Applied Signal Processing – 2/2/2009 9.15-12.15h

Please put your name and student number on all of your answer sheets. The final grade for this exam will $\frac{9}{10.5}\ x\ the\ total\ number\ of\ points\ +1$.

<u>1.</u> Give the Fourier Transform of the following functions:

(a) $f(t)=1\ \ (|t|\leqslant\frac{T}{2});\quad f(t)=0\ \ elsewhere$ **(0.5)**

(b) $f(t)=\delta(t)$ **(0.5)**

(c) $f(t)=\sin(\omega t)$ **(0.5)**

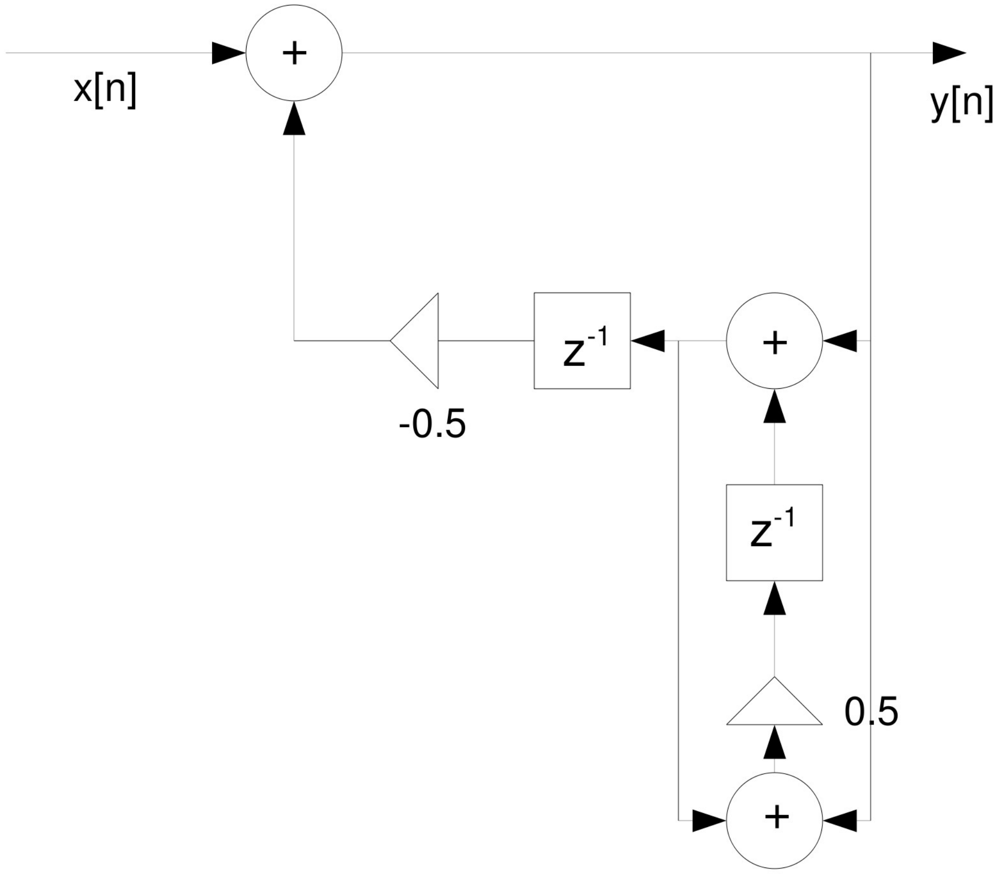
<u>2.</u> The transfer function of a linear system is given by

$$H(z)=\frac{z^2+0.64}{z^2-0.81}$$

(a) Sketch the pole-zero diagram. **(0.5)**
(b) State the reason why this system is stable. **(0.5)**
(c) Determine and sketch the magnitude response of the system. **(1)**
(d) What kind of filter is represented by this system (Lowpass, etc.)? **(0.5)**
(e) Draw a block diagram of this system. **(0.5)**
(f) Find the first 5 terms of the impulse response. **(1)**
(g) How should the transfer be modified if the magnitude response has to be zero at $\omega=\pi/3$? **(0.5)**

<u>3.</u> (a) Perform the circular convolution in the time domain of the sequences {1,1,0,1} and {1,0,0,1}. **(0.5)**
(b) Perform the same circular convolution using DFT's and IDFT's. **(1)**

<u>4.</u> (a) For the block diagram below, introduce some appropriate internal variables, and write the algorithm that tells you how to compute each output sample y[n] from each input sample x[n]. **(1)**



(b) Give an alternative, equivalent realisation of this block diagram. **(0.5)**

(c) Working in the z-domain, show that the transfer function of the block diagram above is given by

$$H(z)=\frac{1-0.5z^{-1}}{1+0.25z^{-2}}$$ **(0.5)**

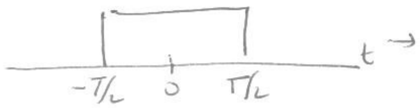
(d) Show that the causal impulse response of this filter is given by

$$h[n]=(0.5)^n\left(\cos\left(\frac{\pi n}{2}\right)-\sin\left(\frac{\pi n}{2}\right)\right)\mu[n]$$ **(1)**

Exam 2/2/09

Answers

1a)



$$\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt = \int_{-T/2}^{T/2} e^{-i\omega t}\, dt = \left[-\frac{1}{i\omega} e^{-i\omega t} \right]_{-T/2}^{T/2}$$

$$= -\frac{1}{i\omega}\left(-i \sin\frac{\omega T}{2} - i \sin\frac{\omega T}{2} \right) = -\frac{2i}{i\omega} \sin\frac{\omega T}{2} =$$

$$= \frac{2\sin\left(\frac{\omega T}{2}\right)}{\omega} = T \operatorname{sinc}\left(\frac{\omega T}{2}\right)$$

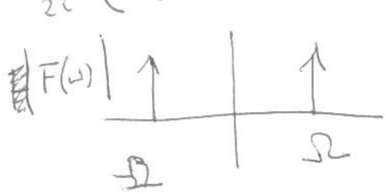
b) $$\int_{-\infty}^{\infty} \delta(t)\, e^{-i\omega t}\, dt = 1$$

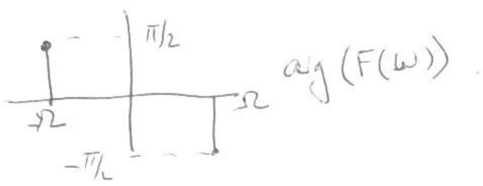
c) $$\int_{-\infty}^{\infty} \sin \Omega t \; e^{-i\omega t}\, dt$$ (I use Ω instead of ω for clarity)

$$= \int_{-\infty}^{\infty} \frac{e^{i\Omega t} - e^{-i\Omega t}}{2i}\, e^{-i\omega t}\, dt =$$

$$\frac{1}{2i}\left[\int_{-\infty}^{\infty} e^{-i(\omega-\Omega)t}\, dt - \int_{-\infty}^{\infty} e^{-i(\omega+\Omega)t}\, dt \right] =$$

$$\frac{1}{2i}\left(\delta(\omega-\Omega) - \delta(\omega+\Omega) \right) = \frac{1}{2}\left(-i\,\delta(\omega-\Omega) + i\,\delta(\omega+\Omega) \right)$$

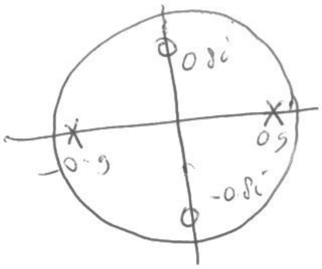




(2)

$$H(z) = \frac{(z - 0.8e^{-\pi/2 i})(z + 0.8 e^{-\pi/2 i})}{(z - 0.9)(z + 0.9)}$$

a) Poles at $z = \pm 0.9$
Zero's at $z = \pm 0.8i$



b) All poles inside the unit circle.

c) $|H(e^{i\omega})|^2 = H(e^{i\omega})H(e^{-i\omega})$

$$= \left(\frac{e^{2i\omega} + 0.64}{e^{2i\omega} - 0.81}\right)\left(\frac{e^{-2i\omega} + 0.64}{e^{-2i\omega} - 0.81}\right)$$

$$= \left(\frac{1 + 0.4096 + 0.64(e^{2i\omega} + e^{-2i\omega})}{1 + 0.6561 - 0.81(e^{2i\omega} + e^{-2i\omega})}\right)$$

$$= \frac{1.496 + 1.28\cos 2\omega}{1.6561 - 1.62\cos 2\omega}$$

Calculate maxima and minima of $|H(e^{i\omega})|^2$

$$\frac{d\,|H(e^{i\omega})|^2}{d\omega} = 0 \qquad -2.56\sin 2\omega\,(1.6561 - 1.62\cos 2\omega)$$
$$\cancel{1.4}\,(1.496 + 1.28\cos 2\omega)\cdot \cancel{1.62}\,3.24\sin 2\omega = 0$$

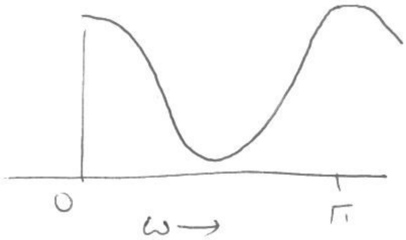
$-\cancel{2.56 \sin 2\omega}$

$-4.240 \sin 2\omega + \cancel{4.147 \sin 2\omega \cos 2\omega}$

$-4.847 \sin 2\omega - \cancel{4.147 \sin 2\omega \cos 2\omega} = 0$

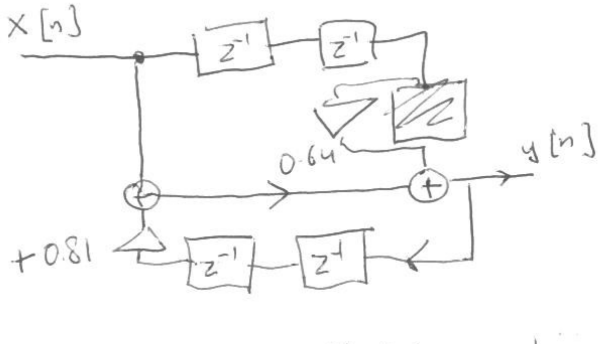
$\sin 2\omega = 0 \quad \vee \quad -9.087 = 0 \quad \rightarrow \quad \omega = \frac{k\pi}{2}$

Drawing tells you that the function is periodic with period π. Maximum at $\omega = k\pi$, Minimum at $\omega = (k + \frac{1}{2})\pi$.



d) This is band stop filter

e) $H(z) = \dfrac{1 + 0.64 z^{-2}}{1 - 0.81 z^{-2}}$



f) $H(z) = H_1(z) + H_2(z)$, waarby

$H_1(z) = \dfrac{1}{1 - 0.81 z^{-2}}$ en $H_2(z) = \dfrac{0.64 z^{-2}}{1 - 0.81 z^{-2}}$

The impulse response of H_1 : $h_1[n]$ is then:

$$h_1[n] = \{ \underset{\uparrow}{1} \quad 0 \quad 0.81 \quad 0 \quad 0.81^2 \quad 0 \quad 0.81^3 \; \cdots \}$$

and $h_2[n] = \{ \underset{\uparrow}{0} \quad 0 \quad 0.64 \quad 0 \quad 0.64\times 0.81 \quad 0 \quad 0.64\times 0.81^2 \; \cdots \}$

So $h[n] = \{ 1 \quad 0 \quad 1.45 \quad 0 \quad 1.17 \; \cdots \}$

g) Add a zero at $\omega = \pi/3$. But to keep the coefficients real, a zero at the conjugate of $\omega = e^{i\pi/3}$, $e^{-i\pi/3}$ has to be introduced as well.

$$H'(z) = H(z)\left(z - e^{-i\pi/3}\right)\left(z - e^{i\pi/3}\right) =$$

$$H(z)\left(z^2 + 1 - z\right)$$

3) a)

$$\begin{array}{ccccccc}
1 & 1 & 0 & 1 & & & x[n] \\
1 & 0 & 0 & 1 & & & y[n] \\
\hline
1 & 1 & 0 & 1 & & & \\
1 & 0 & 1 & 1 & 1 & 0 & 1 \\
\hline
\{2 & 1 & 1 & 2\} & & &
\end{array}$$

$z[n] = x[n] \; ④ \; y[n]$

b) $z[n] = \mathrm{IDFT}\left(\mathrm{DFT}(x[n]) \cdot \mathrm{DFT}(y[n]) \right)$

$$\mathrm{DFT}(x[n]) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

3

$DFT(y[n]) =$

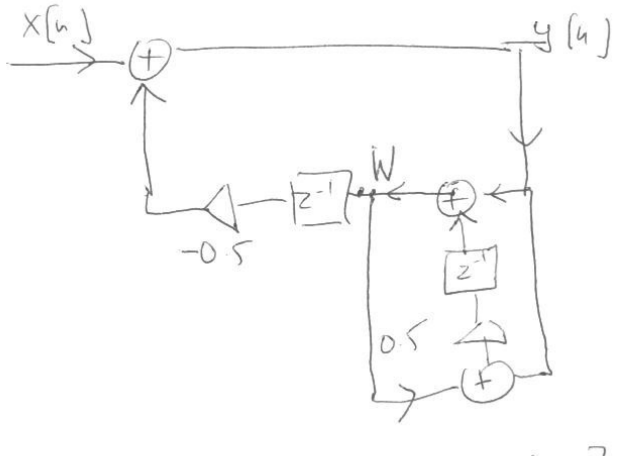
$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & 1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1+i \\ 0 \\ 1-i \end{pmatrix}$$

$$Z[k] = X[k]^T \, Y[k]^T = \begin{pmatrix} 6 \\ 1+i \\ 0 \\ 1-i \end{pmatrix}$$

$$IDFT\left(Z[k]^T\right) = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \begin{pmatrix} 6 \\ 1+i \\ 0 \\ 1-i \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix}$$

Is identical to the response in a)

4



$$W[n] = y[n] + 0.5\, y[n-1] + 0.5\, W[n-1] \quad \text{or} \quad W = Y + 0.5 z^{-1} Y + 0.5 z^{-1} W$$

$$y[n] = x[n] - 0.5\, W[n-1]$$

$$= x[n] - 0.5\, y[n-1] - 0.25\, y[n-2]$$

$$W(1 - 0.5 z^{-1}) = Y(1 + 0.5 z^{-1}$$

$$Y = X - 0.5 z^{-1} W$$

$$W = Y \frac{(1 + 0.5 z^{-1})}{1 - 0.5 z^{-1}}$$

$$\to \cancel{Y} \; Y = X - 0.5z^{-1}\left(Y\frac{1+0.5z^{-1}}{1-0.5z^{-1}}\right)$$

$$Y\left(1+0.5z^{-1}\left(\frac{1+0.5z^{-1}}{1-0.5z^{-1}}\right)\right) = X$$

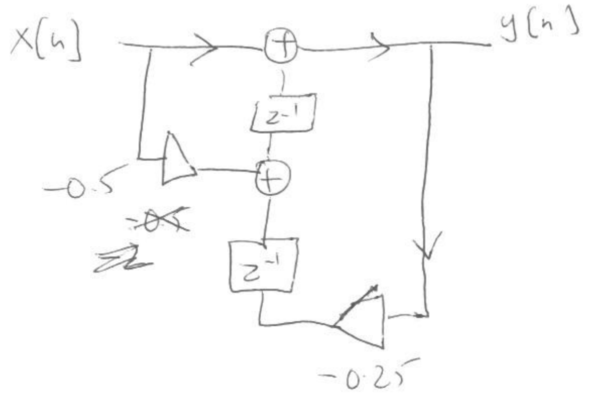
$$Y\left(\frac{1-0.5z^{-1}+0.5z^{-1}+0.25z^{-2}}{1-0.5z^{-1}}\right) = X$$

$$Y\left(\frac{1+0.25z^{-2}}{1-0.5z^{-1}}\right) = X$$

$$Y(1+0.25z^{-2}) = X(1-0.5z^{-1})$$

$$y[n] - 0.25y[n-2] = x[n] - 0.5x[n-1]$$

b)



c) Zie a)

d)

(d) From the lectures, we know the z-transform of

$$a^n \cos \omega_0 n \, \mu[n] \qquad \left(= \frac{1 - a z^{-1} \cos \omega_0}{1 - 2 a z^{-1} \cos \omega_0 + a^2 z^{-2}} \right)$$

and

$$a^n \sin \omega_0 n \, \mu[n] \qquad \left(= \frac{a z^{-1} \sin \omega_0}{1 - 2 a z^{-1} \cos \omega_0 + a^2 z^{-2}} \right)$$

Now, use $\omega_0 = \frac{\pi}{2}$, $a = 0.5$

$$\rightarrow \mathcal{Z}\left(0.5^n \cos\left(\frac{\pi n}{2}\right) \mu[n]\right) = \frac{1}{1 + 0.5^2 z^{-2}}$$

$$\mathcal{Z}\left(0.5^n \sin \frac{\pi n}{2} \mu[n]\right) = \frac{0.5 z^{-1}}{1 + 0.5^2 z^{-2}}$$

$$\rightarrow \mathcal{Z}\left(\left(\left(0.5^n \cos \frac{\pi n}{2}\right) - \left(0.5^n \sin \frac{\pi n}{2}\right)\right) \mu[n]\right) = \frac{1 - 0.5 z^{-1}}{1 + 0.25 z^{-2}}$$

So, the causal impulse response is

$$h[n] = 0.5^n \left(\cos \frac{\pi n}{2} - \sin \frac{\pi n}{2}\right) \mu[n]$$