The orbits of stars

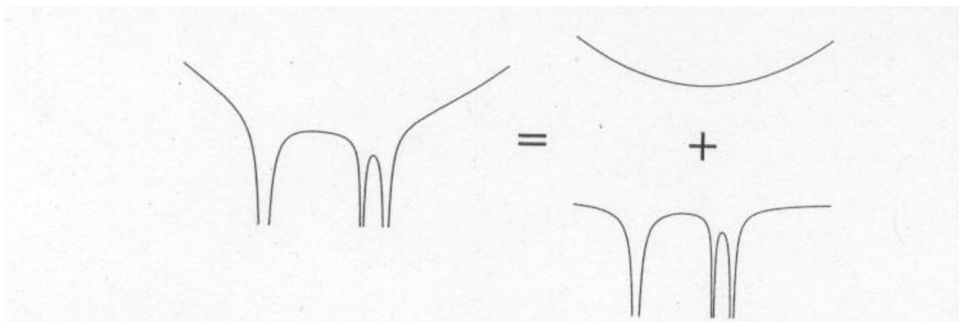
- Stars travel in the Galaxy under the force of gravity. If we know how the mass in the Galaxy is distributed, we can find the gravitational force, and from this calculate how the positions and velocities of stars will change over time.

- Conversely, we can also use the stellar motions to derive where the mass is. This is how we discovered that much of the matter in the Galaxy is not visible.

- Since the orbits of stars take them through different regions of the Galaxy, their motions at the time we observe them have been affected by the gravitational fields through which they travelled earlier. So we can use the equations of motion to infer from the observed motions how the mass is distributed in parts of the Galaxy that we cannot see directly.

- Usually we can consider the stars as point masses, because their sizes are very small compared to the distances between them.

- The gravitational potential of a galaxy can be regarded as the sum of a smooth component (the average over a region containing many stars), and the very deep potential well around each individual star.



- We will see that the motion of stars within a galaxy is determined almost entirely by the smooth part of the force. Two-body encounters (leading to energy transfer between individual stars) are only important within dense clusters.

Motion under gravity

- Newton's law of gravity tells us that a point mass M attracts a second mass m separated by distance **r** causing the velocity **v** of m, to change as $d(m\mathbf{v})/dt = -G\,M\,m/r^3\,\mathbf{r}$

- In a cluster of N stars with masses m_α, at positions $\mathbf{x}_\alpha$, we derive the force acting on star β by adding the contributions from all other stars:

$$d(m_\beta\,\mathbf{v}_\beta)/dt = -\,\Sigma_\alpha\,G\,m_\alpha\,m_\beta/|\mathbf{x}_\beta - \mathbf{x}_\alpha|^3\,(\mathbf{x}_\beta - \mathbf{x}_\alpha)$$

  – Note that the mass m_β drops out of the equation, so that heavy and light stars suffer the same acceleration.

- This equation can also be written in terms of the gradient of the gravitational potential $\Phi(\mathbf{x})$:

$$d(m\mathbf{v})/dt = -\,m\,\nabla\Phi(\mathbf{x}),\ \text{with}\ \Phi(\mathbf{x}) = -\,\Sigma_\alpha\,G\,m_\alpha/|\mathbf{x} - \mathbf{x}_\alpha|$$

where we have chosen an arbitrary integration constant, so that $\Phi(\mathbf{x}) \longrightarrow 0$ at large radii

- If we think of a continuous mass distribution, the potential at point **x** is given by the integral over the density $\rho(\mathbf{x}')$ at all points

$$\Phi(\mathbf{x}) = -\,G \int \rho(\mathbf{x}')/|\mathbf{x} - \mathbf{x}'|\, d^3x'$$

where we have essentially replaced the discrete summation by an integral over a volume, and the masses by $\rho(x)\, d^3x$

- This equation allows one to derive the potential if the density is known. But it is also possible to determine the density from the potential. The relation is given by Poisson's equation:

$$\nabla^2\Phi(\mathbf{x}) = 4\pi G\, \rho(\mathbf{x})$$

Note that not all forms of $\Phi(\mathbf{x})$ give a physically meaningful density, since this should satisfy that $\rho(\mathbf{x}) > 0$ everywhere (mass is always positive).

  – Note as well the similarity to the eq. that relates the potential associated to the electric field ($\nabla\Phi = -\,\mathbf{E}$) and the charge distribution ρ_e: $\nabla^2\Phi = -\,4\pi\kappa\, \rho_e$, where κ is Coulomb's constant. Here ρ_e may be positive or negative, reflecting that the electric force can be repulsive or attractive.

Spherical systems: Newton's theorems

- Theorem 1: A body that is inside a spherical shell of matter experiences no net gravitational force from that shell.

$\delta m_1 = \rho\, r_1^2\, dr_1\, d\Omega_1$ and $\delta m_2 = \rho\, r_2^2\, dr_2\, d\Omega_2$

But $dr_1 = dr_2 = dr$ and $d\Omega_1 = d\Omega_2 = d\Omega$.

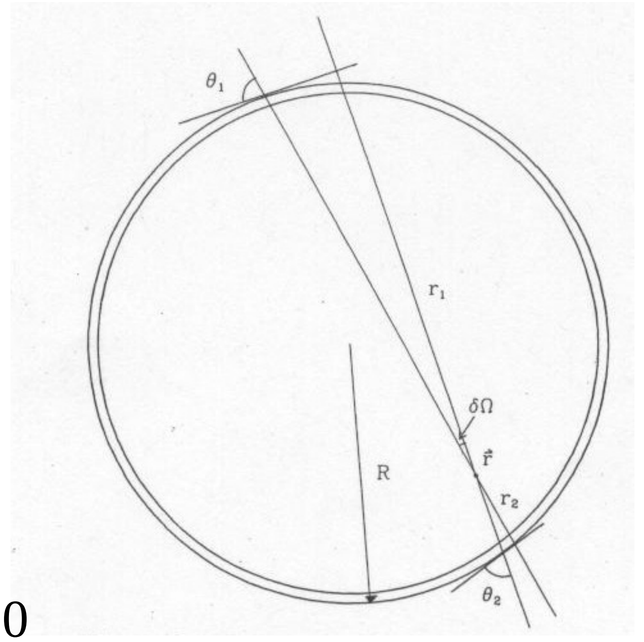
Then $\delta\, m_1/r_1^2 = \delta\, m_2/r_2^2$.

A particle located at **r** experiences a force

$\mathbf{F} = \mathbf{f_1} + \mathbf{f_2}$ where

$\mathbf{f_1} = -\,G\,\delta\, m_1/r_1^2\, \boldsymbol{\varepsilon_1}$ and $\mathbf{f_2} = -G\,\delta\, m_2/r_2^2\, \boldsymbol{\varepsilon_2}$

Since $\boldsymbol{\varepsilon_1} = -\boldsymbol{\varepsilon_2}$, thus $F = -G(\delta\, m_1/r_1^2 - \delta\, m_2/r_2^2) = 0$



Theorem 2: The gravitational force on a body that lies outside a closed spherical shell of matter is the same as it would be if all the shells' mass was concentrated in a point at its centre.

- The second theorem implies that within any spherical object with density $\rho(r)$ the gravitational force a particle feels at a radius R is only due to the mass inside that radius.

- Thus, if a star moves on a circular orbit, its acceleration is given by

$$v_c^2/r = GM(<r)/r^2$$

- For a point mass, the circular velocity $v_c^2 = GM/r$, and so $v_c \propto r^{-1/2}$

- Note that since M generally increases with radius (or is at least constant), the above equation implies that for a spherical galaxy, the circular velocity never falls off more rapidly than the Kepler case $r^{-1/2}$.

Conserved quantities

- Energy:
  - If the mass distribution is static (the galaxy is not collapsing or colliding), the potential at a given position will be time independent (since the potential and the mass distribution are directly related, for example, through Poisson's equation).
  - This implies that the energy of a star moving in such a galaxy will be conserved: $E = \frac{1}{2} m v^2 + m \Phi(\mathbf{x})$
- Angular momentum:
  - The angular momentum of a star located at position $\mathbf{x}$ in a galaxy, and moving with velocity $\mathbf{v}$ is $\mathbf{L} = m\, \mathbf{x} \times \mathbf{v}$. The torque on the star $\boldsymbol{\tau} = d\mathbf{L}/dt = m\, \mathbf{x} \times \mathbf{F}$
  - For a spherical galaxy, the force is purely radial, and thus $\boldsymbol{\tau} = \mathbf{0}$: the total angular momentum is conserved.
  - In an axisymmetric galaxy, the only component that is conserved is L_z (or J_ϕ).

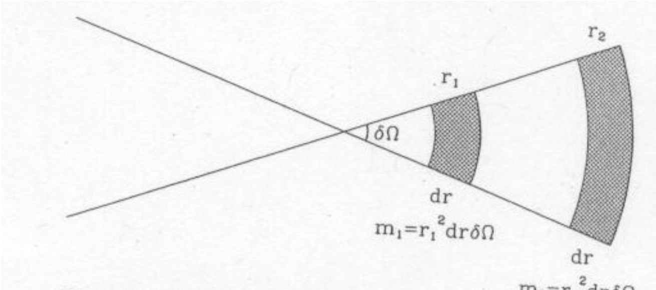
How important are collisions in gravitational systems?

- Given enough time, molecules of air or dust particles in a room will spread themselves out evenly, reaching an equilibrium state. This happens through collisions, where they exchange energy and momentum. The forces between molecules are small unless they are very close to each other, and so typically molecules will be subject to violent and short-lived accelerations, in between long periods when they move at nearly constant speeds. Typically at room temperature, each molecule experiences 10^{11} encounters per second.

- However, the nature of the gravitational force is different: it is long-range, and close encounters in galaxies are less important. The average (smoother) mass distribution will determine the motion of a stars.

- The net gravitational force acting on a star in a galaxy is determined by the gross structure of the galaxy, rather than by whether it is located close to another star.

- Consider the following example:



- The force on a star located at the apex of the cone of constant density:

$dF_1 = G\ m_*\ dm_1/r_1^2 = G\ m_*\ r_1^2\ \rho\ dr\ d\Omega\ /r_1^2 = G\ m_*\ \rho\ dr\ d\Omega$,

while the force from the more distant shell is

$dF_2 = Gm_*\ dm_2/r_2^2 = G\ m_*\ \rho\ dr\ d\Omega$.

- Thus, if the density of stars is constant throughout the cone, the force produced by shells at different distances will be the same (makes explicit the long-range nature of gravity).

Strong encounters

- We will now calculate the time between strong encounters: one star comes so close another that the collision completely changes its speed and direction of motion

- In this case, the change in potential energy should be at least as large as the initial kinetic energy:

$$Gm^2/r > m\,v^2/2, \text{ or } r < r_s = 2\,G\,m/v^2$$

- The distance r_s is the strong encounter radius. Near the Sun, stars have random speeds $v \sim 30$ km/s, and for $m=0.5\,M_\odot$, $r_s \sim 1$ AU

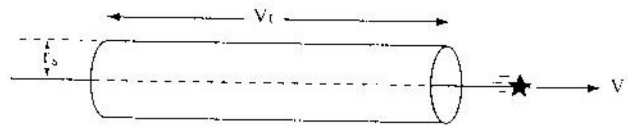


Figure 3.4 During time t, this star will have a strong encounter with any other star lying within the cylinder of radius r_s.

- As a star like the Sun moves relative to nearby stars with velocity v for a time t, it will have a strong encounter with any other star within a cylinder of radius r_s, and volume $\pi r_s^2 v t$ centered on its path.

- If there are n stars per unit volume, a star like the Sun will on average have one close encounter in a time t_{coll} such that $n \pi r_s^2 v t_{coll} = 1$

- The characteristic time between collisions is $t_{coll} \sim 1/(n\pi r_s^2 v)$, or $t_{coll} = v^3/(4\pi G^2 m^2 n)$

- Normalizing to some characteristic values

$$t_{coll} \sim 4\times10^{12}\ \text{yr}\ \ (v/10\ \text{km/s})^3\ \ (m/M_\odot)^{-2}\ \ (n/1\ \text{pc}^{-3})^{-1}$$

- Since $n \sim 0.1\ \text{pc}^{-3}$ near the Sun, $t_{coll} \sim 10^{15}$ years, which far exceeds the age of the Universe.

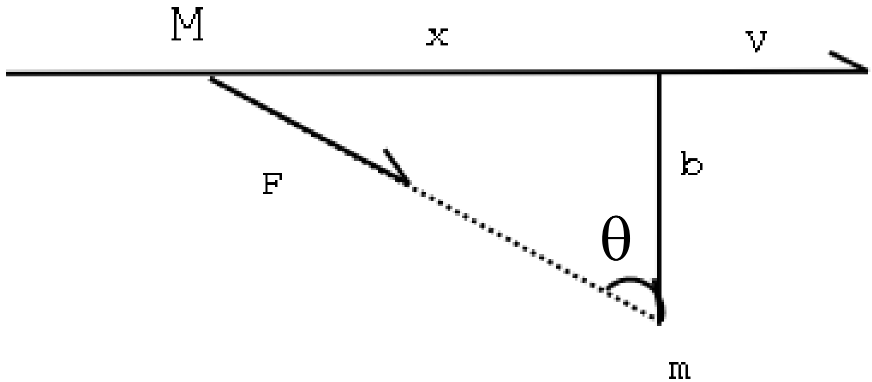
- Strong encounters are only important in the dense cores of globular clusters.

Distant weak encounters

- In a distant encounter, the force of one star on another is so weak that the stars hardly deviate from their original paths after the encounter.

- We will consider the case of a star moving through a system of N identical stars of mass m.

- The goal is to obtain an estimate of the difference between the velocity of the star after crossing the system, and its velocity if the mass had been smoothly distributed throughout the system, rather than concentrated into individual stars.



- The change in velocity will depend on b, the mass of the stars and their relative velocity. We assume that $\delta v/v \ll 1$, and that the perturbing star is stationary (this is known as the impulse approximation).

- The pull by m induces a motion $\delta v_\perp$ perpendicular to the original trajectory. The force is $\mathbf{F} = -GmM/r^2\, \boldsymbol{\varepsilon}_\mathbf{r}$, and $F_\perp = GmM/r^2 \cos\theta$, where $r^2 = x^2 + b^2$ and $\cos\theta = b/r$. Replacing, we find that $F_\perp = GmM/b^2\, (1 + (vt/b)^2)^{-3/2}$

- Since $M\, dv_\perp/dt = F_\perp$, we can compute the change in velocity by integrating over time. We find $\Delta v_\perp = 2\, Gm/(bv)$.

- Therefore the faster the star M passes by m, the smaller the perturbation is.

- Now we compute the cumulative effect of the individual encounters.

- If the surface density of stars in the system is $N/(\pi R^2)$, where R is some characteristic radius, the number of encounters dn_e with impact parameter b, a star suffers when crossing the system is

  $dn_e = N/(\pi R^2)\, 2\pi b\, db = 2N/R^2\, b\, db$.

- Each of these encounters will produce a change in $\delta v_\perp$, but because the perturbations are randomly oriented, the mean **vector** change is null $\langle \delta v_\perp \rangle = 0$.

- But there will be a change in modulus,

$$\delta v_{\perp}^2 = (2\,Gm/bv)^2\, 2\,N/R^2\, b\, db$$

- We can integrate this equation, to obtain $\Delta v_{\perp}^2 = 8\,N\,(Gm/vR)^2 \ln \Lambda$, where $\Lambda = b_{max}/b_{min}$.

- If we define the number of weak encounters that a star has to experience to change its velocity by the same order as its incoming velocity by: $n_{relax}\, \Delta v_{\perp}^2 = v^2$, then $n_{relax} = v^4 R^2 / (8\, G^2 m^2\, N \ln\Lambda)$

- We can define a timescale

$$\boxed{t_{relax} = n_{relax}\, R/v = v^3 R^3 / (8\, G^2 m^2\, N \ln\Lambda)}$$

- This is known as the relaxation timescale: it gives an estimate of the timescale required for a star to change its velocity by the same order, due to weak encounters with a "sea" of stars.

- We can compare the relaxation timescale to the collision timescale derived previously: $t_{coll} = v^3/(4\pi G^2 m^2 n)$. If we use that $n \sim N/(\pi R^3)$, then

$$t_{relax} = t_{coll}/(2 \ln \Lambda)$$

which shows that the relaxation timescale is always shorter than the timescale for 2-body encounters.

- Typically $\ln \Lambda \sim 20$. The exact values of b_{min} and b_{max} are not very important, because of the logarithmic dependence: Typically b_{max} will be the system size, and $b_{min} = r_s$, for example for 300 pc < b_{max} < 30 kpc, and r_s = 1 AU (near the Sun), $\ln \Lambda \sim 18 – 22$.

•For example, for an elliptical galaxy, $N \sim 10^{11}$ stars, $R \sim 10$ kpc, and the average relative velocity of stars is $v \sim 200$ km/s, then $t_{relax} \sim 10^8$ Gyr! which is much longer than the age of the Universe.

•This implies that when calculating the motions of stars like the Sun, we can ignore the pulls of the individual stars, and consider them to move in the smoothed-out potential of the entire Galaxy.

•For stars in a globular cluster like ω Cen, $t_{relax} \sim 0.4$ Gyr, so relaxation will be important over a Hubble time.