DYNAMICS OF GALAXIES

1. Fundamentals

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Winter 2008/9

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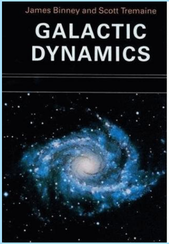

James Binney & Scott Tremaine

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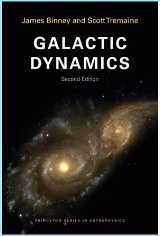

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Fundamental equations

The collisionless Boltzmann equation

Studies of galactic dynamics start with two fundamental equations. The first is the *continuity equation*, also called the *Liouville* or *collisionless Boltzmann equation*.

It states that in any element of phase space the time derivative of the distribution function equals the number of stars entering it minus that leaving it, if no stars are created or destroyed.

Write the distribution function in phase space as $f(x, y, z, u, v, w, t)$ and the potential as $\Phi(x, y, z, t)$.

Now look first for the one-dimensional case at a position x, u. After a time interval dt the stars at $x - dx$ have taken the place of the stars at x, where $dx = udt$.

So the change in the distribution function is

$$df(x,u) = f(x - udt, u) - f(x,u)$$

$$\frac{df}{dt} = \frac{f(x-udt,u) - f(x,u)}{dt} = \frac{f(x-dx,u) - f(x,u)}{dx} u = -\frac{df(x,u)}{dx} u$$

For the velocity replace the positional coordinate with the velocity x with u and the velocity u with the acceleration du/dt. But according to Newton's law we can relate that to the force or the potential. So we get

$$\frac{df}{dt} = -\frac{df(x,u)}{du}\frac{du}{dt} = \frac{df(u,x)}{du}\frac{d\Phi}{dx}$$

The total derivative of the distribution function then is

$$\frac{\partial f(x,u)}{\partial t} + \frac{\partial f(x,u)}{\partial x} u - \frac{\partial f(x,u)}{\partial u} \frac{\partial \Phi}{\partial x} = 0$$

In three dimensions this becomes

$$\frac{\partial f}{\partial t} + u \frac{\partial f}{\partial x} + v \frac{\partial f}{\partial y} + w \frac{\partial f}{\partial z} - \frac{\partial \Phi}{\partial x} \frac{\partial f}{\partial u} - \frac{\partial \Phi}{\partial y} \frac{\partial f}{\partial v} - \frac{\partial \Phi}{\partial z} \frac{\partial f}{\partial w} = 0$$

Usually dynamical systems are assumed to be in equilibrium so that we have

$$u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} - \frac{\partial \Phi}{\partial x}\frac{\partial f}{\partial u} - \frac{\partial \Phi}{\partial y}\frac{\partial f}{\partial v} - \frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial w} = 0.$$

This is the collisionless Boltzmann equation

Usually (especially in disk galaxies) we work in cylindrical coordinates.

The distribution function then is $f(R, \theta, z, V_{\rm R}, V_\theta, V_{\rm z}, t)$ and the collisonless Boltzmann equation becomes

$$\begin{aligned} V_{\rm R}\frac{\partial f}{\partial R} + \frac{V_\theta}{R}\frac{\partial f}{\partial \theta} + V_{\rm z}\frac{\partial f}{\partial z} + \left(\frac{V_\theta^2}{R} - \frac{\partial \Phi}{\partial R}\right)\frac{\partial f}{\partial V_{\rm R}} & \\ - \left(\frac{V_{\rm R} V_\theta}{R} + \frac{1}{R}\frac{\partial \Phi}{\partial \theta}\right)\frac{\partial f}{\partial V_\theta} - \frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial V_{\rm z}} &= 0. \end{aligned}$$

For axial symmetry this reduces to

$$V_{\mathrm{R}}\frac{\partial f}{\partial R}+V_{\mathrm{z}}\frac{\partial f}{\partial z}-\left(\frac{\partial \Phi}{\partial R}-\frac{V_{\theta}^{2}}{R}\right)\frac{\partial f}{\partial V_{\mathrm{R}}}-\frac{V_{\mathrm{R}}V_{\theta}}{R}\frac{\partial f}{\partial V_{\theta}}-\frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial V_{\mathrm{z}}}=0.$$

For spherical symmetry this reduces further to

$$V_{\mathrm{R}}\frac{\partial f}{\partial R}-\left(\frac{\partial \Phi}{\partial R}-\frac{V_{\theta}^{2}}{R}\right)\frac{\partial f}{\partial V_{\mathrm{R}}}=0.$$

Here the velocity V_{θ} corresponds to the angular momentum of the system.

Poisson's equation

The second fundamental equation is Poisson's equation, which says that the gravitational potential derives from the combined gravitational forces of all the matter.

It can be written as

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} \equiv \nabla^2\Phi = 4\pi G\rho(x,y,z)$$

In cylindrical coordinates

$$\frac{\partial^2\Phi}{\partial R^2} + \frac{1}{R}\frac{\partial\Phi}{\partial R} + \frac{1}{R^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2} = 4\pi G\rho(R,\theta,z).$$

For the axisymmetric case

$$\frac{\partial K_{\rm R}}{\partial R} + \frac{K_{\rm R}}{R} + \frac{\partial K_{\rm z}}{\partial z} = -4\pi G\rho(R,z)$$

the spherical case

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial \Phi}{\partial R}\right) = 4\pi G\rho(R).$$

and the plane-parallel case

$$\frac{dK_{\rm z}}{dz} = -4\pi G\rho(z).$$

The collissionless Boltzmann and Poisson equations together completely describe the dynamics of a system.

The Poisson equation always refers to the total mass density distribution ρ. In the Boltzmann equation we may be looking at the distribution function of a sub-component, for which the mass density then is denoted by ν.

In a self-gravitating system of course ρ and ν are the same.

Hydrodynamic equations

In practice we never observe full distribution functions, but only the first three moments of it in the form of density, systematic motion and amount of random motion of velocity dispersion.

The hydrodynamic, moment or Jeans equations are obtained from the collissionless Boltzmann equation by multiplication by a velocity to some power followed by integration over all velocities (as in calculating moments for a distribution).

The Boltzman equation was

$$u\frac{\partial f}{\partial x}+v\frac{\partial f}{\partial y}+w\frac{\partial f}{\partial z}-\frac{\partial \Phi}{\partial x}\frac{\partial f}{\partial u}-\frac{\partial \Phi}{\partial y}\frac{\partial f}{\partial v}-\frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial w}=0.$$

First we change to the often used notation to write this as

$$\frac{\partial f}{\partial t}+v_i\frac{\partial f(\vec{x},\vec{v})}{\partial x_i}-\frac{\partial \Phi}{\partial x_i}\frac{\partial f(\vec{x},\vec{v})}{\partial v_i}=0$$

Implicit is that we sum over all the values for $i=1,2,3$.

Next we take the convention $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dv_1 dv_2 dv_3 \equiv \int d^3v$

Then the zeroth, first and second order moments in velocity become

$$\int f \ d^3v = \nu$$

$$\frac{1}{\nu} \int v_i f \ d^3v = \langle v_i \rangle$$

$$\frac{1}{\nu} \int v_i v_j f \ d^3v = \langle v_i v_j \rangle$$

From now on I write $f = f(\vec{x}, \vec{v})$.

Zeroth order moment of the Boltzmann equation

$$\int \frac{\partial f}{\partial t} d^3v + \int v_i \frac{\partial f}{\partial x_i} d^3v - \frac{\partial \Phi}{\partial x_i} \int \frac{\partial f}{\partial v_i} d^3v = 0$$

This can be rewritten as

$$\frac{\partial \nu}{\partial t} + \frac{\partial}{\partial x_i} \int v_i f \; d^3v - \frac{\partial \Phi}{\partial x_i} \int \left. f(v_i) \right]_{-\infty}^{\infty} d^2v_{\neq i} = 0$$

Then

$$\left. f(v_i) \right]_{-\infty}^{\infty} = 0 \quad \Longrightarrow \quad \frac{\partial \nu}{\partial t} + \frac{\partial}{\partial x_i}(\nu \langle v_i \rangle) = 0$$

First order moment of the Boltzmann equation

$$\int v_j \frac{\partial f}{\partial t} d^3 v + \int v_i v_j \frac{\partial f}{\partial x_i} d^3 v - \frac{\partial \Phi}{\partial x_i} \int v_j \frac{\partial f}{\partial v_i} d^3 v = 0$$

Now

$$\int v_j \frac{\partial f}{\partial v_i} d^3 v = \int f(v_i)]_{-\infty}^{\infty} d^2 v_{\neq i} - \int \left(\frac{\partial v_j}{\partial v_i} \right) f \ d^3 v = 0 - \delta_{ij} \nu$$

so

$$\frac{\partial}{\partial t} \left(\nu \langle v_j \rangle \right) + \frac{\partial}{\partial x_i} \left(\nu \langle v_i v_j \rangle \right) + \nu \frac{\partial \Phi}{\partial x_j} = 0$$

Second order moment of the Boltzmann equation

Similarly we find

$$\nu \frac{\partial \langle v_j \rangle}{\partial t} - \langle v_i \rangle \frac{\partial (\nu v_j)}{\partial x_i} + \frac{\partial (\nu \langle v_i v_j \rangle)}{\partial x_i} + \nu \frac{\partial \Phi}{\partial x_j} = 0$$

This equation is often rewritten using the velocity dispersion tensor:

$$\sigma_{ij}^2 = \langle (v_i - \langle v_i \rangle) \times (v_j - \langle v_j \rangle) \rangle = \langle v_i v_j \rangle - \langle v_i \rangle \langle v_j \rangle = \overline{v_i v_j} - \overline{v_i}.\overline{v_j}$$

Then

$$\frac{\partial (\nu \sigma_{ij}^2)}{\partial x_i} = \frac{\partial (\nu \langle v_i v_j \rangle)}{\partial x_i} - \langle v_j \rangle \frac{\partial (\nu \langle v_i \rangle)}{\partial x_i} - \nu \langle v_i \rangle \frac{\partial \langle v_j \rangle}{\partial x_i}$$

So we can write the second order Bolzmann equation as

$$\nu\frac{\partial\langle v_j\rangle}{\partial t}+\nu\langle v_i\rangle\frac{\partial\langle v_j\rangle}{\partial x_i}+\frac{\partial(\nu\sigma_{ij}^2)}{\partial x_i}+\nu\frac{\partial\Phi}{\partial x_j}=0$$

So we see that the zeroth, first and second order Boltzmann equations describe relations between the density distribution of a component ν, the mean motions $\langle v_i\rangle$ and the random motions $\langle v_i v_j\rangle$ or σ_{ij} with the potential Φ.

Densities, mean and random motions are *in principle* observables.

Jeans equations

These moment equations are also called Jeans equations and are usually applied in equilibrium when f is not a function of time.

In the practical case the velocity dispersion tensor is assumed to have a diagonal form, i.e. there is a velocity ellipsoid with semi-major axes $\sigma_{11}, \sigma_{22}, \sigma_{33}$ and all cross-terms equal to zero.

In general the Jeans equation cannot be solved without additional assumptions.

And in practice we measure only surface density distributions projected onto the plane of the sky and velocities and velocity dispersion projected onto the line-of-sight.

In the axi-symmetric case the Jeans equations are derived in the same way.

For the radial direction we find:

$$\frac{\partial}{\partial R}(\nu\langle V_{\rm R}^2\rangle)+\frac{\nu}{R}\{\langle V_{\rm R}^2\rangle-V_{\rm t}^2-\langle(V_\theta-V_{\rm t})^2\rangle\}+\frac{\partial}{\partial z}(\nu\langle V_{\rm R}V_{\rm z}\rangle)=\nu K_{\rm R}$$

By assumption we have taken here $V_{\rm t}=\langle V_\theta\rangle$ and $\langle V_{\rm R}\rangle=\langle V_{\rm z}\rangle=0$.

This can be rewritten as:

$$\begin{aligned}-K_{\rm R} \quad = \quad & \frac{V_{\rm t}^2}{R}-\langle V_{\rm R}^2\rangle\left[\frac{\partial}{\partial R}(\ln\nu\langle V_{\rm R}^2\rangle)+\frac{1}{R}\left\{1-\frac{\langle(V_\theta-V_{\rm t})^2\rangle}{\langle V_{\rm R}^2\rangle}\right\}\right]+\\ & \langle V_{\rm R}V_{\rm z}\rangle\frac{\partial}{\partial z}(\ln\nu\langle V_{\rm R}V_{\rm z}\rangle)\end{aligned}$$

The last term reduces in the symmetry plane to

$$\langle V_{\rm R} V_{\rm z}\rangle \frac{\partial}{\partial z}(\ln \nu \langle V_{\rm R} V_{\rm z}\rangle) = \frac{\partial}{\partial z}\langle V_{\rm R} V_{\rm z}\rangle$$

and may then be assumed zero.

For the azimuthal direction the moment equation is seldom used, because it only contains cross-terms of the velocity tensor. It reads

$$\frac{2\nu}{R}\langle V_{\rm R} V_\theta\rangle + \frac{\partial}{\partial R}(\nu \langle v_{\rm R} V_\theta\rangle) + \frac{\partial}{\partial z}(\nu \langle V_\theta V_{\rm z}\rangle) = 0$$

In the vertical direction the moment equation becomes

$$\frac{\partial}{\partial z}(\nu\langle V_{\mathrm{z}}^2\rangle) + \frac{\nu\langle V_{\mathrm{R}} V_{\mathrm{z}}\rangle}{R} + \frac{\partial}{\partial R}(\nu\langle V_{\mathrm{R}} V_{\mathrm{z}}\rangle) = \nu K_{\mathrm{z}}$$

For spherical symmetry we have velocities V_{R}, V_θ and V_ϕ

$$\frac{\partial}{\partial R}(\nu\langle V_{\mathrm{R}}^2\rangle) + \frac{\nu}{R}\{2\langle V_{\mathrm{R}}^2\rangle - V_{\mathrm{t}}^2 - \langle (V_\theta - V_{\mathrm{t}})^2\rangle - \langle V_\phi^2\rangle\} = \nu K_{\mathrm{R}}$$

In plane-parallel layers the Jeans equation reduces to

$$\frac{d}{dz}\left\{\nu\langle V_{\mathrm{z}}^2\rangle\right\} = \nu K_{\mathrm{z}}$$

Virial equations

The virial equations are derived from the first-order moment Jeans equation for a self-gravitating system (so $\nu = \rho$) by taking its first order moment over spatial coordinates.

$$\frac{\partial}{\partial t}\left(\rho\bar{v}_j\right) + \frac{\partial}{\partial x_i}\left(\rho\overline{v_i v_j}\right) + \rho\frac{\partial\Phi}{\partial x_j} = 0$$

So we get

$$\int x_k \frac{\partial\left(\rho\bar{v}_j\right)}{\partial t} d^3x = -\int x_k \frac{\partial}{\partial x_i}\left(\rho\overline{v_i v_j}\right) d^3x - \int x_k \rho\frac{\partial\Phi}{\partial x_j} d^3x$$

Moment of inertia tensor

Look at the term on the left

$$\int x_k \frac{\partial\left(\rho \bar{v}_j\right)}{\partial t} d^3x$$

and define the moment of inertia tensor

$$I_{jk} = \int \rho x_j x_k d^3x$$

Take the first derivative of this tensor.

$$\frac{d}{dt} I_{jk} = \int \frac{\partial \rho}{\partial t} x_j x_k d^3 x$$

Now recall the zeroth-order moment Jeans equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho \langle v_i \rangle) = 0$$

Then we can write

$$\frac{d}{dt} I_{jk} = - \int \frac{\partial \left(\rho \bar{v}_i\right)}{\partial x_i} x_j x_k d^3 x$$

This reduces to

$$\frac{d}{dt} I_{jk} = \int \rho \bar{v}_i \left(\delta_{ij} x_k + \delta_{ik} x_j \right) d^3x = \int \rho \left(\bar{v}_j x_k + \bar{v}_k x_j \right) d^3x$$

and

$$\frac{d^2}{dt^2} I_{jk} = \int \left[x_k \frac{\partial}{\partial t} \left(\rho \bar{v}_j \right) + x_j \frac{\partial}{\partial t} \left(\rho \bar{x}_k \right) \right] d^3x$$

The moment of inertia tensor should be symmetric with respect to the coordinates, so

$$\frac{d^2}{dt^2} \left(\tfrac{1}{2} I_{jk} \right) = \int x_k \frac{\partial}{\partial t} \left(\rho \bar{v}_j \right) d^3x$$

Kinetic energy tensor

Now take the first term on the right and use integration by parts.

$$
\begin{aligned}
-\int x_k \frac{\partial}{\partial x_i}\left(\rho\overline{v_i v_j}\right) d^3x &= \int \rho\overline{v_i v_j}\frac{\partial x_k}{\partial x_i} d^3x - \int \frac{\partial}{\partial x_i}\left(x_k \rho\overline{v_i v_j}\right) d^3x \\
&= \int \delta_{ik}\rho\overline{v_i v_j}\frac{\partial x_k}{\partial x_i} d^3x - \int \delta_{ik}\frac{\partial}{\partial x_i}\left(x_k \rho\overline{v_i v_j}\right) d^3x \\
&= \int \rho\overline{v_k v_j} d^3x - 0 = 2K_{kj}
\end{aligned}
$$

where we have defined the kinetic energy tensor.

We can distinguish between the ordered and random motions using

$$\overline{v_k v_j} = \bar{v}_k . \bar{v}_j + \sigma^2_{kj}$$

This gives rize to a motions tensor T_{jk} and a velocity dispersion tensor Π_{jk}

$$\begin{aligned} K_{ij} &= \int \rho \bar{v}_i . \bar{v}_j d^3x + \tfrac{1}{2} \int \rho \sigma^2_{ij} d^3x \\ & T_{ij} + \tfrac{1}{2} \Pi_{ij} \end{aligned}$$

Potential energy tensor

Finally the second term on the right. This we define as the potential energy tensor.

$$W_{jk} = -\int x_j \frac{\partial \Phi}{\partial x_k} d^3x$$

This finally gives

$$\tfrac{1}{2} \frac{d^2}{dt^2} I_{ij} = 2T_{ij} + \Pi_{ij} + W_{ij}$$

The trace of the tensors give the total energies, so the trace of the last equation reduces for the static case to

$$2T + \Pi = 2K = -W$$

Integrals of motion

Recall the collisonless Boltzmanmn equation

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} - \frac{\partial \Phi}{\partial x}\frac{\partial f}{\partial u} - \frac{\partial \Phi}{\partial y}\frac{\partial f}{\partial v} - \frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial w} = 0.$$

Now consider the equations of motion of an individual star:

$$\frac{dx}{dt} = u,\ \frac{dy}{dt} = v,\ \frac{dz}{dt} = w,\ \frac{du}{dt} = -\frac{\partial \Phi}{\partial x},\ \frac{dv}{dt} = -\frac{\partial \Phi}{\partial y},\ \frac{dw}{dt} = -\frac{\partial \Phi}{\partial z}$$

Fill this in and we get

$$\frac{\partial f}{\partial t} + \frac{dx}{dt}\frac{\partial f}{\partial x} + \frac{dy}{dt}\frac{\partial f}{\partial y} + \frac{dz}{dt}\frac{\partial f}{\partial z} + \frac{du}{dt}\frac{\partial f}{\partial u} + \frac{dv}{dt}\frac{\partial f}{\partial v} + \frac{dw}{dt}\frac{\partial f}{\partial w} \equiv \frac{Df}{Dt} = 0.$$

So along the path of any star in phase space the total derivative of the distribution function Df/Dt is zero.

The density in phase space is constant along the path of any star and the flow of stars in phase space is incompressible.

The equations of motion of a star can be rearranged as:

$$dt = \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = \frac{du}{-\partial\Phi/\partial x} = \frac{dv}{-\partial\Phi/\partial y} = \frac{dw}{-\partial\Phi/\partial z}$$

These are 6 independent ordinary differential equations which yield 6 integration constants for each orbit.

These integration constants thus correspond to a set of 6 independent properties with each combination of values related to a particular stellar orbits.

The distribution function f then simply tells which of these orbits are actually populated, so the general solution of the Boltzmann equation can be written as

$$f(x, y, z, u, v, w) = F(I_1, I_2, ..., I_6)$$

The I's are called the integrals of motion.

The question is then to what physical properties (if any!) these integrals of motion correspond.

Summarizing we have:

- Integrals of motion are functions $I_i(\vec{r}, \vec{v}, t)$ that are constant along an orbit (or $DI/Dt = 0$).
- In phase space there are surfaces $I_i(\vec{r}, \vec{v}, t) = \text{constant}$ and the orbit is the intersection of these surfaces.
- There cannot be more than 6 integrals of motion.

Isolating integrals of motion

We see that the distribution function depends *only* on the integrals of motion. So what are these?

One can be identified as the energy, which is always conserved along an orbit:

$$I_1 = E = \frac{1}{2}(u^2 + v^2 + w^2) + \Phi(x, y, z) = \mathrm{constant}$$

This is called an *isolating* integral of motion, because for particular values it isolates hyper-surfaces in phase space.

The others in general are non-isolating and are only implicit in the numerical integration of an orbit.

In an axisymmetric potential there is a second isolating integral: the angular momentum in the direction of the symmetry axis z is also conserved along an orbit.

$$I_2 = J = RV_\theta$$

Then we have

$$f(R, z, V_\mathrm{R}, V_\theta, V_\mathrm{z}) = F(E, J)$$

Actually, in a spherically symmetric potential all three components of the angular momentum are isolating intergals.

In the case of the Galaxy near the plane (at small z) the potential is separable and the R- and z-motions will then be decoupled

$$\Phi(R,z) = \Phi_1(R) + \Phi_2(z)$$

Then the decoupled z-energy is a third integral of motion:

$$I_3 = \frac{1}{2} V_z^2 + \Phi_2(z)$$

I will have much more to say later about the so-called third integral problem, which is related to this.

In general any symmetry in the potential or any coordinate system in which the potential can be separated gives rise to integrals of motion.

The integrals that I mentioned for these specific cases restrict the orbit of a star to certain regions of 6-dimensional phase space.

That is why they are called isolating integrals of motion.

But not all integrals of motion have this property and they are called non-isolating integrals and are not of much use.

The concept isolating versus non-isolating will be illustrated next with a simple example.

Non-isolating integrals of motion

Consider the two-dimensional harmonic oscillator with different periods. The equations of motion are

$$x = X \sin \alpha(t - t_x) \quad ; \quad y = Y \sin \beta(t - t_y)$$

Obviously when α/β is rational the orbit is periodic and has a single path.

What are the integrals of motion? First realise that

$$\frac{dx}{dt} = X\alpha \cos \alpha(t - t_x)$$

From x and dx/dt we can form a time-independent parameter:

$$I_1 = \left(\frac{dx}{dt}\right)^2 + \alpha^2 x^2 = X^2(\alpha^2+1) = \mathrm{constant}$$

This then is an integral of motion and confines x to the interval $(-X < x < X)$.

Similarly we have

$$I_2 = \left(\frac{dy}{dt}\right)^2 + \beta^2 y^2 = Y^2(\beta^2+1)$$

Together these integrals then confine the orbit to the area $(-X < x < X, -Y < y < Y)$.

There is a third time-independent quantity that we can derive as follows.

Eliminate t from the two equations of motion; then we get

$$I_3 = \frac{1}{\alpha} \arcsin\left(\frac{x}{X}\right) + \frac{1}{\beta} \arcsin\left(\frac{y}{Y}\right) = t_x - t_y$$

This can be re-arranged as

$$x = X \sin\left[\alpha I_3 - \frac{\alpha}{\beta} \arcsin\left(\frac{y}{Y}\right)\right]$$

Now $\arcsin(y/Y)$ repeats every interval 2π and therefore the second term repeats every interval $2\pi\alpha/\beta$.

If α/β is rational we then get for any value of y a finite number of values for x between $-X$ and X and therefore the orbit is periodic. Then I_3 can also assume a finite number of values and therefore is an isolating integral of motion.

But if α/β is irrational, the second term can assume an infinity of values and x also is not constrained and I_3 can have an infinite number of values and does not constrain the orbit within the area $(-X < x < X, -Y < y < Y)$.

Then I_3 is a non-isolating integral of motion and of no practicle value.

So we see that:

- The number of isolating integrals of motion depend on *both* the potential and the particular orbit and
- For a particular potential some orbits can have more isolating integrals than others.

A further illustration of non-isolating integrals of motions is phase mixing[1].

Assume that stars move in a potential $\Phi(\vec{r})$ and have closed orbits on $(\vec{r}, \vec{v})$. One integral of motion is the total energy of a star

$$E = \frac{1}{2} v^2 + \phi(\vec{r})$$

The orbital period $T(E)$ depends on E. Take for the starting position $\vec{r_\circ}$.

Then the orbital phase angle ψ of the star at time t is

$$\psi(E, \vec{r}) = \psi(E, \vec{r_\circ}) + 2\pi \frac{t}{T(E)}$$

[1] K.C. Freeman, Stars & Stellar Sytems IX, 409 (1975)

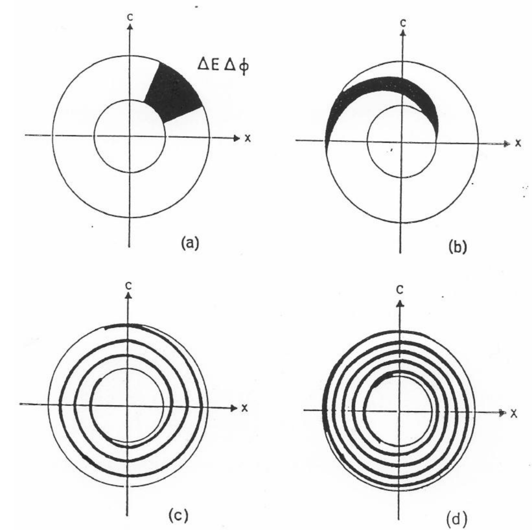
Therefore

$$\psi(E, \vec{r_\circ}) = \psi(E, \vec{r}) - 2\pi \frac{t}{T(E)}$$

is another integral of motion.

So the distribution function can be written as $f(E, \psi - 2\pi t/T)$ and we can follow f in the (E, ψ)-plane.

Say, it intially starts as a distribution limited by values of E and ψ. Then since T is a function of E we find a development as in the following schematic figure.


c
ΔE Δφ
x
(a)
c
x
(b)
c
x
(c)
c
x
(d)

Although initially confined to a small range $\Delta E \Delta \psi$, the distribution function evolves to a distribution over all phases.

So the distribution function looses its dependence on phase angle and the second integral is non-isolating.

The only isolating integral is the energy.

In general, it may be stated that the non-isolating integrals do define surfaces in phase space, they come close in phase space to any point allowed by the isolating integrals and therefore provide no further constraints on the properties of the orbits.

Jeans' theorem

Jeans' theorem is:
Any arbitrary function of the integrals of motions satisfies the collisionless Boltzmann equation

This is so because the distribution function is constant along the path of an orbit, $Df/Dt = 0$. If f is any function of $I_1....I_n$.

$$\frac{Df}{Dt} = \sum_{i=1}^{n} \frac{\partial f}{\partial I_i} \frac{dI_i}{\partial t} = 0$$

However, in order to make a self-consistent system as a solution that resembles a real galaxy, we also need to satisfy the Poisson equation. This is referred to as the self-consistency problem.

Now, the integral of $f(I_i)$ over all integrals I_i at any position is the local density and this must be single valued.

But in general we only know the (single valued) isolating integrals.

Lynden-Bell[2] inferred from this that the distribution function can be completely defined by the isolating integrals only.

E.g. in a system that is spherical in all its properties (so it must depend on the magnitude of the angular momentum, but not its direction) the distribution function is $f = f(E, L^2)$.

Lynden-Bell[3] showed that it is possible for rotating systems to be spherical, while intuitively one expects it to be always oblate.

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[2]D. Lynden-Bell, MNRAS 123, 1 (1962)
[3]D. Lynden-Bell, MNRAS 120, 240 (1960)