The orbits of stars

- To compute the orbit of a star in a galaxy (that is, its trajectory), we will consider only the smooth part of the gravitational potential (i.e. that is due to the large-scale mass distribution). This is justified by the result we derived previously: forces from individual stars or encounters are not important in galaxies.

- In a static spherical potential, we saw that both energy and angular momentum are conserved. The latter implies that the motion of a star is restricted to a plane: the orbital plane.

- In this case, only two coordinates are needed to describe the location of a star. Typically, one uses polar coordinates in the plane (r, ϕ) to describe the motion.

Orbits of stars in the Milky Way disk

- For stars in the Milky Way disk, we use a cylindrical coordinate system (R,ϕ,z), where $z = 0$ corresponds to the mid-plane of the disk, and the centre is at $R=0$.

- Cylindrical coordinates are preferred to spherical ones because of the symmetries of the mass distribution. To first order, the disk is axisymmetric: it is independent of the angular coordinate ϕ.
  - We are neglecting non-axisymmetric features such as the bar, the spiral arms...

- In this case, also the gravitational potential Φ is independent of ϕ. Therefore, $\partial\Phi/\partial\phi = 0$, and the force in the ϕ-direction is zero.

- This implies that a star conserves its angular momentum about the z-axis (as shown below)

- The equations of motion for a star in the disk are

$$d^2\mathbf{r}/dt^2 = -\nabla\Phi,$$

or, in each direction, and using that $\mathbf{r} = R\boldsymbol{\varepsilon}_\mathbf{R} + z\,\boldsymbol{\varepsilon}_\mathbf{z}$,

$$d^2R/dt^2 - R(d\phi/dt)^2 = -\partial\Phi/\partial R \qquad (1)$$

$$d^2z/dt^2 = -\partial\Phi/\partial z \qquad (2)$$

$$d(R^2\, d\phi/dt)/dt = -\,\partial\Phi/\partial\phi = 0 \qquad (3)$$

- Eq. (3) $\quad L_z = R^2\, d\phi/dt = \text{cst.}$

reflecting the conservation of angular momentum about z-axis

- Eq.(1) can also be written as $\quad d^2R/dt^2 = -\partial\Phi_{eff}/\partial R \qquad (4)$

where $\quad \Phi_{eff} = \Phi(R,z) + L_z^2/(2\,R^2).$

- If we multiply Eq. (4) by dR/dt, and integrate wrt t, then

$$\tfrac{1}{2}\,(dR/dt)^2 + \Phi_{eff}(R,z;L_z) = cst.$$

which is like an energy-conservation law.

- The effective potential Φ_{eff} ($= \Phi(R,z) + L_z^2/(2\,R^2)$) behaves like a potential energy for the star's motion in R and z.

- Note that a star that moves on a circular orbit: $dR/dt = 0$ has a constant effective potential, therefore:

∗ $\partial\Phi_{eff}/\partial R = 0$ thus $\partial\Phi/\partial R - L_z^2/R^3 = 0$, and
∗ $\partial\Phi_{eff}/\partial z = \partial\Phi/\partial z = 0$

- The second eq. is satisfied for z=0 (since the disk is symmetric with respect to its mid-plane $\Phi(R,z) = \Phi(R,-z)$).

- Given an angular momentum L_z , the first eq. will be satisfied at radius R_g. At this radius:

$$\partial\Phi/\partial R|_{R_g} = L_z^2/R_g^3 = R_g\,(d\phi/dt)^2$$

- Note that this circular orbit is the orbit that has the least energy for a given angular momentum L_z.

Epicycles

- We will now derive approximate solutions to the eq. of motion for stars on nearly circular orbits. Let us define: $x = R - R_g$, and expand the effective potential around the point $(R_g,0)$:

$$\Phi_{eff}(R,z) \sim \Phi_{eff}(R_g,0) + \tfrac{1}{2}\, \partial^2\Phi_{eff}/\partial R^2|_{R_g,0}\, x^2 + \tfrac{1}{2}\, \partial^2\Phi_{eff}/\partial z^2|_{R_g,0}\, z^2 + \ldots$$

(the linear terms disappear because this expansion is performed around a stationary point of the potential).

- Let us define

$\kappa^2 = \partial^2\Phi_{eff}/\partial R^2|_{R_g,0}$

and

$\nu^2 = \partial^2\Phi_{eff}/\partial z^2|_{R_g,0}$

The eq. of motion become

- $d^2R/dt^2 = -\partial\Phi_{eff}/\partial R$, or $d^2x/dt^2 = -\, \partial^2\Phi_{eff}/\partial R^2|_{R_g,0}\, x$

$$d^2x/dt^2 = -\kappa^2 x$$

- $d^2z/dt^2 = -\partial\Phi_{eff}/\partial z$, or $d^2z/dt^2 = -\, \partial^2\Phi_{eff}/\partial z^2|_{R_g,0}\, z$

$$d^2z/dt^2 = -\nu^2 z$$

- These are the equations of motion of two decoupled harmonic oscillators with frequencies κ and ν.

  κ is the *epicyclic frequency* and

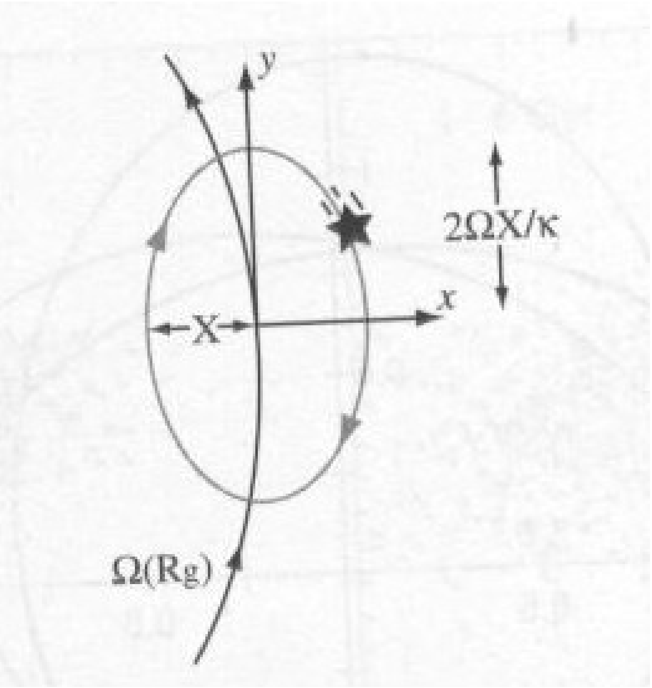
  ν as the *vertical frequency*:

$$\kappa^2(R_g) = \partial^2\Phi/\partial R^2|_{R_g,0} + 3\, L_z^2/R_g^4$$
$$\nu^2(R_g) = \partial^2\Phi/\partial z^2|_{R_g,0}$$

- The solution to the eq. of motion is

$$x = X_o \cos(\kappa t + \Psi) \quad \text{and} \quad z = Z_o \cos(\nu t + \theta) \qquad \text{for } \kappa^2 > 0.$$

- **The motion of a star in the disk can be described as an oscillation about a guiding center that is moving on a circular orbit.**



- Note as well that

$$d\phi/dt = L_z/R^2 = \Omega(R_g)\, R_g^2/(R_g + x)^2 \sim \Omega_g(1 - 2x/R_g)$$

which can be integrated to obtain

$$\phi(t) = \phi_0 + \Omega_g\, t - 2\, \Omega_g/\kappa\, X_o/R_g \sin(\kappa\, t + \Psi)$$

- The first two terms give the guiding center motion. The third represents harmonic motion with the same frequency as the x-oscillation, but 90 deg out of phase, and with a different amplitude.

- This motion is known as the **epicyclic motion**. It is retrograde because it is in the opposite sense of the guiding centre.

- The approximation to 2nd order in z in the effective potential ($\Phi_{eff} \propto z^2$) is only valid if $\rho(z) \sim$ cst (since $\nabla^2\Phi \sim \rho$). However, the disk density decreases exponentially. This means that the approximation can at most be valid for 1 scale-height (z < 300 pc). Since a good fraction of the disk stars move to higher heights, the motion in the z-direction is not well-described as an harmonic oscillation.

- There is a relation between the epicyclic frequency κ and the angular frequency Ω:

$$\kappa^2 = [R\, d\Omega^2/dR + 4\, \Omega^2]_{R_g}.$$

This relation derives from

  - $R\, \Omega^2 = d\Phi/dR$ (centrifugal force = gravitational pull)
  - and $\Omega^2 = L_z^2/R^4$
  - These equations can be replaced in the definition of κ

- In general $\Omega \leq \kappa \leq 2\, \Omega$. For example:
  - for a sphere of uniform density $\Omega(R)$ = cst, and $\kappa = 2\Omega$
  - for the Kepler problem (point mass), $\Omega \propto r^{-3/2}$, and $\kappa = \Omega$

- The epicyclic frequency is related to the Oort constants:

- Recall that
  - $A = -1/2\ R\ d\Omega/dR|_{R_o}$ and $B = -(½\ R\ d\Omega/dR + \Omega)_R$, where R_o is the location of the Sun, and Ω the angular frequency of the LSR motion.

- Therefore, at the Sun $\kappa_o^2 = -4\ B(A - B) = -4\ B\ \Omega_o$

- Using the measured value of B, we find that

$$\kappa_o/\Omega_o \sim 1.3 \pm 0.2$$

Therefore the Sun makes 1.3 radial oscillations in the time it takes to complete an orbit around the Galactic centre.