ$ and this can only be done with objects of known distance such as HII regions). One determination of the Galactic rotation curve:



We see that up to large distances from the center the rotation velocity does not drop.

In a circular orbit around a point mass M we have $M = V^2 R/G$ (as in the Solar System). This is called a Keplerian rotation curve.

One expects that the rotation curve of the Galaxy tends to such a behavior as one moves beyond the boundaries of the disk. We we do see a flat rotation curve.

We also see this in other galaxies. It shows that more matter must be present than what we observe in stars, gas and dust and this is called dark matter.

With the formula estimate the mass within $R_\circ$ as $\sim 9.6 \times 10^{10} M_\odot$.

At the end of the measured rotation curve this enclosed mass becomes $\sim 10^{12} M_\odot$.

Epicycle orbits

For small deviation from the circular rotation, the orbits of stars can be described as epicyclic orbits.

If $R_\circ$ is a fudicial distance from the center and if the deviation $R - R_\circ$ is small compared to $R_\circ$, then we have in the radial direction

$$\frac{d^2}{dt^2}(R-R_\circ) = \frac{V^2(R)}{R} - \frac{V_\circ^2}{R_\circ} = 4B(A-B)(R-R_\circ) = -\kappa^2(R-R_\circ),$$

where the last approximation results from making a Taylor expansion of $V(R)$ at $R_\circ$ and ignoring higher order terms.

This equation is of the form $\ddot{x} = -\kappa^2 x$ and is easily integrated

$$R - R_\circ = \frac{V_{\mathrm{R},\circ}}{\kappa} \sin \kappa t,$$

In the tangential direction we have

$$\frac{d\theta}{dt} = \frac{V(R)}{R} - \frac{V_\circ}{R_\circ} = -2\frac{A-B}{R_\circ}(R - R_\circ),$$

where θ is the angular tangential deviation seen from the Galactic center. Then

$$\theta R_\circ = -\frac{V_{\mathrm{R},\circ}}{2B} \cos \kappa t$$

The orbital velocities are

$$V_{\mathrm{R}} = V_{\mathrm{R},\circ} \cos \kappa t,$$

$$V_{\theta} - V_{\theta,\circ} = \frac{V_{\mathrm{R},\circ}\kappa}{-2B} \sin \kappa t.$$

The period in the epicycle equals $2\pi/\kappa$ and κ is the epicyclic frequency

$$\kappa = 2\{-B(A-B)\}^{1/2}.$$

In the solar neighborhood $\kappa \sim 36$ km s^{-1} kpc^{-1}.

For a flat rotation curve we have

$$\kappa = \sqrt{2}\frac{V_{\circ}(R)}{R}.$$

Through the Oort constants and the epicyclic frequency, the parameters of the epicycle depend on the local forcefield, because these are all derived from the rotation velocity and its radial derivative.

The direction of motion in the epicycle is opposite to that of galactic rotation.

The ratio of the velocity dispersions or the axis ratio of the velocity ellipsoid in the plane for the stars can be calculated as

$$\frac{\langle V_{\mathrm{R}}^2\rangle^{1/2}}{\langle V_{\theta}^2\rangle^{1/2}} = \sqrt{\frac{-B}{A-B}}.$$

For a flat rotation curve this equals 0.71.

With this result the hydrodynamic equation can then be reduced to the so-called asymmetric drift equation. Recall

$$-K_{\mathrm{R}} = \frac{V_{\mathrm{t}}^2}{R} - \langle V_{\mathrm{R}}^2\rangle \left[\frac{\partial}{\partial R}(\ln \nu\langle V_{\mathrm{R}}^2\rangle) + \frac{1}{R}\left\{1 - \frac{\langle (V_{\theta} - V_{\mathrm{t}})^2\rangle}{\langle V_{\mathrm{R}}^2\rangle}\right\}\right] + \langle V_{\mathrm{R}} V_{\mathrm{z}}\rangle \frac{\partial}{\partial z}(\ln \nu\langle V_{\mathrm{R}} V_{\mathrm{z}}\rangle)$$

For the case the cross-dispersion in the last term is zero, we can now write

$$V_{\rm rot}^2 - V_{\rm t}^2 =$$

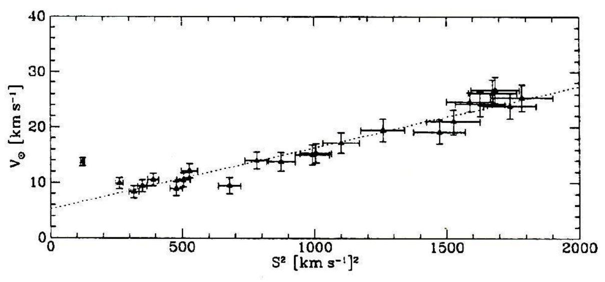
$$-\langle V_{\rm R}^2 \rangle \left\{ R \frac{\partial}{\partial R} \ln \nu + R \frac{\partial}{\partial R} \ln \langle V_{\rm R}^2 \rangle + \left[1 - \frac{B}{B-A} \right] \right\}.$$

Here $V_{\rm rot}$ is the 'circular' velocity that corresponds directly to a centrifigal force $V_{\rm rot}^2/R$ equal to the gravitational force $K_{\rm R}$.

If the asymmetric drift $(V_{\rm rot} - V_{\rm t})$ is small, the left-hand term can be approximated by

$$V_{\rm rot}^2 - V_{\rm t}^2 \sim 2V_{\rm rot}(V_{\rm rot} - V_{\rm t}).$$

The term asymmetric drift comes from the observation that objects in the Galaxy with larger and larger velocity dispersion lag more and more behind in the direction of Galactic rotation.



Vertical motion

For the vertical motion the equivalent approximation is also that of a harmonic oscillator.

For a constant density the hydrodynamic equation reduces to

$$K_{\mathrm{z}} = \frac{d^2 z}{dt^2} = -4\pi G \rho_0 z.$$

Integration gives

$$z = \frac{V_{\mathrm{z},\circ}}{\lambda} \sin \lambda t \quad ; \quad V_{\mathrm{z}} = V_{\mathrm{z},\circ} \cos \lambda t.$$

The period equals $2\pi/\lambda$ and the vertical frequency λ is

$$\lambda = (4\pi G \rho_0)^{1/2}.$$

For the solar neighbourhood we have $\rho_0 \sim 0.1\ \mathrm{M}_\odot\ \mathrm{pc}^{-3}$.

With the values above for $R_\circ$, $V_\circ$, A and B, the epicyclic period $\kappa^{-1} \sim 1.7 \times 10^8$ yrs and the vertical period $\lambda^{-1} \sim 8 \times 10^7$ yrs.

This should be compared to a period of rotation of 2.4 $\times 10^8$ yrs.

The Sun moves with $\sim$20 km s^{-1} towards the Solar Apex at Galactic longitude $\sim 57^\circ$ and latitude $\sim +27^\circ$.

From the curvature of the ridge of the Milky Way the distance of the Sun from the Galactic Plane is estimated as 12 pc.

The axes of the solar epicycle are about $\sim$0.34 kpc in the radial direction and $\sim$0.48 kpc in the tangential direction.

The amplitude of the vertical motion is $\sim$85 pc.

Resonances

The most important ones are between epicyclic frequency and some other frequency that we will call *pattern speed* $\Omega_{\rm p}$.

The *inner Lindblad resonance* occurs for

$$\Omega_{\rm p} = \Omega_{\rm rot}(R) - \frac{\kappa}{2}$$

where $\Omega_{\rm rot}(R)$ is the angular rotation speed.

This resonance occurs at the radius, where –in a rotating frame with angular velocity $\Omega_{\rm p}$– the particle goes through 2 epicycles in the same time is it goes once around the centre. The resulting orbit in that frame then is closed and has an oval shape.

It goes back to Lindblad's discovery that the property $\Omega_{\rm rot}(R) - \kappa/2$ in the inner Galaxy is roughly constant with R.

The pattern speed may be identified with that of the rotating frame in which the spiral *pattern* (not the spiral arms as physical structures themselves) is stationary or with the body rotation of a bar or oval distortion.

Equivalently we have the *outer Lindblad resonance*

$$\Omega_{\rm p} = \Omega_{\rm rot}(R) + \frac{\kappa}{2}$$

and *co-rotation*

$$\Omega_{\rm p} = \Omega_{\rm rot}(R)$$

Higher order Lindblad resonances (involving κ/n) sometimes also play a role.

Instabilities

Jeans instability

We then start with the Jeans instability in a homogeneous medium.

There are various ways of describing it to within an order of magnitude.

The first is to make use of the virial theorem

$$2\ T_{\mathrm{kin}} + \Omega = 0$$

for stability against gravitational contraction.

In a uniform, isothermal sphere the kinetic energy is

$$T_{\mathrm{kin}} = 1/2\ M\langle V^2\rangle$$

and the potential energy

$$\Omega = -\frac{3}{5}\frac{GM^2}{R}$$

So the sphere will contract when its mass M is larger than the value required by the virial theorem.

This is called the Jeans mass M_{Jeans}, which then comes out as

$$M_{\mathrm{Jeans}} = \left(\frac{5}{3G}\right)^{3/2}\left(\frac{3}{4\pi}\right)^{1/2}\left(\frac{\langle V^2\rangle^3}{\rho}\right)^{1/2}$$

A method that gives roughly the same result starts by calculating the free-fall time of a homogeneous sphere.
Anywhere the equation of motion is

$$\frac{d^2 r}{dt^2} = -\frac{G\ M(r)}{r^2} = -\frac{4\pi}{3} G \rho r$$

Solve this and apply for $r = 0$, then

$$t_{\rm ff} = \left(\frac{3\pi}{32 G \rho} \right)^{1/2}$$

The free-fall time is independent of the initial radius and depends only on the density. Now, if there were no gravity a star will move out to the radius of the sphere R in a time

$$t = \frac{R}{\langle V^2 \rangle^{1/2}}$$

For marginal stability the two have to be equal and it follows that the Jeans length is

$$R_{\rm Jeans} = \left(\frac{3\pi}{32} \frac{\langle V^2 \rangle}{G\rho} \right)^{1/2}$$

Sometimes in the literature the Jeans length is taken as the *diameter* of the sphere.

Toomre criterion for local stability

Next we need to consider Toomre's[1] criterion for local stability:

$$Q = \frac{\langle V_{\mathrm{R}}^2 \rangle^{1/2} \kappa}{3.36 G \sigma}$$

$\langle V_{\mathrm{R}}^2 \rangle^{1/2}$ is the stellar velocity dispersion in the R-direction, σ is the local disk surface density and κ is the epicyclic frequency.

An approximate derivation of Toomre's criterion can be made for an infinitesimally thin disk.

1. At small scales the Jeans instability needs to be considered.

[1] A. Toome, Ap.J. 139, 1217 (1964)

Take an area with radius R and surface density σ. The equation of motion is

$$\frac{d^2R}{dt^2} = -\pi G\sigma$$

Solve this and apply for $R = 0$; this gives the free-fall time

$$t_{\rm ff} = \left(\frac{2R}{\pi G\sigma}\right)^{1/2}$$

A star moves out to radius R in a time

$$t = \frac{R}{\langle V^2\rangle^{1/2}}$$

and this must for marginal stability be equal to the free-fall time.

This then gives the Jeans length

$$R_{\mathrm{Jeans}} = \frac{2\langle V^2\rangle}{\pi G \sigma}$$

2. At large scale we need to consider stability resulting from differential rotation.

Take an area with radius $R_\circ$; the angular velocity from differential rotation is

$$\Omega = B$$

The centrifugal force is then

$$F_{\mathrm{cf}} = R_\circ \Omega^2$$

Let it contract to radius R, then the angular velocity becomes

$$\Omega = \frac{R_\circ^2 B}{R^2}$$

and the centrifugal force

$$F_{\rm cf} = R\Omega^2 = \frac{R_\circ^4 B^2}{R^3}$$

If the contraction is dR then

$$\frac{dF_{\rm cf}}{dR} = -\frac{3 R_\circ^4 B^2}{R^4}$$

Now look at the gravitational force

$$F_{\rm grav} = -\frac{G\pi R_\circ^2 \sigma}{R^2}$$

This is correct to within a factor 2 for a flat distribution. Then

$$\frac{dF_{\rm grav}}{dR} = \frac{2\pi G R_\circ^2 \sigma}{R^3}$$

At $R = R_\circ$ these two must compensate each other, so

$$R_{\rm crit} = \frac{2\pi G \sigma}{3B^2}$$

and the disk is stable for all $R > R_{\rm crit}$.

3. Toomre's stability criterion then follows by considering that the disk is stable at all scales if the minimum radius for stability by differential rotation is equal to or smaller than the maximum radius for stability by random motions (the Jeans radius).

Thus

$$\langle V^2 \rangle_{\rm crit}^{1/2} = \frac{\pi}{\sqrt{3}} \frac{G\sigma}{B}$$

In practice $B \approx -A$ (for flat rotation curves), so we can write

$$\langle V^2 \rangle_{\rm crit}^{1/2} \sim 2\pi \left(\frac{2}{3} \right)^{1/2} \frac{G\sigma}{\kappa} = 5.13 \frac{G\sigma}{\kappa}$$

Toomre in his precise treatment found a constant of 3.36.

Goldreich–Lynden-Bell criterion

This can be extended to the criterion, that Goldreich and Lynden-Bell[2] derived for stability of gaseous disks of finite thickness against sheared instabilities:

$$\frac{\pi G\overline{\rho}}{4B(B-A)} \lesssim 1$$

This follows from the result for the Toomre criterion above as follows.

From the vertical oscillation above we find that the maximum distance from the plane is

$$z_\circ = \frac{V_{\mathrm{z},\circ}}{(4\pi G\rho_\circ)^{1/2}}$$

[2] R. Goldreich & D. Lynden-Bell, MNRAS 193, 189 (1965)

Equate the critical velocity dispersion in our derivation of the Toomre criterion to $V_{\mathrm{z},\circ}$, then

$$\frac{12}{\pi} G^{-1} z_\circ^2 \rho_\circ \frac{B^2}{\sigma^2}$$

Now take a mean density $\overline{\rho}$ equal to $\sigma / z_\circ$ and to $\frac{1}{2}\rho_0$ and using $(B - A) \approx 2B$, we get

$$\frac{\pi}{3} G \frac{\overline{\rho}}{B(B-A)} \sim 1$$

These sheared instabilities were proposed by Goldreich & Lynden-Bell as a possible mechanism for the formation of spiral structure.

More recently, Toomre[3] has studied the process in stellar disks and finds an instability based on shear due to differential rotation, that he called *swing amplification*. This process is prevented when

$$X = \frac{R\kappa^2}{2\pi mG\sigma} \gtrsim 3$$

where m is the number of arms. For $-B \approx A$ (a flat rotation curve) this can be written as

$$\frac{QV_{\rm rot}}{\langle V_{\rm R}^2\rangle^{1/2}} \gtrsim 3.97\ m$$

This is Toomre's local stability citerion if the velocity dispersion is replaced by $0.22\ V_{\rm rot}/m$.

[3] A. Toomre, Normal Galaxies, ed. S.M. Fall & D. Lynden-Bell, 111 (1981)

Global stability

For global stability there is a global condition due to Efstathiou, Lake & Negroponte[4] from numerical experiments, which reads

$$Y = V_{\mathrm{rot}} \left(\frac{h}{GM_{\mathrm{disk}}} \right)^{1/2} \gtrsim 1.1$$

For a pure exponential disk with surface density $\sigma(R) = \exp(-R/h)$ without any dark halo $Y = 0.59$.

For a flat rotation curve it is then easy to show that the condition implies that within the disk radius of 4 to 5 scalelengths h the mass in the halo should exceed that of the disk by a factor of about 3.5.

[4] G. Efstathiou, G. Lake & J. Negroponte, MNRAS 199, 1069 (1982)

For a flat rotation curve and an exponential disk Y can be rewritten as

$$Y = 0.615 \left\{ \frac{QRV_{\mathrm{rot}}}{h\langle V_{\mathrm{R}}^2\rangle^{1/2}} \right\}^{1/2} \exp\left(\frac{R}{2h}\right)$$

and this gives

$$\frac{QV_{\mathrm{rot}}}{\langle V_{\mathrm{R}}^2\rangle^{1/2}} \gtrsim 7.91$$

Comparing this to the equation for swing amplification we see that for spirals that are stable against global modes, swing amplification is possible for all modes with $m \geq 2$, at least at those radii where the rotation curve is flat.

Ostriker & Peebles[5] have also found from numerical experiments a general condition for global stability.

Stability occurs only when the ratio of kinetic energy in rotation S to the potential energy Ω

$$t = \frac{S}{|\Omega|} \lesssim 0.14$$

The virial theorem says that $2S + 2R + \Omega = 0$, where S is the kinetic energy in random motions.

Since $R/S > 0$, we would have expected t to have the range $0 - 0.5$ available.

[5] J.P. Ostriker & P.J.E. Peebles, Ap.J. 186, 467 (1973)

The criterion translates into $R/S \gtrsim 2.5$, while for the local Galactic disk it is about 0.15.

So disk galaxies require additional material with high random motion in order to conform to the criterion, either in the disk itself (e.g. the stars in the central region) or in the dark halo.

Tidal radius

Globular clusters have tidal radii due to the force field of the Galaxy. These radii can be estimated as follows.

Assume two point masses M (the Galaxy) and m (the cluster) and a separation R in a circular orbit (the following can be adapted to elliptical orbits as well with R the smallest separation).

Kepler's third law says

$$\frac{T^2}{a^3} = \frac{4\pi^2}{G(M+m)}$$

For a circular orbit we can find the angular velocity of the globular cluster around the center of gravity is

$$\Omega = \left[\frac{G(M+m)}{R^3} \right]^{1/2}$$

The center of gravity is at a distance $MR/(M+m)$ from the cluster.

Take a star at distance r from the center of the cluster in the direction of M and calculate where the total force on that star is zero. Thus in terms of accelleration (after dividing by G)

$$\frac{M}{(R-r)^2} - \frac{m}{r^2} - \frac{M+m}{R^3}\left(\frac{MR}{M+m} - r \right) = 0$$

Since r is much less than R we may expand the first term

$$\frac{M}{(R-r)^2} \approx \frac{M}{R^2}\left(1 + 2\frac{r}{R}\right)$$

Since m is small compared to M the third term can be reduced to

$$\frac{M+m}{R^3}\left(\frac{MR}{M+m} - r\right) = \frac{M}{R^2} - \frac{mr}{R^3}$$

Then the equation reduces to

$$\frac{3Mr}{R^3} - \frac{m}{r^2} = 0$$

The tidal radius then is the solution for r of this equation:

$$r_{\rm tidal} \sim R \left(\frac{m}{3M} \right)^{1/3}$$

For $M = 10^{12}$ M$_\odot$, $m = 10^5$ M$_\odot$ and $R = 10$ kpc we get $r_{\rm tidal} \approx 30$ pc.

Observed tidal radii can be used to constrain the mass distribution in the Galaxy.

Ellipsoidal velocity distribution

The Schwarzschild distribution

The distribution of space velocities of the local stars can be described with the so-called ellipsoidal distribution.

This was first introduced by Karl Schwarzschild and is therefore also called the Schwarzschild distribution.

The distribution is Gaussian along the principal axes, but has different dispersions. This anisotropy was Schwarzschild's explanation of the "star-streams" that were discovered by Kapteyn.

The general equation for the Schwarzschild distribution is

$$f(R,z,V_{\mathrm{R}},V_\theta,V_{\mathrm{z}}) = \frac{8\langle V_{\mathrm{R}}^2\rangle\langle V_\theta^2\rangle\langle V_{\mathrm{z}}^2\rangle}{\pi^{3/2}}\nu \exp\left[-\frac{V_{\mathrm{R}}^2}{2\langle V_{\mathrm{R}}^2\rangle}-\frac{(V_\theta-V_{\mathrm{t}})^2}{2\langle V_\theta^2\rangle}-\frac{V_{\mathrm{z}}^2}{2\langle V_{\mathrm{z}}^2\rangle}-\frac{V_{\mathrm{R}}V_\theta}{2\langle V_{\mathrm{R}}V_\theta\rangle}-\frac{V_{\mathrm{R}}V_{\mathrm{z}}}{2\langle V_{\mathrm{R}}V_{\mathrm{z}}\rangle}-\frac{(V_\theta-V_{\mathrm{t}})V_{\mathrm{z}}}{2\langle V_\theta V_{\mathrm{z}}\rangle}\right]$$

There is an interesting deduction that can be made from this ellipsoidal velocity distribution, which was done by Oort in the same paper in which he discovered differential rotation, defined the Oort constants and laid the foundations for "stellar dynamics"[6].

Take the asymmetric drift equation, insert this distribution and add the condition that $z = 0$ is a plane of symmetry.

Then you get an equation in terms of velocities and multiplications thereof that has to be identical, so that all terms need to be zero.

This is a lot of algebra (see Oort's paper).

[6] J.H.Oort, B.A.N. 4, 269 (1928), see also his chapter in Stars & Stellar Systems V, Galactic Structure, ed. Adriaan Blaauw & Maarten Schmidt, 455 (1965)

$$\Pi\frac{\partial f}{\partial\varpi}+\frac{\Theta}{\varpi}\left(\Theta\frac{\partial f}{\partial\Pi}-\Pi\frac{\partial f}{\partial\Theta}\right)+Z\frac{\partial f}{\partial z}+K_\varpi\frac{\partial f}{\partial\Pi}+K_z\frac{\partial f}{\partial Z}=0 \quad (6)$$

This equation is generally solvable *), but at present I shall only consider some particular solutions, which take account of the fact that the distribution of the peculiar motions of the stars has been found to approximate very closely to a function of the following form :

We shall assume, then, that the velocity distribution is of the ellipsoidal type and that the centre of symmetry of this distribution has a velocity $\Theta_\circ$ with respect to the stationary co-ordinate system. The directions of the axes of the Schwarzschild ellipsoid will be left undetermined for the present, so that we find a distribution function of the following form :

$$f=f_\circ e^{-h^2\Pi^2-k^2(\Theta-\Theta_\circ)^2-l^2Z^2-m\Pi(\Theta-\Theta_\circ)-n\Pi Z-p(\Theta-\Theta_\circ)Z} \quad (8)$$

in which h, k, l, m, n, p, $f_\circ$ and $\Theta_\circ$ are functions of ϖ and z. Inserting (8) in equation (6) we get after dividing by $-f$ and arranging according to powers of Π, Θ, Z:

$$\begin{aligned}
&\Pi^3\frac{\partial h^2}{\partial\varpi}+\Pi^2\Theta\left(\frac{\partial m}{\partial\varpi}-\frac{m}{\varpi}\right)+\Pi^2Z\left(\frac{\partial h^2}{\partial z}+\frac{\partial n}{\partial\varpi}\right)+\Pi\Theta^2\left(\frac{\partial k^2}{\partial\varpi}+\frac{2h^2-2k^2}{\varpi}\right)+\Pi\Theta Z\left(\frac{\partial m}{\partial z}+\frac{\partial p}{\partial\varpi}-\frac{p}{\varpi}\right)+\\
&+\Pi Z^2\left(\frac{\partial l^2}{\partial\varpi}+\frac{\partial n}{\partial z}\right)+\Theta^3\frac{m}{\varpi}+\Theta^2Z\left(\frac{\partial k^2}{\partial z}+\frac{n}{\varpi}\right)+\Theta Z^2\frac{\partial p}{\partial z}+Z^3\frac{\partial l^2}{\partial z}-\Pi^2\frac{\partial(m\Theta_\circ)}{\partial\varpi}-\\
&2\Pi\Theta\left\{\frac{\partial(k^2\Theta_\circ)}{\partial\varpi}-\frac{k^2\Theta_\circ}{\varpi}\right\}-\Pi Z\left\{\frac{\partial(m\Theta_\circ)}{\partial z}+\frac{\partial(p\Theta_\circ)}{\partial\varpi}\right\}-\Theta^2\frac{m\Theta_\circ}{\varpi}-2\Theta Z\frac{\partial(k^2\Theta_\circ)}{\partial z}-Z^2\frac{\partial(p\Theta_\circ)}{\partial z}+\\
&+\Pi\left\{\frac{\partial(k^2\Theta_\circ^2)}{\partial\varpi}+2h^2K_\varpi+nK_z-\frac{1}{f_\circ}\frac{\partial f_\circ}{\partial\varpi}\right\}+\Theta(mK_\varpi+pK_z)+Z\left\{\frac{\partial(k^2\Theta_\circ^2)}{\partial z}+2l^2K_z+nK_\varpi-\frac{1}{f_\circ}\frac{\partial f_\circ}{\partial z}\right\}-\\
&m\Theta_\circ K_\varpi-p\Theta_\circ K_z=0
\end{aligned} \quad (9)$$

As this equation must hold for all values of Π, Θ and Z, the co-efficients of the different powers must vanish separately. We thus get the following conditions:

$$m = p = 0 \tag{10}$$

$$\frac{\partial h^2}{\partial \varpi} = \frac{\partial l^2}{\partial z} = 0 \tag{11}$$

$$\frac{\partial h^2}{\partial z} + \frac{\partial n}{\partial \varpi} = 0;\ \frac{\partial l^2}{\partial \varpi} + \frac{\partial n}{\partial z} = 0;\ \frac{\partial k^2}{\partial z} + \frac{n}{\varpi} = 0 \tag{12}$$

$$\frac{\partial k^2}{\partial \varpi} = \frac{2(k^2 - h^2)}{\varpi} \tag{13}$$

$$\frac{\partial (k^2 \Theta_0)}{\partial \varpi} = \frac{k^2 \Theta_0}{\varpi} \tag{14}$$

$$\frac{\partial (k^2 \Theta_0)}{\partial z} = 0 \tag{15}$$

$$\frac{1}{f_0}\frac{\partial f_0}{\partial \varpi} = \frac{\partial (k^2 \Theta_0^2)}{\partial \varpi} + nK_z + 2h^2 K_\varpi \tag{16}$$

$$\frac{1}{f_0}\frac{\partial f_0}{\partial z} = \frac{\partial (k^2 \Theta_0^2)}{\partial z} + nK_\varpi + 2l^2 K_z \tag{17}$$

stars considered. In our present notation we have thus:

$$A = \frac{1}{2}\left(\frac{\Theta_0}{\varpi} - \frac{\partial \Theta_0}{\partial \varpi}\right)$$

and similarly for the quantity derived from proper motions:

$$B = \frac{1}{2}\left(-\frac{\Theta_0}{\varpi} - \frac{\partial \Theta_0}{\partial \varpi}\right)$$

Thus, inserting these in (19):

$$h^2/k^2 = -B/(A-B) \tag{26}$$

The result is

$$
\begin{aligned}
2\langle V_{\mathrm{R}}^2\rangle &= C_1 + \tfrac{1}{2}C_5 z^2 \\
2\langle V_{\theta}^2\rangle &= C_1 + C_2 R^2 + \tfrac{1}{2}C_5 z^2 \\
2\langle V_{\mathrm{z}}^2\rangle &= C_4 + \tfrac{1}{2}C_5 z^2 \\
2\langle V_{\mathrm{R}} V_{\mathrm{z}}\rangle &= -C_5 R z \\
\langle V_{\mathrm{R}} V_{\theta}\rangle &= \langle V_{\theta} V_{\mathrm{z}}\rangle = 0 \\
V_{\mathrm{t}} &= \frac{C_3 R}{C_1 + C_2 R^2 + \tfrac{1}{2}C_5 z^2}
\end{aligned}
$$

The constants C_1 to C_5 are positive constants.

The density distribution at $z = 0$ follows from

$$\frac{\partial \ln \nu}{\partial R} = 2C_1 K_{\rm R} + \frac{C_2^2 R^2 + (2C_1 C_3^2 - C_1 C_2) R}{(C_2 R^2 + C_1)^2} - \frac{C_1 R}{C_5 R^2 + 2C_4}$$

and the vertical gradient from

$$\frac{\partial \ln \nu}{\partial z} = (C_5 R^2 + 2C_4) K_{\rm z} - C_5 z \left[R K_{\rm R} + \frac{2(C_2 + 2C_3^2) R^2 + C_5 z^2 + 2C_1}{(2C_2 R^2 + C_5 z^2 + 2C_1)^2} + \frac{1}{C_5 z^2 + 2C_1} \right]$$

Oort's derivation only holds if the stellar velocity distribution is exactly Gaussian.

It is too restrictive (e.g. it does not allow high-velocity stars) and therefore, it cannot be used for a description of galactic dynamics.

In reality, the velocity distributions are not precisely Gaussian and are better seen as a superposition of Gaussians (such as of groups of stars with similar ages).

So, these equations are of historical interest only. However, it is interesting to see that Oort assumed that $C_5 = 0$. This uncoupled the radial and vertical motion (as for a third integral).

Properties of the velocity ellipsoid

For the solar neighbourhood, but probably anywhere in galactic disks, the velocity distribution of the stars is very anisotropic.

- The *ratio of the radial versus tangential velocity dispersions* is determined by the local differential rotation and can be derived using the epicycle approximation.

  The axis ratio of the epicycles depend on the local Oort constants and therefore axis ratio of the velocity ellipsoid is

$$\frac{\langle V_\theta^2 \rangle}{\langle V_{\mathrm{R}}^2 \rangle} = \frac{-B}{(A-B)}$$

- The *ratio of the vertical to radial velocity dispersion* is unconstrained, as a result of the third integral.

  However, the existence of a third integral does not necessarily imply that the velocity distribution has to be anisotropic.

  If no third integral would exist, the velocity distribution would have to be isotropic, according to Jeans.

- The *long axis of the velocity ellipsoid in the plane* should point to the center.

  However, it does not in practice. This is called the "deviation of the vertex" and presumably is due to local irregularities in the Galactic gravitational field.

- The *long axis of the velocity ellipsoid outside the plane* has an unknown orientation.

  This has been a longstanding problem, also sometimes referred to as the "tilt" of the velocity ellipsoid.

  Oort assumed the long axis to be parallel to the Galactic plane ($C_5 = 0$), but later assumed it to be pointing always towards the Galactic center.

There is an interesting consequence in this respect of flat rotation curves[7].

Take the Poisson equation for the axisymmetric case

$$\frac{\partial K_{\mathrm{R}}}{\partial R} + \frac{K_{\mathrm{R}}}{R} + \frac{\partial K_{\mathrm{z}}}{\partial z} = -4\pi G\rho(R,z)$$

For a flattened disk, it can be shown that the first two terms in or near the plane $z = 0$ are

$$\frac{\partial K_{\mathrm{R}}}{\partial R} + \frac{K_{\mathrm{R}}}{R} \approx 2(A - B)(A + B)$$

[7] P.C. van der Kruit & K.C. Freeman, Ap.J. 303, 556 (1986)

In 1965, Oort[8] estimated that the first two terms are in the solar neighborhood and in the plane of the Galaxy about 34 times smaller than the third term.

For a flat rotation curve we have $A = -B$, so the equation reduces to that for a plane-parallel case.

On this basis one may expect for small distances from the plane that the long axis is parallel to the plane.

So with flat rotation curves the plane-parallel case turns out to be a much better description of reality than may expected on the basis of the form of the Poisson equation.

[8] J.H. Oort, Stars & Stellar Systems V, Galactic Structure, ed. Adriaan Blaauw & Maarten Schmidt, p. 455 (1965)

The closure problem

The hydrodynamical equations were obtained by multiplication of the Liouville equation with velocities and then integrating over all velocity space.

This system is not complete (there is a "closure problem"): there are only three equations for eight unknowns (the density, rotation velocity, three velocity dispersions and three "cross-dispersions" as a function of position).

In principle one could take higher order moments (by multiplying the Jeans equations with velocities once more and again integrating over all velocities), but this produces more extra unknowns than extra equations.

However, with reasonable assumptions[9] there has been some progress.

It works as follows. In analogy to the second moment

$$\sigma_{\rm ab}(R,z) = \langle V_{\rm a} V_{\rm b}\rangle = \frac{1}{\nu}\int (V_{\rm a} - \langle V_{\rm a}\rangle)(V_{\rm b} - \langle V_{\rm b}\rangle) f d^3 V$$

one defines the third and fourth moments as

$$S_{\rm abc}(R,z) = \langle V_{\rm a} V_{\rm b} V_{\rm c}\rangle = \frac{1}{\nu}\int (V_{\rm a} - \langle V_{\rm a}\rangle)(V_{\rm b} - \langle V_{\rm b}\rangle)(V_{\rm c} - \langle V_{\rm c}\rangle) f d^3 V$$

$$\begin{aligned} T_{\rm abcd}(R,z) &= \langle V_{\rm a} V_{\rm b} V_{\rm c} V_{\rm d}\rangle \\ &= \frac{1}{\nu}\int (V_{\rm a} - \langle V_{\rm a}\rangle)(V_{\rm b} - \langle V_{\rm b}\rangle)(V_{\rm c} - \langle V_{\rm c}\rangle)(V_{\rm d} - \langle V_{\rm d}\rangle) f d^3 V \end{aligned}$$

[9] P.O. Vandervoort, Ap.J. 195, 333 (1975); and in particular P. Amendt & P. Cuddeford, Ap.J. 368, 79 (1991); P. Cuddeford & P. Amendt, MNRAS 256, 166 (1992)

The third moment correponds to the "skewness" (e.g. $S_{\mathrm{RRR}}/(\sigma_{\mathrm{RR}})^{3/2}$). It is zero for a Gaussian, since this is completely symmetric.

The fourth moment corresponds to the "kurtosis" (e.g. $T_{\mathrm{RRRR}}/(\sigma_{\mathrm{RR}})^{2}$), which decribes how peaked the distribution is; a Gaussian has a kurtosis of 3.

The assumptions of Amendt & Cuddeford were

- All parameters can be expanded in terms of a small parameter ϵ, which is the ratio of the radial velocity dispersion to the rotation velocity.
- The ordering scheme of these remains such that only terms in the leading order have to be taken. Thus e.g. in

$$S_{\mathrm{abc}} = \sum_{n=0}^{\infty} \epsilon^{n+3} S_{\mathrm{abc}}^{n+3}$$

  the higher order components of S_{abc} become smaller with n.
- The velocity distributions are Gaussian (Schwarzschild) up to one more order than required by the equations. This happens to translate e.g. for the kurtosis into

$$\frac{T_{\mathrm{Rzzz}}}{\sigma_{\mathrm{RZ}}^2 \sigma_{\mathrm{zz}}^2} = 3 + O(\epsilon^3)$$

These assumptions mean that we have to do with a cool, highly flattened and quasi-isothermal system.

Then the system can be closed and four more equations result after a lot of algebra. Here they are from the publication

Since it is rare for $\overline{v_\phi}(R, z)$ to be known *a priori*, particularly its z-behavior, we use equations (A1) and (A4) to eliminate $\overline{v_\phi}$ and ν, and after some tedious algebra arrive at a system of four partial differential equations for the four components of the velocity dispersion tensor, in terms of the potential Φ. The first three of these equations are

$$\Phi_{,z}(\sigma_{zz}^2\,\partial_z\,\sigma_{Rz}^2 + \sigma_{Rz}^2\,\partial_R\,\sigma_{Rz}^2) = \frac{1}{3}\left[\frac{(R^3\Phi_{,R})_{,R}}{R^3} - \Phi_{,zz}\right]\sigma_{Rz}^2\,\sigma_{zz}^2 - \frac{\Phi_{,z}}{R}\,\sigma_{Rz}^4 + \frac{\Phi_{,Rz}}{3}\,\sigma_{zz}^2(\sigma_{zz}^2 - \sigma_{RR}^2)\,, \tag{79}$$

$$\sigma_{zz}^2\,\partial_z\,\sigma_{zz}^2 + \sigma_{Rz}^2\,\partial_R\,\sigma_{zz}^2 = 0\,, \tag{80}$$

$$\begin{aligned}\Phi_{,z}\left[4\,\sigma_{Rz}^2\,\partial_z\,\sigma_{Rz}^2 + \sigma_{zz}^2\,\partial_z\,\sigma_{RR}^2 + 4\,\sigma_{RR}^2\,\partial_R\,\sigma_{Rz}^2 + \frac{4}{\sigma_{zz}^2}\,(\sigma_{RR}^2\,\sigma_{zz}^2 - \sigma_{Rz}^4)\partial_z\,\sigma_{zz}^2 + \sigma_{Rz}^2\,\partial_R\,\sigma_{RR}^2\right] &= 2\Phi_{,Rz}\,\sigma_{Rz}^2\,\sigma_{zz}^2 - \frac{1}{R}\,(6\Phi_{,z} + 4R\Phi_{,Rz})\sigma_{Rz}^2\,\sigma_{RR}^2 \\ &\quad + \frac{1}{R}\,(6\Phi_{,R} + 2R\Phi_{,RR})\sigma_{zz}^2\,\sigma_{RR}^2 - 2\Phi_{,zz}\,\sigma_{Rz}^4 + 2\Phi_{,Rz}\,\frac{\sigma_{Rz}^6}{\sigma_{zz}^2} - \frac{8}{R}\,\Phi_{,R}\,\sigma_{zz}^2\,\sigma_{\phi\phi}^2 + \frac{8}{R}\,\Phi_{,z}\,\sigma_{Rz}^2\,\sigma_{\phi\phi}^2\,.\end{aligned} \tag{81}$$

The new form of the fourth equation, equation (C1), is cumbersome and is included in Appendix C.

For ease of notation we make the substitutions $\sigma_{Rz}^2 = t$; $\sigma_{RR}^2 = u$; $\sigma_{\phi\phi}^2 = v$; $\sigma_{zz}^2 = w$, and use subscript comma notation to denote partial differentiation on the potential. When we eliminate the mean velocity and number density from equation (78) we obtain the final equation of our closed system of partial differential equations:

$$
\begin{aligned}
&\Phi_{,z}\Big\{(2t^4 + u^2w^2 - 3uwt^2)(\partial_z w)^2 + (2t^3w - utw^2)(\partial_z w\,\partial_z t) + (4t^4 - 3uwt^2)(\partial_z w\,\partial_R t)\\
&\quad - 2t^2w^2(\partial_z t)^2 - 3t^3w(\partial_R t\,\partial_z t) - \frac{1}{2}w^2t^2(\partial_R t\,\partial_z u) - uwt^2(\partial_R t)^2 - \frac{1}{2}t^3w(\partial_R t\,\partial_R u) - \frac{1}{2}w^2t^2(\partial_z w\,\partial_z u) - \frac{1}{2}t^3w(\partial_z w\,\partial_R u)\Big\}\\
&\quad + [\Phi_{,Rz}(t^3 - uwt)(w^2 - uw + t^2) - \Phi_{,zz}t^2w(t^2 - uw)]\partial_z w\\
&\quad + \left\{\Phi_{,Rz}(2w^2 - 2uw + t^2) + \left[\frac{1}{R^3}(R^3\Phi_{,R})_{,R} - 2\Phi_{,zz}\right]tw - \frac{3}{R}\Phi_{,z}t^2\right\}t^2w\,\partial_z t\\
&\quad + \left[2\Phi_{,Rz}t^3(w^2 + t^2) - \frac{3}{R}(R\Phi_{,z})_{,R}t^3uw - 2\Phi_{,zz}t^4w + \frac{1}{R^3}(R^3\Phi_{,R})_{,R}uw^2t^2\right]\partial_R t\\
&\quad + \left[\frac{1}{2R^3}(R^3\Phi_{,R})_{,R}w^2 - \frac{1}{2R^3}(R^3\Phi_{,z})_{,R}tw\right]t^2w\,\partial_z u + \left[\frac{1}{2R^3}(R^3\Phi_{,R})_{,R}tw - \frac{1}{2R^3}(R^3\Phi_{,z})_{,R}t^2\right]t^2w\,\partial_R u\\
&\quad - \frac{2}{R}(\Phi_{,z}t - \Phi_{,R}w)t^2w^2\,\partial_z v - \frac{2}{R}(\Phi_{,z}t - \Phi_{,R}w)t^3w\,\partial_R v = \frac{4}{R^2}(\Phi_{,z}t - \Phi_{,R}w)t^3wv\,. \qquad \text{(C1)}
\end{aligned}
$$

These equations can be used to derive further information on the velocity ellipsoid in cool, flattened galaxies (i.e. in disks).

There are a few applications.

The tilt of the velocity ellipsoid.

From the equations it can be found that

$$\frac{\partial \langle V_{\rm R} V_{\rm z} \rangle}{\partial z}(R,0) = \lambda(R) \left(\frac{\langle V_{\rm R}^2 \rangle - \langle V_{\rm z}^2 \rangle}{R} \right)(R,0)$$

with

$$\lambda(R) = \left[R^2 \frac{\partial^3 \Phi}{\partial R \partial z^2} \left(3\frac{\partial \Phi}{\partial R} + R\frac{\partial^2 \Phi}{\partial R^2} - 4R\frac{\partial^2 \Phi}{\partial z^2} \right)^{-1} \right](R,0)$$

For a flat rotation curve this gives

$$\lambda(R,0) = \left(\frac{2\pi G R^3}{V_{\rm t}^2 - 8\pi G R^2 \rho} \frac{\partial \rho}{\partial R} \right)(R,0)$$

The radial dependence velocity dispersions.

A solution of the equations has the following form

$$f_1(R)\left(\frac{\partial\langle V_{\mathrm{R}}^2\rangle}{\partial R}\right)(R,0)+f_2(R)\langle V_{\mathrm{R}}^2\rangle(R,0)=f_3(R)$$

The functions f have complicated forms and are related to the local potential and kinematics through parameters α, β and γ.

$$\alpha=-\left(\frac{\partial^2\Phi}{\partial z^2}\right)(R,0)=-\lambda^2$$

where λ is the vertical frequency.

$$\begin{aligned}\beta &= \left(\frac{\partial^2\Phi}{\partial R^2}\right)(R,0) + \frac{3}{R}\left(\frac{\partial\Phi}{\partial R}\right)(R,0)\\ &= \frac{1}{R^3}\left(\frac{\partial(R^2 V_{\mathrm{t}}^2)}{\partial R}\right)(R,0) = -\kappa^2\end{aligned}$$

with κ the epicyclic frequency.

$$\begin{aligned}\gamma &= \frac{1}{4}\left\{R\left(\frac{\partial^2\Phi}{\partial R^2}\right)\left(\frac{\partial\Phi}{\partial z}\right)^{-1} + 3\right\}(R,0)\\ &= \left(\frac{\langle V_\theta^2\rangle}{\langle V_{\mathrm{z}}^2\rangle}\right)(R,0)\end{aligned}$$

which is the anisotropy in the velocity distribution.

This can be solved for a given potential; the most realistic solution is with a logarithmic-exponential potential

$$\Phi(R,z) = A \ln R - BR - Cz^2 \exp\left(-\frac{R}{h}\right),$$

which has

$$\left(\frac{\partial^2\Phi}{\partial z^2}\right)(R,0) = 2C \exp\left(-\frac{R}{h}\right)$$

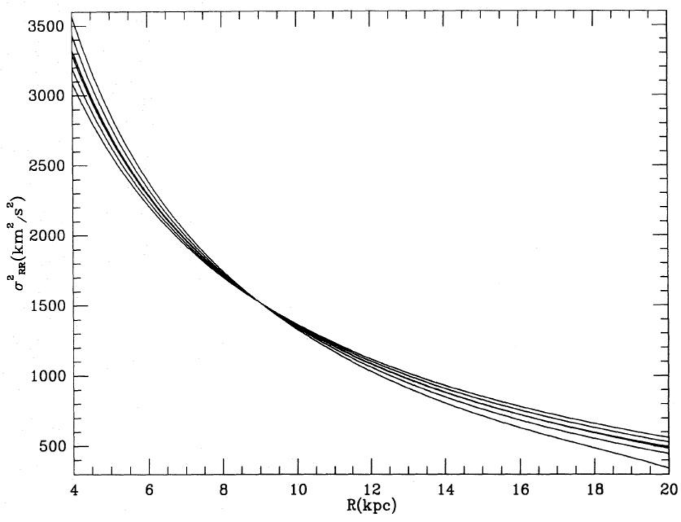
and thus an exponential density profile (as has been observed for the surface brightness distribution).

The resulting distributions show

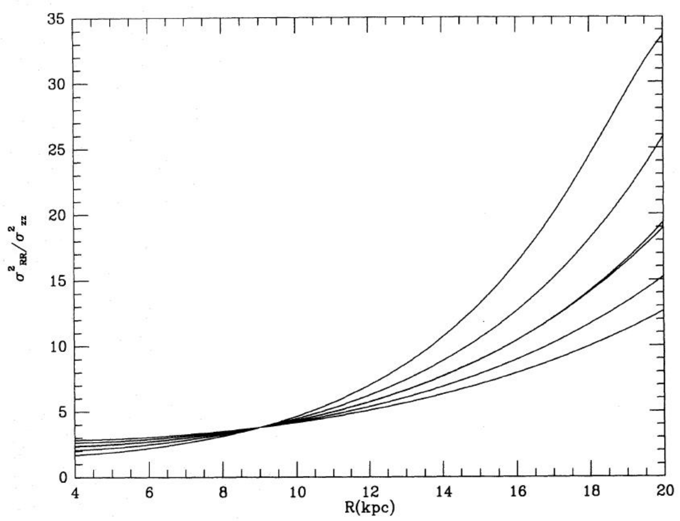
- The radial velocity dispersion $\langle V_{\rm R}^2 \rangle$ decreases more or less exponentially with radius
- The velocity anisotropy $\langle V_{\rm R}^2 \rangle / \langle V_{\rm z}^2 \rangle$ is roughly constant (in the inner regions at least)
- Toomre Q is constant with radius, except near the center.

The following graphs show this for a number of combinations of values for C and h.

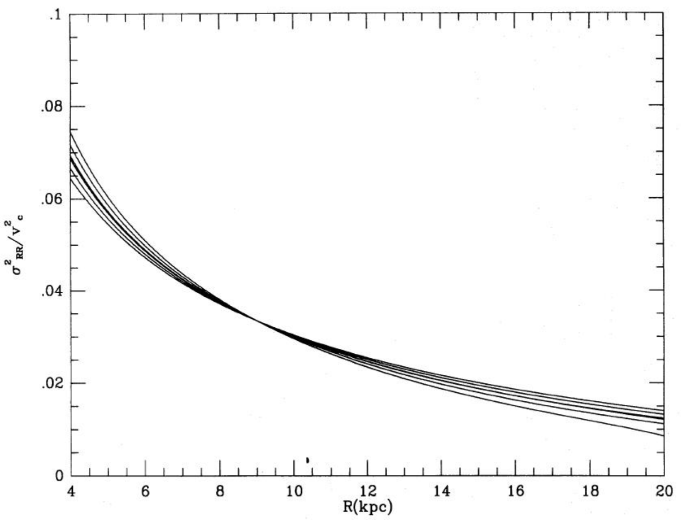
The (square of the) radial velocity dispersion $\langle V_{\mathrm{R}}^2 \rangle$.



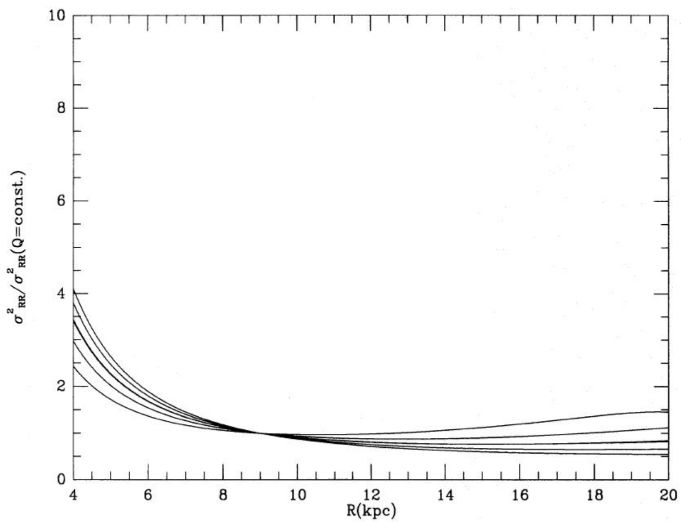
The axis ratio of the velocity ellipsoid $\langle V_{\mathrm{R}}^2 \rangle / \langle V_{\mathrm{z}}^2 \rangle$.



The temperature parameter $\langle V_{\mathrm{R}}^2 \rangle / V_{\mathrm{t}}^2$.



The axis ratio of the velocity ellipsoid w.r.t. Q = constant.



A further application is the following new equation

$$\left(\frac{\partial^2\langle V_{\rm z}^2\rangle}{\partial z^2}\right)(R,0) = -\lambda(R)\left[\left(\frac{\langle V_{\rm R}^2\rangle - \langle V_{\rm z}^2\rangle}{R}\right)\frac{\partial \ln\langle V_{\rm z}^2\rangle}{\partial R}\right](R,0)$$

Since $\lambda(R) > 0$, $\langle V_{\rm R}^2\rangle > \langle V_{\rm z}^2\rangle$ and $\langle V_{\rm z}^2\rangle$ decreasing with R, the righthand side of the equation has to be positive.

That means that $\langle V_{\rm z}^2\rangle$ has a minimum in the plane.

So disks are not strictly isothermal in z and numerical values suggest less peaked in density than the exponential function.

The final application gives a more accurate estimate of the velocity anisotropy in the plane through

$$\frac{\langle V_\theta^2\rangle}{\langle V_\mathrm{R}^2\rangle} = \frac{1}{2}\left\{1+\frac{\partial \ln V_\mathrm{t}}{\partial \ln R} - \frac{S_{\theta\theta\theta}}{V_\mathrm{t}\langle V_\mathrm{R}^2\rangle} + \frac{1}{\nu R V_\mathrm{t}\langle V_\mathrm{R}^2\rangle}\frac{\partial R^2 \nu S_{\mathrm{RR}\theta}}{\partial R} + \frac{R}{V_\mathrm{t}\langle V_\mathrm{R}^2\rangle}\frac{\partial S_{\mathrm{R}\theta\mathrm{z}}}{\partial z} + \frac{V_\mathrm{t}^2 - V_\mathrm{rot}^2}{V_\mathrm{t}\langle V_\mathrm{R}^2\rangle^2} S_{\mathrm{RR}\theta} + \frac{T_{\mathrm{RR}\theta\theta}}{\langle V_\mathrm{R}^2\rangle^2}\right\}$$

In practice this can be approximated as

$$\frac{\langle V_\mathrm{R}^2\rangle}{\langle V_\theta^2\rangle} = \frac{1}{2}\left(1+\frac{\partial \ln V_\mathrm{t}}{\partial \ln R} + \frac{T_{\mathrm{RR}\theta\theta}}{\langle V_\mathrm{R}^2\rangle^2}\right)$$

This constitutes a small correction to the classical result

$$\frac{\langle V_\mathrm{R}^2\rangle}{\langle V_\theta^2\rangle} = \frac{1}{2}\left(1+\frac{\partial \ln V_\mathrm{t}}{\partial \ln R}\right) = \frac{-B}{A-B}$$