Formula Sheet

Fourier Transform	$X(\omega)=F(x(t))=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$ $x(t)=F^{-1}(X(\omega))=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$
Fourier Sequence	$f(x)=\sum_{k=-\infty}^{k=\infty} A_n e^{jnx}$ $A_n=\frac{1}{2}\pi\int_{-\pi}^{\pi} e^{-jnx}dx$
Discrete-Time Fourier Transform	$X(e^{j\omega})=\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ $x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{i\omega})e^{j\omega n}d\omega$
Discrete Fourier Transform (DFT)	$X_k=DF(x)=\sum_{l=0}^{N-1} x_l e^{-2\pi j\frac{lk}{N}}$ $x_l=DF^{-1}(X)=\frac{1}{2\pi}\sum_{k=0}^{N-1} X_k e^{2\pi j\frac{lk}{n}}$
Principal properties of the Fourier Transform	$y=a_1x_1+a_2x_2\rightarrow F(y)=a_1F(x_1)+a_2F(x_2)$ $y(t)=x(t-\tau)\rightarrow F(y)=e^{-j\omega\tau}F(x)$ $y(t)=x(t)e^{j\omega_0 t}\rightarrow F(y)=X(\omega-\omega_0)$
Z-transform	$Z(x)=\sum_{k=-\infty}^{\infty} x_k z^{-k}$ $F(f)=Z(x)\vert_{z=e^{2\pi j\frac{f}{f_s}}}$
Convolution	$y(t)=\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau=x(t)*h(t)$ $y(t)=\sum_k x(t_k)h(t-t_k)$
Group Delay	$\tau(\omega)=-d\frac{(arg\,H(\omega))}{d\omega}$
Bilinear/Spectral Transform	$s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$

Table 3.2: Symmetry relations of the discrete-time Fourier transform of a real sequence.

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega}) = X_{\text{re}}(e^{j\omega}) + jX_{\text{im}}(e^{j\omega})$
$x_{\text{ev}}[n]$	$X_{\text{re}}(e^{j\omega})$
$x_{\text{od}}[n]$	$jX_{\text{im}}(e^{j\omega})$
Symmetry relations	$X(e^{j\omega}) = X^*(e^{-j\omega})$ $X_{\text{re}}(e^{j\omega}) = X_{\text{re}}(e^{-j\omega})$ $X_{\text{im}}(e^{j\omega}) = -X_{\text{im}}(e^{-j\omega})$ $\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ $\arg\{X(e^{j\omega})\} = -\arg\{X(e^{-j\omega})\}$

Note: $x_{\text{ev}}[n]$ and $x_{\text{od}}[n]$ denote the even and odd parts of $x[n]$, respectively.

Table 3.3: Commonly used discrete-time Fourier transform pairs.

Sequence	Discrete-Time Fourier Transform
$\delta[n]$	1
$1, \ (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$\mu[n]$	$\dfrac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$e^{j\omega_o n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi k)$
$\alpha^n \mu[n], \quad (\lvert\alpha\rvert < 1)$	$\dfrac{1}{1 - \alpha e^{-j\omega}}$
$(n+1)\alpha^n \mu[n], \ (\lvert\alpha\rvert < 1)$	$\frac{1}{(1-\alpha e^{-j\omega})^2}$
$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n}, \ (-\infty < n < \infty)$	$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \le \lvert\omega\rvert \le \omega_c, \\ 0, & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$

The z Transform:

$$G(z) = \sum_{n=-\infty}^{\infty} g[n] z^{-n}$$

$$z = \mathrm{Re}(z) + j\,\mathrm{Im}(z)$$

Table 6.2: Some useful properties of the z-transform.

Property	**Sequence**	**z -Transform**	**ROC**
	$g[n]$	$G(z)$	$\mathcal{R}_g$
	$h[n]$	$H(z)$	$\mathcal{R}_h$
Conjugation	$g^*[n]$	$G^*(z^*)$	$\mathcal{R}_g$
Time-reversal	$g[-n]$	$G(1/z)$	$1/\mathcal{R}_g$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G(z) + \beta H(z)$	Includes $\mathcal{R}_g \cap \mathcal{R}_h$
Time-shifting	$g[n - n_o]$	$z^{-n_o} G(z)$	$\mathcal{R}_g$, except possibly the point $z = 0$ or ∞
Multiplication by an exponential sequence	$\alpha^n g[n]$	$G(z/\alpha)$	$\vert\alpha\vert\mathcal{R}_g$
Differentiation of $G(z)$	$ng[n]$	$-z\dfrac{dG(z)}{dz}$	$\mathcal{R}_g$, except possibly the point $z = 0$ or ∞
Convolution	$g[n] \circledast h[n]$	$G(z)H(z)$	Includes $\mathcal{R}_g \cap \mathcal{R}_h$
Modulation	$g[n]h[n]$	$\frac{1}{2\pi j}\oint_C G(v)H(z/v)v^{-1}\,dv$	Includes $\mathcal{R}_g\mathcal{R}_h$
Parseval's relation		$\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi j}\oint_C G(v)H^*(1/v^*)v^{-1}\,dv$	

Note: If $\mathcal{R}_g$ denotes the region $R_{g-} < |z| < R_{g+}$ and $\mathcal{R}_h$ denotes the region $R_{h-} < |z| < R_{h+}$, then $1/\mathcal{R}_g$ denotes the region $1/R_{g+} < |z| < 1/R_{g-}$ and $\mathcal{R}_g\mathcal{R}_h$ denotes the region $R_{g-}R_{h-} < |z| < R_{g+}R_{h+}$.

The z Transform:

$$G(z) = \sum_{n=-\infty}^{\infty} g[n] z^{-n}$$

$$z = \mathrm{Re}(z) + j\,\mathrm{Im}(z)$$

Table 6.1: Some commonly used z-transform pairs.

Sequence	**z-Transform**	**ROC**
$\delta[n]$	1	All values of z
$\mu[n]$	$\dfrac{1}{1-z^{-1}}$	$\|z\| > 1$
$\alpha^n \mu[n]$	$\dfrac{1}{1-\alpha z^{-1}}$	$\|z\| > \|\alpha\|$
$n\,\alpha^n \mu[n]$	$\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\|z\| > \|\alpha\|$
$(n+1)\,\alpha^n \mu[n]$	$\dfrac{1}{(1-\alpha z^{-1})^2}$	$\|z\| > \|\alpha\|$
$(r^n \cos \omega_o n)\mu[n]$	$\dfrac{1-(r\cos\omega_o)z^{-1}}{1-(2r\cos\omega_o)z^{-1}+r^2 z^{-2}}$	$\|z\| > \|r\|$
$(r^n \sin \omega_o n)\mu[n]$	$\dfrac{(r\sin\omega_o)z^{-1}}{1-(2r\cos\omega_o)z^{-1}+r^2 z^{-2}}$	$\|z\| > \|r\|$

USEFUL EXPRESSIONS

The following expressions are often useful in calculating convolutions of analytical discrete signals

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, \quad |a| < 1$$

$$\sum_{n=k}^{\infty} a^n = \frac{a^k}{1-a}, \quad |a| < 1$$

$$\sum_{n=m}^{N} a^n = \frac{a^m - a^{N+1}}{1-a}, \quad a \neq 1$$

$$\sum_{n=0}^{N-1} a^n = \begin{cases} \frac{1-a^N}{1-a}, & |a| \neq 1 \\ N, & a = 1 \end{cases}$$