## Spatial distribution of stars in the Milky Way

- What kinds of stars are present in the Solar neighborhood, and in what numbers?
- How are they distributed spatially? How do we know? How can we measure this?
- Does the spatial distribution vary with age, stellar type? If so, why?
- How does this reflect the evolution of the Milky Way?


## Spatial distribution of stars in the Milky Way

- We use a statistical approach to determine and describe the spatial distribution of stars in the Galaxy
- This approach allows us to derive the structure of the disk and spheroidal components (density distribution and extent).
- This information can then be coupled to the age and chemical composition of the stars, as well as their kinematics, to study for example, the correlations between age, location and motion of the stars.
- This enables us to understand how the Milky Way is built up by stars of different ages, abundances and dynamical properties. This, in turn, gives us hints as to how our Galaxy formed.


## The apparent distribution of stars: star counts

- Star counts are used to understand the structure of the Galaxy and the contribution of different types of stars.
- The simplest, most straightforward approach is to count stars of different brightness (apparent magnitudes) on different directions on the sky:

Let $A(m, b)$ dm be the number of stars with apparent magnitude $(m+d m, m)$ per square degree at a given Galactic latitude b.

Since the brightness $l$ of a star of magnitude $m$ located at latitude $b$ is given by $\mathrm{m}=-2.5 \log l+\mathrm{C}$, or $l=\mathrm{C} * 10^{-0.4 \mathrm{~m}}$

The total brightness received per apparent magnitude bin, per sq. deg. at latitude $b$ is given by the number of stars in that range times their brightness: $l(\mathrm{~m}, \mathrm{~b})=\mathrm{A}(\mathrm{m}, \mathrm{b}) 10^{-0.4 \mathrm{~m}}$

Table 4-1. Star Counts in the Galactic Plane and North Galactic Pole
Star counts for $b=0$ and for $b=90$

| $m_{\text {pg }}$ | $\log A\left(m, 0^{\circ}\right)$ | $\Delta \log A\left(m, 0^{\circ}\right)$ | $\log A\left(m, 90^{\circ}\right)$ | $\Delta \log A\left(m, 90^{\circ}\right)$ | $A\left(m, 0^{\circ}\right) / A\left(m, 90^{\circ}\right)$ | $f\left(\mathrm{~m}, 0^{\circ}\right)$ | $\ell\left(m, 90^{\circ}\right)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  | 0.68-3 |  | 3.3 | 4.0 | 1.2 |
| 4 | 0.2-2 | 0.52 | 0.68-3 | 0.50 | 3.5 | 5.2 | 1.5 |
| 5 | 0.72-2 | 0.47 | 0.64-2 | 0.46 | 3.5 | 6.1 | 1.7 |
| 6 | 0.19-1 | 0.44 | 0.64-2 | 0.44 | 3.6 | 6.7 | 1.9 |
| 7 | 0.63-1 | 0.47 | 0.08-1 | 0.42 | 4.0 | 7.9 | 2.0 |
| 8 | 0.10 | 0.48 | $0.50-1$ $0.92-1$ | 0.42 | 4.6 | 9.6 | 2.1 |
| 9 | 0.58 | 0.46 | 0.92-1 | 0.36 | 5.8 | 11.0 | 1.9 |
| 10 | 1.04 | 0.46 | 0.28 0.63 | 0.35 | 7.4 | 12.6 | 1.7 |
| 11 | 1.50 | 0.44 | 0.63 0.98 | 0.35 | 8.7 | 13.8 | 1.5 |
| 12 | 1.94 | 0.41 | 0.98 1.29 | 0.31 | 11.5 | 14.1 | 1.2 |
| 13 | 2.35 | 0.41 | 1.29 1.57 | 0.26 | 15.5 | 14.4 | 0.9 |
| 14 | 2.76 | 0.35 |  | 0.23 | 22.4 | 14.1 | 0.6 |
| 15 | 3.15 | 0.31 | $\begin{aligned} & 1.80 \\ & 2.06 \end{aligned}$ | 0.26 | 25.0 | 11.5 | 0.5 |
| 16 | 3.46 | 0.38 |  | 0.22 | 36.4 | 11.0 | 0.3 |
| 17 | 3.84 | 0.36 | 2.28 | 0.22 | 50 | 10.0 | 0.2 |
| 18 | 4.2 | 0.3 | $\begin{aligned} & 2.50 \\ & 2.7 \end{aligned}$ | 0.2 | 63 | 7.9 | 0.1 |
| 19 | 4.5 | 0.2 | $\begin{aligned} & 2.1 \\ & 2.8 \end{aligned}$ | 0.1 | 80 | 5.0 | 0.06 |
| 20 | 4.7 |  | $\begin{aligned} & 2.8 \\ & 2.9 \end{aligned}$ | 0.1 | 100 | 3.1 | 0.03 |
| 21 | 4.9 |  | 2.9 |  |  |  |  |

SOURCE: (A1, 245), by permission

- The table (Mihalas \& Binney) shows that bright stars (m<9) are uniformly distributed in latitude, but that there are many more faint stars in the plane than towards the poles.
- Bright stars contribute most of the light at $b=90$, while faint stars contribute mostly to $\mathrm{b}=0$, with the peak coming from stars of $\mathrm{m} \sim 14^{\mathrm{m}}$.
- The integrated star count gives the cumulative number of stars per sq.deg. with apparent magnitudes less than $m$ at latitude $b$ : $N(\mathrm{~m}, \mathrm{~b})=\int^{\mathrm{m}}{ }_{-\infty} \mathrm{A}\left(\mathrm{m}^{\prime}, \mathrm{b}\right) \mathrm{dm}{ }^{\prime}$
- There are clear differences in the distribution of stars according to their spectral type:
- O-B stars are strongly concentrated towards the plane
- A stars are also concentrated towards the plane.
- The fact that they are older on average, implies that they will have dispersed over a larger spatial volume and are not located as close to their formation site as O-B stars.
- Note as well, that the average A star with $\mathrm{m}<8.5$ mag, will be closer than the average O,B stars (which are intrinsically brighter), so that even if they have the same spatial distribution, A stars would be observable with a wide range of Galactic latitudes b.


Figure 4-1. Apparent distribution of stars of various types as a function of galactic latitude. Ordinate gives average number of stars per square degree. $[\operatorname{From}(\mathrm{N} 2)$.

- F, G,M stars show a slight concentration to the plane
- They are in the plane but are very faint intrinsically
- K stars appear to be more concentrated than F stars
-This may seem anomalous because they are later type than F
-But the sample is dominated by giants, which can be observed to (on average) large distances in the disk


Note that the faintest stars, M dwarfs, are restricted to the inner 20 pc or so, while progressively brighter stars are seen fill progressively larger spheres. There are only a handful of the earliest types in the sphere ( $\mathrm{O}, \mathrm{B}$ and A ) stars. These are in black, and there is an indication that they are more restricted to the plane than the other types.

## Star counts in a uniform Galaxy. Olbers' paradox

- Let us consider a simple model to understand what we might observe when performing star counts in our Galaxy. Let us assume that

1. stars are uniformly distributed in space, with a density $D$ of stars per $\mathrm{pc}^{3}$
2. the Galaxy is infinite in extent
3. there is no absorption by interstellar material

- Suppose we observe a field subtending a solid angle W


The number of stars in this volume is

$$
N(\mathrm{r})=\int \mathrm{d}^{3} \mathrm{r} \rho(\mathrm{r})=D \mathrm{w} \int \mathrm{r}^{2} \mathrm{dr}=D \mathrm{w} / 3 \mathrm{r}^{3}
$$

If we relate the distance $r$ to the apparent magnitude $m$ of the stars: $\mathrm{r}=10^{0.2(\mathrm{~m}-\mathrm{M})+1}[\mathrm{pc}]$, then

$$
\log N(\mathrm{~m})=0.6 \mathrm{~m}+C(D, \mathrm{w}, \mathrm{M})
$$

To derive this expression, we have assumed that all stars have the same intrinsic luminosity (or the same abs. magnitude M), but a similar result applies otherwise. The only difference is that the constant $C$ depends on M for each subgroup of stars.

Thus $N(\mathrm{~m})=\mathrm{C} * 100^{0.6 \mathrm{~m}}$, while the number of stars per unit apparent magnitude is $\mathrm{A}(\mathrm{m}): \quad \mathrm{A}(\mathrm{m})=\mathrm{d} N / \mathrm{dm}=\mathrm{C}_{2} 100.6 \mathrm{~m}$

Recall then, that the total brightness received from all stars of magnitude m is $l(\mathrm{~m})=\mathrm{A}(\mathrm{m}) * 10^{-0.4 \mathrm{~m}}$, and so

$$
l(\mathrm{~m})=\mathrm{C}_{3} * 10^{0.2 \mathrm{~m}}
$$

Finally, the total light from all stars of magnitude less than $m$ is (integrating)

$$
l_{\mathrm{tot}}(\mathrm{~m})=\mathrm{K} * 10^{0.2 \mathrm{~m}}
$$

Therefore, as $m$ increases the total light we observe $l_{\text {tot }}(m)$ diverges exponentially!

Thus for an infinite, homogeneous, transparent Galaxy, the sky should be a very bright surface. This result is known as Olbers' paradox, and it shows that one of the assumptions we have made must be wrong.

Clearly, it is not correct to assume that the Galaxy has an infinite extent.
A similar problem arises in discussions of the whole Universe: if it is infinite, homogeneous and transparent then the night sky cannot possibly be dark. In that case, the problem lies in the fact that one needs to take into account the effects of the expansion and finite age of the Universe.

## From star counts to the true spatial distribution

- The number of stars observed per unit apparent magnitude depends upon the true three-dimensional spatial distribution of stars in the Galaxy, but it also depends on the distribution of intrinsic luminosities, or absolute magnitudes of the stars.
- If $\Phi(\mathrm{M})$ is the luminosity function and $D(\mathrm{r})$ the spatial density, then

$$
\mathrm{A}(\mathrm{~m})=\int \Phi(\mathrm{M}) D(\mathrm{r}) \mathrm{dV}
$$

because $A(m)$ gives the number of stars with apparent magnitude ( $\mathrm{m}, \mathrm{m}+\mathrm{dm}$ ); and this quantity is simply given by the density of stars integrated over the volume containing all stars which satisfy $M=M(m, r)$.

Essentially one is counting all stars with (intrinsic) absolute magnitudes which are located at distances r from the Sun such that their apparent magnitude is m .

- To derive the density distribution from the star counts, the luminosity function needs to be determined. In principle one can then invert the integral equation for $D(r)$. In practice, one assumes certain functional forms for $D(\mathrm{r})$, and tries to determine which one provides the best fit.


## The luminosity function and Malmquist bias

- Typically stars counts are made down a certain limiting magnitude, i.e. A(m) is only available for $\mathrm{m}<\mathrm{m}_{\mathrm{lim}}$. We then say that $\mathrm{A}(\mathrm{m})$ is magnitude limited: it then measures the number of stars brighter than $\mathrm{m}_{\text {lim }}$ in a given field in the sky.
- In such cases, it turns out that the mean absolute magnitude of objects $<\mathrm{M}_{\text {samp }}>$ in the sample is brighter than the $<\mathrm{M}_{\text {pop }}>$ of the population as a whole: the volume within which we can see the most luminous objects is larger than that within which we see the faintest ones.
- Thus, luminous objects are over-represented in a magnitude limited sample. This effect is known as Malmquist bias.
- It is easy to correct for this effect when the luminosity function of the sample is a Gaussian.


## The luminosity function in the Solar neighborhood

1. Most stars are intrinsically faint (peak at $\mathrm{M}_{\mathrm{V}} \sim 14$ )

Table 3.16 The general luminosity function (per $10^{4} \mathrm{pc}^{3}$ )
2. All the light comes from the intrinsically most luminous stars
3. Most of the mass comes from large numbers of low luminosity stars
4. The average mass-to-light ratio near the Sun is $\langle\mathrm{M} / \mathrm{L}\rangle_{\mathrm{V}}=0.67$ $M_{\odot} / L_{\odot}$.

| $M_{V}$ | $\Phi\left(M_{V}\right)$ | $\delta L / L_{\odot}$ | $\delta \mathcal{M} / \mathcal{M}_{\odot}$ | $M_{V}$ | $\Phi\left(M_{V}\right)$ | $\delta L / L_{\odot}$ | $\delta \mathcal{M} / \mathcal{M}_{\odot}$ |
| ---: | :---: | :---: | :---: | ---: | ---: | ---: | ---: |
| -6 | 0.0001 | 2.6 | 0.005 | 7 | 29 | 4.0 | 21.3 |
| -5 | 0.0006 | 5.1 | 0.020 | 8 | 33 | 1.8 | 21.8 |
| -4 | 0.0029 | 9.4 | 0.060 | 9 | 42 | 0.90 | 24.2 |
| -3 | 0.013 | 17.1 | 0.17 | 10 | 70 | 0.60 | 35.0 |
| -2 | 0.05 | 28.2 | 0.05 | 11 | 90 | 0.30 | 36.0 |
| -1 | 0.25 | 53.9 | 1.6 | 12 | 127 | 0.17 | 36.3 |
| 0 | 1 | 95.8 | 4.0 | 13 | 102 | 0.055 | 20.8 |
| 1 | 3 | 111 | 7.4 | 14 | 102 | 0.022 | 16.3 |
| 2 | 5 | 64 | 8.7 | 15 | 127 | 0.011 | 16.3 |
| 3 | 12 | 66 | 17.3 | 16 | 102 | 0.0035 | 10.5 |
| 4 | 17 | 36 | 19.4 | 17 | 51 | 0.0007 | 4.3 |
| 5 | 29 | 25 | 28.1 | 18 | 22 | 0.0001 | 1.6 |
| 6 | 30 | 10 | 24.7 | 19 | 13 | 0.0000 | 0.7 |
|  |  |  |  | Total | 1008 | 532 | 356 |

sOURCE: For $M_{V} \leq 0$ from data published in Allen (1973); for $M_{V}>0$ from data published in Jahreiss \& Wielen (1983) and Kroupa, Tout \& Gilmore (1990)

Item 3) shows that it is dangerous to assume that the luminosity density traces the mass density! The only robust way to derive masses is using dynamics.

Item 4) is a lower limit, since objects such as white dwarfs, neutron stars, etc, do contribute to the total mass budget, but virtually no light.

## The distribution of stars in the Galaxy



This image shows that our Galaxy is a disk galaxy. Its spiral structure can be derived from optical data, but suitable tracers are needed.
Note that the central component, the bulge, is not spherical and is asymmetric.

## Tracers of the spiral arms should satisfy:

- known to be associated with spiral arms in external galaxies
- young, so that during their lifetime they have not drifted away from their birthplace
- intrinsically luminous, to be seen at large distances, and thereby trace sizeable portions of the arms
-intrinsic brightness well known: to correct for interstellar absorption and reddening and derive the true spatial structure of the arms
-Good tracers are O-B associations, HII regions, Cepheids, young clusters...
-The most recent studies suggest that the Milky Way has four spiral arms.


Figure 4-11. Spiral arms in the solar neighborhood as inferred from optical evidence. Individual points are at positions of young star clusters and H II regions. Three arms are clearly indicated, separated from one another by about 1.5 kpc : the outer Perseus arm, the local Orion-Cygnus arm, and the inner Sagittarius arm. The Sun is near the inner edge of the Orion-Cygnus arm. [From (B5, 205), by permission.]

## Distribution of stars perpendicular to the plane

- The space density of stars as a function of distance from the Galactic plane is a dynamically interesting quantity.
- For example, a measurement of this quantity allows us to determine the gravitational pull produced by the Galactic disk. This means, we can measure the amount of mass responsible for that pull, and determine whether all the light we observe from the disk is enough to account for its mass. Or whether there is evidence for some "missing mass".
- One of the most noticeable features in the distribution of stars perpendicular to the plane of the Galaxy is how strongly this distribution varies with spectral type (as we have discussed in the beginning of this lecture).
- Gilmore \& Reid (1986) attempted to do star counts to estimate how quickly the density of stars decreases as function of distance from the Galactic plane. And they found an interesting result.
- They observed fields towards the Galactic pole, and measured (V-I) for the stars, down to some limiting magnitude (because of an additional color selection, most of the stars in their sample should be dwarfs). For some of the stars in this sample, trigonometric parallaxes had been measured. This enabled them to measure the absolute magnitude $\mathrm{M}_{\mathrm{V}}$ for these stars.
- They found a relation between the $\mathrm{M}_{\mathrm{V}}$ and (V-I), that was then used to derive $\mathrm{M}_{\mathrm{V}}$ for those stars which did not have a distance estimate.
Most stars are MS dwarfs and hence such a relation is not very surprising. Contamination due to giant stars starts to become important at (V-I) ~ 2.3, which for dwarfs on the main sequence, this corresponds to $\mathrm{M}_{\mathrm{V}} \sim 6.5 \mathrm{mag}$. So their analysis can be plagued with errors at those (and fainter) absolute magnitudes.
- Finally, since they could measure the luminosity function of the stars in the field, they could derive the density distribution as function of height above the plane.

The space density of stars $D(\mathrm{z})$ for stars with $\mathrm{M}_{\mathrm{v}}$ in [4,5] magnitude range cannot be fitted by a single exponential!

Good fit from the superposition of two exponentials with different scaleheights: $\mathrm{z}_{0}=300 \mathrm{pc}$ and $\mathrm{z}_{0}{ }^{\prime}=1350 \mathrm{pc}$

Two possible conclusions:

- The functional form is not correct (why should $D(\mathrm{z})$ be fit by an exponential?),
or,


Figure 10.25 The space density as a function of distance $z$ from the plane of MS stars with absolute magnitudes $4 \leq M_{V} \leq 5$. The full lines are exponentials with scale heights $z_{0}=300 \mathrm{pc}$ (at left) and $z_{0}=1350 \mathrm{pc}$ (at right). The dashed curve shows the sum of these two exponentials. [From data published in Gilmore \& Reid (1983)]

- There are two physically distint components in our Galaxy: a thin and a thick disk.

To justify the last solution, one needs to prove that the two disks are made up of different kinds of stars (for example, of different characteristic age or chemical abundance). Otherwise, such a distinction is not physically meaningful.

## Spatial distribution of stars in the spheroid: halo and bulge

We have studied the spatial distribution of the majority of stars in the Solar neighbourhood, and found that these are distributed in two disk components with different scale heights: the thin disk (young and old) and the thick disk.

In a similar way we may proceed to determine the spatial distribution of globular clusters, or of the stars in our Galaxy. In this case, we find that the spatial density is a very rapidly declining function of distance from the Galactic centre

$$
D(r)=D_{0}\left(r / r_{0}\right)^{-3.5}
$$



- The amount of heavy elements or metals in stars relative to the amount in the Sun is called the metallicity
- The "iron abundance"

$$
[\mathrm{Fe} / \mathrm{H}]=\log (\mathrm{n}(\mathrm{Fe}) / \mathrm{n}(\mathrm{H}))_{\text {star }}-\log (\mathrm{n}(\mathrm{Fe}) / \mathrm{n}(\mathrm{H}))_{\text {Sun }}
$$

is often referred to as the metallicity.

- This defines the solar metallicity as $[\mathrm{Fe} / \mathrm{H}]=0.0$. A star with a metallicity of $[\mathrm{Fe} / \mathrm{H}]=-1.0$ has 10 times less metals than the Sun relative to Hydrogen, while a star with twice the metallicity of the Sun has $[\mathrm{Fe} / \mathrm{H}]=0.3$.


## The stellar halo

-extends out to $\sim 80 \mathrm{kpc}$ from the Galactic centre
-total luminosity of $\sim 10^{9}$ Lsun
-very low density in the Solar Neighbourhood: $0.2 \%$ of the thin disk's central density
-very strongly concentrated: half of total luminosity contained within sphere of $\sim 3 \mathrm{kpc}$ radius around the Galactic centre.
-It contains important clues about the formation of the Galaxy: -its stars are very metal-poor (implying they were formed in pristine environments, like the early Universe)
-very old (the halo globular clusters contain the oldest stars)

- Even today, we observe the stellar halo "in formation"
- dwarf galaxies and globular clusters are disrupted and add their stars to this component.



## The central regions of the Galaxy

- The central regions of the Galaxy are heavily obscured by dust ~ 30 mag in V
- But the dust is patchy, and there are holes which we can peek through, such as Baade's window. The line of sight through Baade's window passes within 500 pc of the galactic center, so we can study bulge stars.
- Bulge stars are mostly old (ages $>9 \mathrm{Gyr}$ ) and have a range of metallicities $(-1<$ $[\mathrm{Fe} / \mathrm{H}]<0.5)$ with an average metallicity a little under solar $([\mathrm{Fe} / \mathrm{H}]=-0.2)$. There is evidence for a complex formation history.
- Observations in the near infrared are particularly useful: stars dominate in this region, while dust emits mostly in the mid \& far infrared.
- Even though extinction is less important in the near IR, it still needs to be taken into account when modeling the data
-The density distribution and the structure of the bulge of our Galaxy is derived from its surface brightness distribution $\sum(\mathrm{l}, \mathrm{b})$, rather than from star counts.
-To derive the underlying spatial distribution or structure of the bulge, one needs to take into account the effects of dust in the measured $\sum(1, b)$.
-The effect of the dust (its emissivity and spatial distribution) is modeled and convolved with sets of models for the underlying stellar distribution to try and match the data best.
-The model that provides the best fit may be considered a good approximation to the actual density distribution of the bulge.
-Recent results indicate that the bulge is not spheroidal, but is more likely bar shaped:
-First, the COBE infrared data showed that the bulge is bigger on one side.


Fig. 2.- $2.2 \mu \mathrm{~m}$ angular scale heights at fixed longitude. Scale heights for $l<0^{\circ}$ are represented by asterisks, whereas diamonds are for scale heights at positive Galactic longitudes. The error bars represent $1 \sigma$ errors on the computed scale height.
-Secondly, if you look at the distances to stars on either side of the bulge, on average stars on one side of the bulge are closer than stars on the other. This can't happen if the bulge is axisymmetric!

This can be explained if the light is dominated by a small bar, whose major axis is ~20deg away from line defined by the Sun and the Galactic centre.

The luminosity of the bulge in the near IR is $\mathrm{L}_{\mathrm{H}} \sim 3.3 * 10^{10} \mathrm{~L}_{\odot}$

It is estimated that it has $1 / 6$ to $1 / 3$ the mass of the Galactic disk.


## Population types and the Galaxy

- The concept of population types was proposed by Baade in 1944. In studies of the Andromeda galaxy he found that the disk stars in M31 were like nearby disk stars in our Galaxy, while the bulge stars resembled those of the Galaxy's globular clusters.
- Population I: made up by young, metal rich stars in the disk of the Galaxy
- Population II: old, metal weak stars in the spheroid.
- The fundamental difference is probably when and where/how they were formed; although a strict distinction between the two populations may eventually become obsolete.
- In particular, since the original classification more population types have appeared, such as an intermediate type II population, corresponding to the thick disk stars.


## Properties of the populations

## Population I

- The Young Disk: O-B stars, young associations and clusters, as well as much of the gas layer, are concentrated towards the galactic plane, with a scale height of $\sim 100 \mathrm{pc}$.
- The Old Disk: stars in the disk older than a few Gyr, (i.e. A-type stars or later). The iron abundance of old disk stars range from about $[\mathrm{Fe} / \mathrm{H}]=-0.5$ to $[\mathrm{Fe} / \mathrm{H}]=0.3$.

Metallicity distribution for K dwarfs near the Sun. Note the sharp peak at $[\mathrm{Fe} / \mathrm{H}] \sim-0.2$ dex (Kotoneva and Flynn, 1999).


Population II stars are old (12-15 Gyr), includes the stellar halo and the bulge. The stellar halo and bulge differ in their metallicities --- the halo is metal weak $([\mathrm{Fe} / \mathrm{H}]<-1)$ while the bulge is generally metal rich (with a metallicity distribution quite like that of the disk).

|  | Population I | Population II |
| :---: | :---: | :---: |
| Age | Young disk : < 1 Gyr <br> Old disk : 1-10 Gyr | 12-15 Gyr |
| Metallicity [ $\mathrm{Fe} / \mathrm{H}$ ] | $-0.5<[\mathrm{Fe} / \mathrm{H}]<0.3$ | $\begin{aligned} & \text { Halo : }[-3.0,-1.0] \\ & \text { Bulge : }[-0.5,+0.5] \end{aligned}$ |
| Rotation, km/s | Young disk: 220 <br> Old disk : 180-200 | Halo : 0 km/s <br> Bulge : ~ 100 |
| Velocity dispersion in (U,V,W) km/s | Young disk : (40, 30,10) <br> Old disk : $(80,60,20)$ | Halo : (150, 100,100) <br> Bulge : ~ $130 \mathrm{~km} / \mathrm{s}$ |

